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ON AXIOMATIC SYSTEMS IN MATHEMATICS AND THEORIES IN PHYSICS

by

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PREFACE

A possible subtitle for this dissertation would be: "Studies in the technique and application of the simple theory of types". This may suggest that the interest of the work is strictly limited: but I hope on the contrary that the persevering reader will find - and perhaps be surprised to find - that the range of topics discussed is extremely wide; and that type theory provides an elegant and adequate symbolism for all those discussions. And I think that even the purely technical parts have a significance which goes beyond the particular system used.

There are several reasons for this fruitfulness of type theory. Firstly it is natural, in almost all branches of mathematics, to distinguish the different logical types of the quantities involved; and it is therefore right and proper that a system of mathematical logic which is to be generally useful should recognise those distinctions. Secondly the particular system of logic here used follows normal mathematical practice in several other important ways: namely, in its use of, though not in its notation for, functional abstraction; in its admission of a descriptions operator; in its extensional character; and in its employment of the deduction theorem. This system of logic is due to Alonso Church, and it has played an indispensable part in the clarification of my ideas.

Chapter I is concerned with the formal development of type theory. In Section 1 an account is given of Church's system, and also of an alternative system which I invented to facilitate the proofs of some of the theorems of later sections. I think I may have overestimated the degree of this facilitation, but the system has some intrinsic interest. The two systems considered are logically equivalent; I intended to give a demonstration of this equivalence in Appendix II, but my proof, though in principle straightforward, was rather tedious and inelegant, so I have omitted it.

In Section 2 the effect of maps of one type into another on objects of higher type is considered, and it is shown that the logical constants are all invariant under permutations of the type of individuals. This section represents the first steps in what, for want of a better term, I will call abstract structure theory; this is on the borderline of mathematics and symbolic logic, and it is an open question whether it is better to use the techniques of logic or of ordinary mathematics in discussing it. I think possibly the best answer is to use a formal logical notation, but to forgo the formalities of logical proof.

In Section 3 a method for consistently introducing new types - virtual types - is elaborated. As an example of its application the problem of forming quotient structures is discussed; this is another piece of abstract structure theory.

In Section 4 the making of inner models in the theory of types is discussed, and a rather general set of sufficient conditions for the existence of a model is given. This result is then used to give a short proof of the independence of the selection axiom.

It is clear to me, and I hope it will also become clear to the reader, that there is a fundamental distinction between elements of the system which can be described by closed formulae (i.e. formulae without free variables) and those which cannot be so described. In Section 5 an attempt is made to describe this distinction within the system itself. Theorem VIII shows that the description can be made by means of an infinite list of formulae. Since here it is the final result rather than the details which is important, I have proceeded rather informally, and have omitted a large number of formal proofs; I hope this treatment will make the work easy to follow, while yet convincing the reader that the result is correct.

Chapter II is concerned with applications; in view of the length of Chapter I, I have confined myself to giving a rough outline; I hope it will not prove too sketchy to be of value. Section 1 is philosophical; in it I dispute the popular opinion that propositions must either be synthetic or analytic, and also give a theory of names. Section 2 is concerned with the definition of the notion of mathematical structure; the definition given can be regarded as the ultimate generalisation of the ideas of Klein's Erlanger programme. Section 3 deals with the logical analysis of theories. I show how the nature of the concepts of a theory is revealed by a consideration of the way in which the corresponding logical objects occur in a formal statement of the theory; and I give examples of the application of the kind of analysis proposed.

There is not much in this dissertation which is really new. Some of the more important results of Chapter I - the invariance of the logical constants, the independence of the selection axiom in the theory of types, and the possibility of giving a formula for 'n-Clo' - have previously been obtained by members of the Polish school; in fact I had convinced myself of the truth of these results before I became aware of their work. Theorem V, the method by which it is proved, and the rather similar method used in proving Theorem III are, I think, original. The idea of Theorem III, and the notion of nonsense elements are due to A.M. Turing.

The debt which I owe to Bourbaki and to Philip Hall for the development of abstract structure theory is obvious; what is new here is perhaps the technique of extending the usual definitions to objects of arbitrarily high type. Similarly my debt to Klein and Weyl will be apparent. From the many writers on mathematical and natural philosophy who have influenced me I single out Poincaré, Russell and Ramsey.

Finally I must try and show the extent of my debt to A.M. Turing. He first called my somewhat unwilling attention to the system of Church, and to the importance of the deduction theorem. Much of the work on permutations and invariance, and on the form of theories was done in conjunction with him. Without his encouragement I should long ago have given way to despair; without his criticism my ideas would have remained shallow and obscure.

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CHAPTER I.

Section 1. The system of logic.

