A note on Fréchet diffrentiation under Lebesgue integrals

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Abstract. It is useful to interchange differentiation with integration. I was not able to easily find the result allowing to interchange Fréchet differentiation and Lebesgue integration. If you are familiar with a reference, please contact me.

Theorem 1. *Let $\mathbb{B}$ be a Banach space, U an open subset of $\mathbb{B}$, $\Omega = \langle |\Omega|, \Sigma_\Omega, \mu \rangle$ a measure space, A measurable set in Ω, and $f : \Omega \times \mathbb{B} \to \overline{\mathbb{R}}$ a (partial) function.*

Write $f_w(x)$ for $f(w, x)$.

Assume f satisfies the following conditions:

1. *For almost every $w \in A$, the function $f_w : \mathbb{B} \to \overline{\mathbb{R}}$ is Fréchet differentiable on U.*
2. *For all $x \in U$, $u \in \mathbb{B}$, the functions $f_-(x)$ and $Df_-(x)(u) : \Omega \to \overline{\mathbb{R}}$ are integrable on A.*
3. *There is an integrable function $\theta : \Omega \to \overline{\mathbb{R}}$ on A such that for almost every $w \in A$, for every $x \in U$:*
$$\left\| Df_w(x) \right\| \leq \theta(w)$$

Then the function $\int_{w \in A} f_w(-)\,\mathrm{d}\mu$ is Fréchet differentiable on U, and its derivative is given for every $x \in U$ and $u \in \mathbb{B}$ by:

$$\left(D \int_{w \in A} f_w(-)\,\mathrm{d}\mu \right)(x_0)(u) = \int_{w \in A} Df_w(x_0)(u)\,\mathrm{d}\mu$$

Proof. Consider any $x_0 \in U$. Take any $h_n \to 0$ in $\mathbb{B}$. We will use the dominated convergence theorem (DCT) to show that:

$$\int_{w \in A} \left| \frac{f_w(x_0 + h_n) - f_w(x_0) - Df_w(x_0)(h_n)}{\|h_n\|} \right| \to 0 \tag{1}$$

For every n, the integrand is integrable, hence measurable, and the constantly 0 function is measurable, and therefore we may use the variant of the DCT in which the conditions only hold almost surely.

By the definition of Fréchet differentiability, the integrand converges pointwise to 0. For the domination condition, we use the following fact:

If $f : \mathbb{B}_1 \to \mathbb{B}_2$ is Fréchet differentiable on U, then for every $x, y \in U$ for which U contains the set

$$[x,y] := \left\{ \lambda x + (1-\lambda y) \middle| 0 \le \lambda \le 1 \right\}$$

we have:

$$\left\| f(x) - f(y) \right\| \le \sup_{z\in[x,y]} \left\| Df(z) \right\| \cdot \|x-y\|$$

This fact holds in the more general case where f is *Gâtteux* differentiable on U (Ambrosetti and Prodi, 1995, Theorem 1.8).

Choose a ball contained in U and n large enough from which every $x_0 + h_n$ is in this ball. This ball contains $[x_0, x_0 + h_n]$ for every n, and so by the quoted fact, for almost every $w \in \Omega$:

$$\begin{aligned}\frac{\left\| f_w(x_0+h_n) - f_w(x_0) \right\|}{\|h_n\|} &\le \frac{1}{\|h_n\|} \cdot \sup_{z\in[x_0,x_0+h]} \left\| Df_w(z) \right\| \cdot \|h_n\| \\ &\le \sup_{z\in U} \left\| Df_w(z) \right\| \le \theta(w)\end{aligned}$$

Therefore, for almost every $w \in \Omega$:

$$\begin{aligned}&\left| \frac{f_w(x_0+h_n) - f_w(x_0) - Df_w(x_0)(h_n)}{\|h_n\|} \right| \\ &\qquad \le \frac{\left\| f_w(x_0+h_n) - f_w(x_0) \right\|}{\|h_n\|} + \frac{\left| Df_w(x_0)(h_n) \right|}{\|h_n\|} \\ &\qquad \le \theta(w) + \left\| Df_w(x_0) \right\| \le 2\theta(w)\end{aligned}$$

Therefore, by the DCT, (1) above holds.

We have thus shown that:

$$\lim_{h\to 0} \int_{w\in A} \left| \frac{f_w(x_0+h) - f_w(x_0) - Df_w(x_0)(h)}{\|h\|} \right| = 0$$

and therefore

$$\lim_{h\to 0} \int_{w\in A} \frac{f_w(x_0+h) - f_w(x_0) - Df_w(x_0)(h)}{\|h\|} \,\mathrm{d}\mu = 0$$

Consider the mapping:

$$\begin{aligned}\mathbf{T} : \mathbb{B} &\to \overline{\mathbb{R}} \\ u &\mapsto \textstyle\int_{w\in A} Df_w(x_0)(u) \,\mathrm{d}\mu\end{aligned}$$

By assumption 2, $\mathbf{T}$ is well-defined. It is linear in u, and we have:

$$\begin{aligned}\left| \mathbf{T}(u) \right| &\le \int_{w\in A} \left| Df_w(x_0)(u) \,\mathrm{d}\mu \right| \\ &\le \int_{w\in A} \left\| Df_w(x_0) \right\| \cdot \|u\| \,\mathrm{d}\mu \\ &\le \|u\| \cdot \int_{w\in A} \theta(w) \,\mathrm{d}\mu\end{aligned}$$

Therefore $\mathbf{T} : \mathbb{B} \to \mathbb{R}$ is a bounded linear operator.

Finally, for every h close enough to x_0, we have:

$$\begin{aligned}
\lim_{h\to 0} \frac{\left\| \int_{w\in A} f_w(x_0+h)\,\mathrm{d}\mu - \int_{w\in A} f_w(x_0)\,\mathrm{d}\mu - \mathbf{T}(h) \right\|}{\|h\|} \\
&= \lim_{h\to 0} \left| \int_{w\in A} \frac{f_w(x_0+h) - f_w(x_0) - Df_w(x_0)(h)}{\|h\|} \right| \\
&= 0
\end{aligned}$$

Therefore

$$\int_{w\in A} f_w(-)\,\mathrm{d}\mu(x_0)(u)$$

is Fréchet differentiable at x_0 and $\mathbf{T}$ is its derivative. $\square$

Bibliography

Ambrosetti, A. and Prodi, G. (1995). *A Primer of Nonlinear Analysis*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.