# A universal characterisation of locally determined $\omega$-colimits

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***Abstract*—Characterising colimiting $\omega$-cocones of projection pairs in terms of least upper bounds of their embeddings and projections is important to the solution of recursive domain equations. We present a universal characterisation of this local property as $\omega$-cocontinuity of locally continuous functors. We present a straightforward proof using the enriched Yoneda embedding. The proof can be generalised to Cattani and Fiore's notion of locality for adjoint pairs.**

## 15 MINUTE TALK OUTLINE

In the category theoretic solution of recursive domain equations [SP82], several technical results hinge upon the fact that the universality of $\omega$-cocones of projection pairs can be characterised *locally* in terms of least upper bounds (lubs) of their embeddings and projections. To fix terminology and notation, consider an $O$-category $K$. Let $K_{\mathrm{PR}}$ be the $O$-category consisting of *projection pairs* $f : A \to B$ given by $f = \langle f^L : A \to B, f^R : B \to A \rangle$ where $f^R \circ f^L = \mathrm{id}_A$ and $f^L \circ f^R \leq \mathrm{id}_B$.

**Definition** ([SP82, Definition 8]). *We say that a cocone $\langle C, c \rangle$ for an $\omega$-chain of projection pairs is* locally determined *when* $\sqcup_{n\in\mathbb{N}} c_n^L \circ c_n^R = \mathrm{id}_C$.

*When all colimiting $\omega$-cocones of projection pairs are locally determined, we say that the $O$-category* has locally determined $\omega$-colimits of projection pairs.

For example, the category $\omega\mathbf{CPO}$ of (not necessarily pointed) $\omega$-cpos and continuous functions has locally determined $\omega$-colimits.

The importance of these cocones lies in the fact that every locally determined cocone is colimiting. As any locally continuous functor $F : K \to L$ gives a continuous functor $F_{\mathrm{PR}} : K_{\mathrm{PR}} \to L_{\mathrm{PR}}$, given by $F_{\mathrm{PR}} f \coloneqq \langle F f^L, F f^R \rangle$, and locally determined $\omega$-cocones are preserved by these functors. Our contribution is to show the converse:

**Theorem.** *An $\omega$-colimiting cocone of projection pairs is locally determined if and only if it is preserved by every locally continuous functor.*

Let $\widehat{K}$ be the $O$-category of $O$-presheaves, namely locally continuous functors and natural transformations from $K^{\mathrm{op}}$ to $\omega\mathbf{CPO}$. Let $\mathbf{y} : K \to \widehat{K}$ be the enriched Yoneda embedding $\mathbf{y}x \coloneqq \omega\mathbf{CPO}(-, x)$. Then, following from general principles [Kel82, Section 2.4], $\mathbf{y}$ is locally continuous and fully faithful.

As is well-known, lubs and colimits in $O$-functor categories are given pointwise. The same argument shows that $\omega$-colimits of projection pairs are also given componentwise in $O$-functor categories. Therefore:

**Proposition.** *If $K$, $L$ are $O$-categories and $L$ has locally determined $\omega$-colimits of projection pairs, then so does the $O$-functor category $L^K$. In particular, every $O$-presheaf category $\widehat{K}$ has locally determined $\omega$-colimits.*

We complete the proof of our theorem. Let $\langle C, c \rangle$ be any colimiting cocone that is preserved (in particular) by the locally continuous Yoneda embedding. As $\widehat{K}$ has locally determined $\omega$-colimits:

$$\mathbf{y}\left(\sqcup_n c_n^L \circ c_n^R\right) = \sqcup_n \mathbf{y}(c_n^L) \circ \mathbf{y}(c_n^R) = \mathbf{y}(\mathrm{id})$$

By the faithfulness of the Yoneda embedding we deduce that $\langle C, c \rangle$ is locally determined.

**Corollary.** *An $O$-category has locally determined $\omega$-colimits of projection pairs if and only if every locally continuous functor from it yields an $\omega$-cocontinuous functor on projection pairs.*

Much of the theory of recursive domain equations generalises to *adjoint pairs* $\langle f^L, f^R \rangle$ where $f^L \circ f^R \leq \mathrm{id}$ and $\mathrm{id} \leq f^R \circ f^L$. Cattani et al. [CFW98], [CF07] generalised locally determined cocones as follows:

**Definition** (cf. [CF07, Theorem 1.5]). *We say that a cocone $\langle C, c \rangle$ for an $\omega$-chain $\Delta$ of adjoint pairs is* locally determined *when $\sqcup_{n\in\mathbb{N}} c_n^L \circ c_n^R = \mathrm{id}_C$ and, for all $n \in \mathbb{N}$:*

$$\sqcup_{m\geq n} \Delta^R_{n\leq m} \circ \Delta^L_{m\geq n} = c_n^R \circ c_n^L$$

*When all colimiting $\omega$-cocones of adjoint pairs are locally determined, we say that the $O$-category* has locally determined $\omega$-colimits of projection pairs.

As $\omega\mathbf{CPO}$ has locally determined $\omega$-colimits of adjoint pairs, almost identical proofs show the following:

**Theorem.** *An $\omega$-colimiting cocone of adjoint pairs is locally determined if and only if it is preserved by every locally continuous functor.*

**Corollary.** *An $O$-category has locally determined $\omega$-colimits of adjoint pairs if and only if every locally continuous functor from it yields an $\omega$-cocontinuous functor on adjoint pairs.*

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## References


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