We are going to consider certain kinds of theoretical system, and so we wish to be able to characterise and to exemplify such systems as clearly as possible; and this is most conveniently done by introducing a system of formal logic in which the formation of expressions and the inference from one expression to another are governed by definite rules. A theoretical system may then be described in terms of some particular class of expressions. The systems of logic must be sufficiently wide and flexible, so that any argument of classical mathematics may be represented within it; in fact, the less specialised it is, the better will it suit our purpose. But we must choose one particular system - 'pour fixer les idées'; and having made the choice we shall use all the technical facilities it provides. This means that some of our results will be theorems about the particular system used, and all the results will only be proved for that system: but the most important results will also be true for any other system that is capable of bearing the same - intuitive - interpretation as does the chosen system.

There seem to be two general kinds of system suitable for our purpose: the set theoretical kind - for example that used by Gödel in 'The consistency of the continuum hypothesis', or that used by Quine in his 'Mathematical logic'; and the type theory kind - for example that used in 'Principia Mathematica'. The advantages of the second kind are many, its disadvantages few. For, firstly, we are primarily interested in the applications of symbolic logic to mathematics and to theoretical science; and it is then important to preserve the distinctions between objects of different logical type - for example, the distinction between functions and functionals. Secondly, the axioms and the rules of the system we adopt are closer to normal mathematical argument than are those of any of the set-theoretic systems, so that translation into and from the formal system can be done with little effort. Thirdly, we shall see that by introducing different basic types - that is, different types of 'individuals' - some of the disadvantages of type theory may be overcome. (This process is analogous to the representation of any given axiomatic system within the functional calculus of the first order.) The fact that, in type theory, many definitions and theorems have to be stated separately in each type, has sometimes been urged as an objection; but in practice it does not lead to much complication, because one can use symbols to stand for arbitrary types in just the same way that one uses a free variable; so that, for example, $Q_{o\alpha\alpha}$ is interpreted as the identical relation between objects in <u>any</u> type α. Lastly - for what it is worth - the consistency of a type theoretical system seems less open to that used by Quine in his 'Mathematical logic'; and the type theory kind - for example that used in 'Principia Mathematica'. The advantages of the second kind are many, its disadvantages few. For, firstly, we are primarily interested in the applications of symbolic logic to mathematics and to theoretical science; and it is then important to preserve the distinctions between objects of different logical type - for example, the distinction between functions and functionals. Secondly, the axioms and the rules of the system we adopt are closer to normal mathematical argument than are those of any of the set-theoretic systems, so that translation into and from the formal system can be done with little effort. Thirdly, we shall see that by introducing different basic types - that is, different types of 'individuals' - some of the disadvantages of type theory may be overcome. (This process is analogous to the representation of any given axiomatic system within the functional calculus of the first order.) The fact that, in type theory, many definitions and theorems have to be stated separately in each type, has sometimes been urged as an objection; but in practice it does not lead to much complication, because one can use symbols to stand for arbitrary types in just the same way that one uses a free variable; so that, for example, $Q_{o\alpha\alpha}$ is interpreted as the identical relation between objects in <u>any</u> type α. Lastly - for what it is worth - the consistency of a type theoretical system seems less open to doubt than does the consistency of the various set-theoretic systems.

The system we shall use is substantially the same as that introduced by Church in Church (1). We shall actually describe two different systems, and show that they are equivalent. The first is Church's with some very minor modifications; the second is useful because the proofs of some metalogical theorems are shorter for it than for the first system, while the theorems themselves can be taken over from one system to the other. Church's system is a version of the <u>simple</u> theory of types; but we shall see that it is possible to make definitions which are rather analogous to the definitions of the 'orders' in the branched theory of types, and which, like those, serve to show that contradictions will not arise from paradoxes similar to Grelling's or Richard's.

(A) <u>Type symbols</u>.

These are made up from lower case greek letters, and '(', and')'. Letters, other than particularly designated ones, are used as type-symbol variables; in particular α, β, γ, δ, ε, are used in this way. Any one particular type symbol may be substituted for each of the occurrences of a type-symbol variable in a logical or metalogical statement.

The rules for the formation of type symbols are as follows:

1) If α and β are type symbols, then $(\alpha\beta)$ is a <u>complex</u> type symbol; and the <u>parts</u> of $(\alpha\beta)$ are α, β, the parts of α and the parts of β. $(\alpha\beta)$ designates the type of functions whose arguments range over type β, and whose values lie in type α.

2) o is a <u>basic</u> type symbol. (It designates the type of propositions, in which there are just two objects, representing truth and falsehood.)

3) ι is a basic type symbol. (It designates the type of individuals.) Sometimes we shall need further basic type symbols κ, λ; and we shall then suppose that the definitions and statements made for type ι are extended to the types κ, λ.

4) Certain further type symbols (e.g. ν and ρ) will be introduced by the method of virtual types (see section 3).

5) No expression is a type symbol unless it is one in virtue of 1) - 4).

Brackets may be conventionally omitted from type symbols. By 'a pair of brackets' we mean a left and a right bracket between which there are an equal number of left and right brackets. A pair of brackets may be omitted from a type symbol if there is a left bracket or nothing at all immediately to the left of the left bracket of the pair. This convention allows omitted brackets to be restored in an unambiguous way. Thus we write $\alpha\beta(\gamma\delta\varepsilon)$ instead of $((\alpha\beta)((\gamma\delta)\varepsilon))$.

(B) Well formed formulae.

The expressions of the two systems (C) and (G) which we are describing are made up from the following symbols:

(a) Improper symbols: (and λ.

(b) Constant symbols: N_{oo}, A_{ooo}, $\iota_{o(oo)}$, $\iota_{\iota(o\iota)}$, C_o, C_ι; these are common to both systems. Also (C) has the symbols $\Pi_{o(o\alpha)}$, and (G) has the symbols $Q_{o\alpha\alpha}$.

(c) Symbols for variables: $\underline{a}_\alpha, \ldots, \underline{z}_\alpha, \underline{a}'_\alpha, \ldots$.

The meaningful expressions of the systems are the well formed formulae; the rules of formation of these are as follows:

1) Any constant or variable symbol standing on its own is a well formed formula, and its type is that designated by its suffix.

2) If $\underset{\sim}{F}_{\alpha\beta}$, $\underset{\sim}{A}_\beta$ are well formed formulae of types $\alpha\beta$ and β, then $(\underset{\sim}{F}_{\alpha\beta}\underset{\sim}{A}_\beta)$ is a well formula of type α; and the parts of $(\underset{\sim}{F}_{\alpha\beta}\underset{\sim}{A}_\beta)$ are $\underset{\sim}{F}_{\alpha\beta}$ and its parts, and $\underset{\sim}{A}_\beta$ and its parts.

3) If $\underset{\sim}{A}_\alpha$ is a well formed formula, then $(\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha)$ is a well formed formula of type $\alpha\beta$, and all the occurrences of the variable $\underset{\sim}{x}_\beta$ in it are bound occurrences. The parts of $(\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha)$ are $\underset{\sim}{A}_\alpha$ and its parts.

4) A formula is well formed only if it is so in virtue of 1) - 3); and the occurrencesof a variable are bound only if they are so in virtue of 3). Occurrences of a variable which are not bound are free; a variable is called bound or free according to the nature of its occurrences.

The process described in 2) is to be interpreted as the application of the function represented by $\underset{\sim}{F}_{\alpha\beta}$ to the argument represented by $\underset{\sim}{A}_{\beta}$, giving the value represented by $(\underset{\sim}{F}_{\alpha\beta}\,\underset{\sim}{A}_{\beta})$. The process described in 3) is to be interpreted as the functional abstraction of the formula $\underset{\sim}{A}_{\alpha}$ with respect to the variable $\underset{\sim}{x}_{\beta}$; i.e. $(\lambda\underset{\sim}{x}_{\beta}\underset{\sim}{A}_{\alpha})$ represents the function whose values for a given argument are represented by the expression obtained by substituting for $\underset{\sim}{x}_{\beta}$ in $\underset{\sim}{A}_{\alpha}$ an expression representing the given argument. We shall in future use 'formula' to mean 'well formed formula'.

(C) <u>Conventions and abbreviations</u>.

As in the preceding paragraph, when making statements about the system we use bold face capitals $\underset{\sim}{A}_{\alpha}$, $\underset{\sim}{B}_{\beta}$, ..., to stand for arbitrary well formed formulae, and lower case bold face letters to stand for arbitrary variable symbols. We allow the other symbols of the system to stand for themselves, as also do such further symbols as are introduced as abbreviations. We always omit the suffix from all the occurrences of a bound variable except from the binding (i.e. the leftermost) occurrence. We often omit the suffix from constant symbols and from those introduced as abbreviations.

We omit brackets, with association to the left, in exactly the same way as described for type symbols. Further we omit a pair of brackets if the left bracket occurs immediately to the right of the binding occurrence of a variable and the scope of the brackets is the same as the scope of the binding occurrence. We omit 'λ' when it occurs immediately to the right of a binding occurrence of a variable which has the same scope. When there are one or more consecutive binding occurrences of variables, we place a '.' immediately to the right of the rightermost such occurrences. Thus we write

$$(\lambda \underline{x}_{\alpha}\underline{y}_{\alpha\alpha}.\underline{f}_{\alpha\alpha\alpha}\underline{x}(\underline{f}_{\alpha\alpha\alpha}(\underline{y}\underline{x})\underline{x}))\underline{z}_{\alpha}\underline{w}_{\alpha\alpha}$$

for $(((\lambda \underline{x}_{\alpha}(\lambda \underline{y}_{\alpha\alpha}((\underline{f}_{\alpha\alpha\alpha}\underline{x}_{\alpha})((\underline{f}_{\alpha\alpha\alpha}(\underline{y}_{\alpha\alpha}\underline{x}_{\alpha}))\underline{x}_{\alpha}))))\underline{z}_{\alpha})\underline{w}_{\alpha\alpha})$.

We now introduce a number of abbreviations. The metalogical sign ' $\longrightarrow$ ' stands for 'is an abbreviation for'; any formula containing abbreviations can be expanded into a formula containing only the symbols of the system. But, for the most part (and certainly whenever it has a type suffix), the newly introduced symbol represents a particular (constant) element in the interpretation of the system; and so, for example, '$\mathrm{Num}_{o\alpha'} \longrightarrow \underset{\sim}{A}_{o\alpha'}$ ' may be read as 'the element represented by 'Num' is defined by the formula $\underset{\sim}{A}_{o\alpha'}$'. Elements introduced in this way will be represented by single roman capitals, or by three letter combinations which are intended to bear some relation to the nature of the element introduced; for example 'Num' is short for 'number'. A dictionary of these signs is given at the back.

$$\sim \underset{\sim}{P}_o \rightarrow N_{oo}\underset{\sim}{P}_o$$

$$\underset{\sim}{P}_o \mathbin{\&} \underset{\sim}{Q}_o \rightarrow A_{ooo}\underset{\sim}{P}_o\underset{\sim}{Q}_o$$

$$\underset{\sim}{P}_o \vee \underset{\sim}{Q}_o \rightarrow \sim(\sim\underset{\sim}{P}_o \mathbin{\&} \sim\underset{\sim}{Q}_o)$$

$$\underset{\sim}{P}_o \supset \underset{\sim}{Q}_o \rightarrow (\sim\underset{\sim}{P}_o \vee \underset{\sim}{Q}_o)$$

$$\underset{\sim}{P}_o \equiv \underset{\sim}{Q}_o \rightarrow (\underset{\sim}{P}_o \supset \underset{\sim}{Q}_o) \mathbin{\&} (\underset{\sim}{Q}_o \supset \underset{\sim}{P}_o)$$

$$T_o \rightarrow C_o \equiv C_o$$

$$F_o \rightarrow \sim T_o$$

$$(\underset{\sim}{x}_\alpha)(\underset{\sim}{A}_o) \rightarrow \Pi_{o(o\alpha)}(\lambda\underset{\sim}{x}_\alpha.\underset{\sim}{A}_o) \qquad \text{System (C)}$$

$$(\underset{\sim}{x}_\alpha)(\underset{\sim}{A}_o) \rightarrow Q_{o(o\alpha)(o\alpha)}(\lambda\underset{\sim}{x}_\alpha.\underset{\sim}{A}_o)(\lambda\underline{y}_\alpha.T_o) \qquad \text{System (G)}$$

$$(E\underset{\sim}{x}_\alpha)(\underset{\sim}{A}_o) \rightarrow \sim(\underset{\sim}{x}_\alpha)(\sim\underset{\sim}{A}_o)$$

$$\Sigma_{o(o\alpha)} \rightarrow \lambda\underline{f}_{o\alpha}.(E\underline{x}_\alpha)(\underline{fx})$$

$$\underset{\sim}{A}_\alpha = \underset{\sim}{B}_\alpha \rightarrow (\underline{f}_{o\alpha})(\underline{f}\underset{\sim}{A}_\alpha \supset \underline{f}\underset{\sim}{B}_\alpha) \qquad \text{System (C)}$$

$$\underset{\sim}{A}_\alpha = \underset{\sim}{B}_\alpha \rightarrow Q_{o\alpha\alpha}\underset{\sim}{A}_\alpha\underset{\sim}{B}_\alpha \qquad \text{System (G)}$$

$$\iota^{\alpha\beta}_{\alpha\beta(o(\alpha\beta))} \rightarrow \lambda\underline{f}_{o(\alpha\beta)}\underline{x}_\beta.\,\iota^\alpha_{\alpha(o\alpha)}(\lambda\underline{y}_\alpha.(E!\underline{g}_{\alpha\beta})(\underline{gx} = \underline{y} \mathbin{\&} \underline{fg}))$$

$$C_{\alpha\beta} \rightarrow \lambda\underline{x}_\beta.C_\alpha$$

(Since ι and C are constants of the system for the basic types, the two definitions above yield $\iota^\alpha_{\alpha(o\alpha)}$ and C_α for every type symbol α).

$$(\imath\underset{\sim}{x}_\alpha)(\underset{\sim}{A}_o) \rightarrow \iota^\alpha_{\alpha(o\alpha)}(\lambda\underset{\sim}{x}_\alpha.\underset{\sim}{A}_o)$$

$$(E!\underset{\sim}{x}_\alpha)(\underset{\sim}{A}_o) \rightarrow (E\underset{\sim}{x}_\alpha)((\underline{y}_\alpha)((\lambda\underset{\sim}{x}_\alpha.\underset{\sim}{A}_o)\underline{y} \supset \underline{y} = x) \mathbin{\&} \underset{\sim}{A}_o)$$

$$J^\alpha_{o(o\alpha)} \rightarrow \lambda\underline{f}_{o\alpha}.(E!\underline{x}_\alpha)(\underline{fx})$$

$$I^\alpha_{\alpha\alpha} \rightarrow \lambda\underline{x}_\alpha.\underline{x}$$

$$K_{\alpha\beta\alpha} \rightarrow \lambda\underline{x}_\alpha\underline{y}_\beta.\underline{x}$$

$$W_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)} \rightarrow \lambda\underline{f}_{\alpha\beta\gamma}\underline{g}_{\beta\gamma}\underline{x}_\gamma.\underline{fx}(\underline{gx})$$

$$V_{o(o\alpha\alpha)} \rightarrow \lambda\underline{q}_{o\alpha\alpha}.(\underline{x}_\alpha,\underline{y}_\alpha,\underline{z}_\alpha)(\underline{qxx} \;.\&.\; \underline{qxy} \supset \underline{qyx} \;.\&.\; \underline{qxy} \mathbin{\&} \underline{qyz} \supset \underline{qxz})$$

$$0_{\alpha'} \rightarrow \lambda \underline{f}_{\alpha\alpha} \underline{x}_{\alpha} . \underline{x}$$

$$1_{\alpha'} \rightarrow \lambda \underline{f}_{\alpha\alpha} \underline{x}_{\alpha} . \underline{fx}$$

$$2_{\alpha'} \rightarrow \lambda \underline{f}_{\alpha\alpha} \underline{x}_{\alpha} . \underline{f}(\underline{fx})$$

where α' is an abbreviation for $\alpha\alpha(\alpha\alpha)$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_{\alpha'\alpha'} \rightarrow \lambda \underline{m}_{\alpha'} \underline{f}_{\alpha\alpha} \underline{x}_{\alpha} . \underline{f}(\underline{mfx})$$

$$\mathrm{Num}_{o\alpha'} \rightarrow \lambda \underline{m}_{\alpha'} . (\underline{f}_{o\alpha'})(\underline{f}0_{\alpha'} \mathbin{\&} (\underline{n}_{\alpha'})(\underline{fn} \supset \underline{f}(\underline{Sn})) . \supset \underline{fm})$$

Unless otherwise stated all the above definitions apply to both systems. The introduction of binary connectives which stand between the formulae they connect complicate the conventions for omission of brackets, and we shall not attempt to introduce strict conventions (which would probably be forgotten as soon as made). Instead we shall rely on common sense, normal usage, and the meanings of formulae, to make it clear how the parts of a formula are connected together. It is more important (especially with long formulae) that the interpretation should be clear, than that the reintroduction of brackets should be a purely mechanical process. We treat the first occurrence of a quantified variable, or a variable in a description - i.e. '$(\underline{x}_{\alpha})$', '$(E\underline{x}_{\alpha})$', or '$(\iota\underline{x}_{\alpha})$' - as a binding occurrence, preserving the suffix there, and dropping it from the other related occurrences. We always omit a pair of brackets the left bracket of which occurs between two such binding occurrences of variables. We always write '$(\underline{x}_{\alpha})(\underline{y}_{\alpha})(\underset{\sim}{A}_{o})$' for '$(\underline{x}_{\alpha})((\underline{y}_{\alpha})(\underset{\sim}{A}_{o}))$' where '$(\underline{x}_{\alpha})$' and '$(\underline{y}_{\alpha})$' are such binding occurrences; if they are of the same kind we may run them both together, writing '$(E\underset{\sim}{x}_\alpha, \underset{\sim}{y}_\alpha)(\underset{\sim}{A}_o)$' for '$(E\underset{\sim}{x}_\alpha)(E\underset{\sim}{y}_\alpha)(\underset{\sim}{A}_o)$'.

We regard the logical connectives '&', 'v', '$\supset$', '$\equiv$', as being stronger than any others, so that for example, we write $\sim F \ \& \ T = T$,

instead of $(\sim F) \ \& \ (T = T)$;

and we regard '$\supset$' and '$\equiv$' as stronger than '&' and 'v', so that we write $A \ \& \ B \supset B \vee D$

instead of $(A \ \& \ B) \supset (B \vee D)$;

we emphasise this fact by placing dots beside a 'strong' connective. Further the associativity of '&' and 'v', and the fact that expressions on either side of a logical connective must be of type o, imply further possible omissions of brackets.

(D) <u>Rules of inference</u>.

Rules I, II, III, V, are the same for both (C) and (G).

I To replace any part $\underset{\sim}{M}_\alpha$ of a formula by the result of substituting a variable $\underset{\sim}{y}_\beta$ for $\underset{\sim}{x}_\beta$ throughout $\underset{\sim}{M}_\alpha$, provided that $\underset{\sim}{x}_\beta$ is not a free variable of $\underset{\sim}{M}_\alpha$ and $\underset{\sim}{y}_\beta$ does not occur in $\underset{\sim}{M}_\alpha$. (i.e. to infer from a given formula the formula obtained by this replacement).

II To replace any part $((\lambda \underset{\sim}{x}_\beta \underset{\sim}{M}_\alpha)\underset{\sim}{N}_\beta)$ of a formula by the result of substituting $\underset{\sim}{N}_\beta$ for $\underset{\sim}{x}_\beta$ throughout $\underset{\sim}{M}_\alpha$, provided that the bound variables of $\underset{\sim}{M}_\alpha$ are distinct both from $\underset{\sim}{x}_\beta$, and from the free variables of $\underset{\sim}{N}_\beta$.

III Where A_α is the result of substituting N_β for x_β throughout M_α, to replace any part A_α of a formula by $((\lambda x_\beta M_\alpha)N_\beta)$, provided that the bound variables of M_α are distinct both from x_β and from the free variables of N_β.

V From $A_o \supset B_o$ and A_o to infer B_o.

The remaining rules are different in the two systems, and so are given a prefix 'C' or 'G'.

C.IV From a formula M_o to infer the result of substituting a formula A_β for the free occurrences of x_β throughout M_o, provided that the bound variables of M_o are distinct from x_β and the free variables of A_β.

C.VI. From M_o to infer $(x_\alpha)(M_o)$.

G.IV From $A_\beta = B_\beta$ to infer $F_{\alpha\beta} A_\beta = F_{\alpha\beta} B_\beta$.

G.VI From $A_\alpha = B_\alpha$ to infer $\lambda x_\beta . A_\alpha = \lambda x_\beta . B_\alpha$.

G.VII From A_o to infer $A_o = T$, and vice versa.

Rules I-III are the rules of λ-conversion; rules C.IV, C.VI, and G.VI, are, respectively, the rules for the substitution for, the quantification of, and the abstraction of, the variable x_β.

(E) Axioms.

(P) 1) $p \vee p \supset p$;

2) $p \supset p \vee q$;

3) $p \vee q \supset q \vee p$;

4) $(p \supset q) \supset (r \vee p \supset r \vee q)$;

5) $C_o = C_o$. (System (G) only.)

These are the axioms of the propositional calculus; p, q, r, are all variables of type o.

(D) 1) $(E!x_\eta)(f_{o\eta}x) \supset f(\iota_{\eta(o\eta)}\, f_{o\eta})$;

2) $\sim(E!x_\eta)(f_{o\eta}x) \supset \iota_{\eta(o\eta)} f_{o\eta} = C_\eta$.

These are the axioms of description; here η is either o or ι, so that there are altogether four of the axioms.

(E) $(x_\beta)(f_{\alpha\beta}x = g_{\alpha\beta}x) \supset (f_{\alpha\beta} = g_{\alpha\beta})$.

This represents an infinite list of axioms - the axioms of extensionality - there being one for every complex type.

(T) $p_o \equiv q_o . \supset p_o = q_o$.

This axiom asserts that there are only two elements in type o, viz. T and F; it may also be regarded as a further axiom of extensionality.

(A) $\Pi_{o(o\alpha)} f_{o\alpha} \supset f_{o\alpha} x_\alpha$; System (C)

(Q) $x_\alpha = y_\alpha . \supset . f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha$ System (G)

These infinite lists are the axioms of universality and equality respectively.

(S) $(Ej_{\alpha(o\alpha)})(f_{o\alpha})(\Sigma f \supset f(jf))$.

These are the axioms of selection; we shall not often use them, and when we do we shall always make express mention of the fact.

(I) 1) $(Ex_\iota)(Ey_\iota)(x \neq y)$;

2) Num $x_{\iota'}$ & Num $y_{\iota'}$ & $S_{\iota'\iota'}x_{\iota'} = S_{\iota'\iota'}y_{\iota'} . \supset x_{\iota'} = y_{\iota'}$.

These are the axioms of infinity for the type ι ; Newman and Turing (1) have shown that the corresponding propositions for any type α whose parts are not all o, may be proved from (I).

(F) <u>Notes and comments</u>.

1) We have said that the above axioms contain some infinite lists; this is the usual view, and according to it the rules must also be regarded as infinite lists. But it is not necessary to accept it. For if we distinguish between the constant type symbols o and ι, and the variable type symbols $\alpha, \beta, \ldots$, and add as further rule: 'From A_o to infer B_o where B_o is obtained from A_o by substituting a given type symbol for a particular variable type symbol wherever that variable type symbol occurs in A_o', then the infinite lists are avoided.

2) The constants C do not appear in the system (C') of Church (1); they may be described as 'nonsense elements'. They were first introduced, I believe, by Turing in Turing (1). C_o could of course be defined as, say, F (whose definition can be made independently of C); but it is convenient not to do this, for then it remains indeterminate whether C is equal to T, or equal to F. This means that although the logic represented by the type o is strictly two-valued, it is possible to express ignorance of the truth or falsity of a proposition P_o by asserting '$P_o \equiv C_o$'. Of course this is not

entirely satisfactory since given two such dubious propositions p_o and q_o, one can infer '$p_o \equiv q_o$'; but one cannot make use (with modus ponens) of this equivalence, since if one of the propositions ceases to be dubious (e.g. by the discovery of a proof for it), the equivalence ceases to be provable. The reason for introducing the nonsense elements lies of course in the axiom (D2) (where, it should be noted, implication, not equivalence, is asserted). This axiom makes $\iota_{\iota(o\iota)}$ invariant under those permutations of the individuals which leave C_ι invariant. (See section 2.) It is shown in Appendix I that the C's may also be defined in the system (C').

3) In system (G) it will be noted that there is no rule of substitution, although of course such a rule can be derived from the given rules (see subsection (G) below); it is in this derivation that the slightly absurd looking G.VII is required.

In either system the following proposition is provable:

$$A_{ooo} = \lambda \underline{p}_o\ \underline{q}_o.(\underline{f}_{ooo})(\underline{fpq} = \underline{f}TT)\ ;$$

but in system (G) none of the abbreviations occurring in the expression on the right hand side involve A_{ooo}, so that the above could in fact be taken as a definition, and A_{ooo} omitted from the list of constants of (G). It should then, I suppose, be possible to produce rules and axioms involving only T_o, N_{oo}, Q_{ooo}, and involving them in a simpler way than do rule V and the axioms (P); but I have not been able to find a set of such rules and axioms of sufficient elegance to be worth reproducing. It is of course well known that the propositional calculus on its own cannot be founded on N, T, and Q; but here we have the higher types to play with.

4) It is shown in Appendix I that the axiom

$$(\underline{x}_\alpha)(\underline{p}_o \vee \underline{f}_{o\alpha}\underline{x}) \supset \underline{p}_o \vee (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x}),$$

of Church's system (C') follows from the other axioms, provided these include (E) and (T).

5) It might be thought that rules G.IV and G.VI made the axioms (Q) and (E) unnecessary in system (G). But firstly G.VI is not as strong as (E); for example, we have:

$$C_o = C_o \qquad \text{by (P) and (T)}$$

$$(\lambda \underline{p}_o . \underline{f}_{o\alpha}\underline{x}_\alpha) C_o = (\lambda \underline{p}_o . \underline{f}_{o\alpha}\underline{x}_\alpha) C_o \qquad \text{by G.IV}$$

$$\underline{f}_{o\alpha}\underline{x}_\alpha = \underline{f}_{o\alpha}\underline{x}_\alpha \qquad \text{by II}$$

$$\underline{f}_{o\alpha}\underline{x}_\alpha = (\lambda \underline{y}_\alpha . \underline{f}_{o\alpha}\underline{y})\underline{x}_\alpha \qquad \text{by III}$$

$$\lambda \underline{x}_\alpha . \underline{f}_{o\alpha}\underline{x} = \lambda \underline{x}_\alpha . (\lambda \underline{y}_\alpha . \underline{f}_{o\alpha}\underline{y})\underline{x} \qquad \text{by G.VI.}$$

But we cannot prove

$$\lambda \underline{x}_\alpha . \underline{f}_{o\alpha}\underline{x} = \underline{f}_{o\alpha}$$

without using (E). Secondly, both (Q) and (E) are necessary if we require the deduction theorem to hold for system (G).

6) In both systems $(\underline{p}_o = \underline{q}_o) \equiv (\underline{p}_o \equiv \underline{q}_o)$ is provable, and therefore we shall use either ' = ' or ' $\equiv$ ' between propositions, according as to which is most convenient.

7) In order to show that the two systems are equivalent, we have to define a method of translation from one to the other.

We denote the translation of (C) into (G) by T', and that of (G) into (C) by T'', and use T to stand for either of these. $(A_\alpha)^T$ is the translation into one system of the formula A_α belonging to the other; $(A_\alpha)^T$ is defined inductively as follows:

1) If A is a variable, or a constant other than $\Pi_{o(o\alpha)}$ or $Q_{o\alpha\alpha}$, then $(A_\alpha)^T$ is A_α.

2) $(E_{\alpha\beta}A_\beta)^T$ is $(E_{\alpha\beta})^T(A_\beta)^T$

3) $(\lambda x_\beta A_\alpha)^T$ is $\lambda x_\beta (A_\alpha)^T$

4) $(\Pi_{o(o\alpha)})^{T'}$ is $\lambda f_{o\alpha}(Q_{o(o\alpha)(o\alpha)}\, f\, (\lambda x_\alpha T_o))$

5) $(Q_{o\alpha\alpha})^{T''}$ is $\lambda x_\alpha y_\alpha (\Pi_{o(o(o\alpha))}(\lambda f_{o\alpha}(fx \supset fy)))$

In Appendix II it is shown that:

a) If A_o is an axiom of one system, then $(A_o)^T$ is provable in the other.

b) If A_o can be inferred from B_o by one of the rules, then $(A_o)^T$ can be inferred from $(B_o)^T$. Hence provable propositions are translated into provable propositions.

c) $((A_\alpha)^{T'})^{T''} = A_\alpha$, and $((A_\alpha)^{T''})^{T'} = A_\alpha$ are provable.

d) $(A_\alpha = B_\alpha)^T \equiv ((A_\alpha)^T = (B_\alpha)^T)$, and $((x_\alpha)(A_o))^T \equiv (x_\alpha)((A_o)^T)$, are provable. It follows that it is a matter of indifference whether we regard a formula written in ordinary logical notation as belonging to one system or the other.

The equivalence expressed in a) - d) is based on, but is rather stronger than that introduced by Turing in Turing (1). He shows there that his definition defines an equivalence relation between systems, and so it follows that system (G) is equivalent, in his sense, to his nested type system; for he has proved that the latter is equivalent to (C).

8) We write '$\underset{\sim}{A}_o \vdash \underset{\sim}{B}_o$' as an abbreviation for '$\underset{\sim}{A}_o$ can be derived from $\underset{\sim}{B}_o$ by applying the rules and axioms'; and '$\vdash \underset{\sim}{A}_o$' for '$\underset{\sim}{A}_o$ is provable'. (A rather more accurate version of the meaning of these signs is due to Russell; '$\vdash \underset{\sim}{A}_o$' means that if $\underset{\sim}{A}_o$ is not provable then the author stands convicted of error.) The proofs we shall give will be of different kinds:

a) True formal proofs;

b) Proofs of propositions that involve a variable type symbol, and which proceed by an induction over the construction of this type symbol; such proofs may be regarded as either showing how a formal proof - for any given type symbol - could be constructed, or as constituting a formal proof in a system containing the additional rule: 'From $\underset{\sim}{A}^{o}_{o}$ & $\underset{\sim}{A}^{\iota}_{o}$ & $(\underset{\sim}{A}^{\alpha}_{o}$ & $\underset{\sim}{A}^{\beta}_{o} . \supset . \underset{\sim}{A}^{\alpha\beta}_{o})$ to infer $\underset{\sim}{A}^{\gamma}_{o}$' (where $\underset{\sim}{A}^{\gamma}_{o}$ represents the given proposition for the type γ);

c) Proofs of propositions of a given general form; these proceed using metalogical symbols, and can be regarded as establishing schemes for formal proofs, or as establishing derived rules of inference.

We shall usually set out proofs - of whichever kind - line by line. On the left hand side there appears a consecutive numbering of the steps of the proof, with a prefixed letter to indicate the nature of the step; the letters we use are: 'H' to indicate the making of an hypothesis; 'A' to indicate the introduction of an abbreviation; and 'P' to indicate a proposition which we desire to prove; steps without prefixed letters are propositions which are consequences of previous steps (excluding, of course previous steps having a prefix P).

On the right hand side appears some indication of the way in which the proposition occurring in the middle is derived. The most important method of proof is the deduction theorem; this is, in effect, a derived rule of inference: 'If from the <u>hypothesis</u> $\underset{\sim}{A}_o$ one can infer $\underset{\sim}{B}_o$ by application of the rules and axioms, but without generalising on, or substituting for, or abstracting the free variables of $\underset{\sim}{A}_o$, then one can infer $\underset{\sim}{A}_o \supset \underset{\sim}{B}_o$'. The free variables of $\underset{\sim}{A}_o$ are said to be <u>restricted</u> by hypothesis; for each hypothesis made in the course of a proof we indicate on the right hand side the variables which are restricted by it, and until the deduction theorem has been applied these variables appear without suffixes; this convention (which, like the similar one concerning bound variables, is due to Turing) serves to indicate those variables which may <u>not</u> be substituted for, etc. The step at which we apply the deduction theorem, and so pass from conditional to provable propositions is called 'the elimination of the hypothesis' and is indicated by placing the number of the hypothesis in brackets on the right hand side. Thus a simple application of the deduction theorem might appear as follows:

H.1 $\quad \underset{\sim}{A}_o[\underline{x}_\alpha, \underline{y}_\beta] \quad (\underline{x}, \underline{y})$

....

n $\quad \underset{\sim}{B}_o[\underline{x}]$

n + 1 $\quad \underset{\sim}{A}_o[\underline{x}_\alpha, \underline{y}_\beta] \supset \underset{\sim}{B}_o[\underline{x}] \quad$ (H.1).

(We write $\underset{\sim}{A}_o[\underline{x}_\alpha, \underline{y}_\beta]$ etc. to indicate that the variables $\underline{x}_\alpha$ and $\underline{y}_\beta$ occur free in the proposition $\underset{\sim}{A}_o$; of course both or neither might also occur in $\underset{\sim}{B}_o$). Another kind of argument which is very frequent is of this kind:

1 $\quad (E\underline{x}_\alpha)(\underset{\sim}{A}_o[\underline{x}])$

H.2 $\quad \underset{\sim}{A}_o[\underline{x}_\alpha] \quad (\underline{x})$

....

n $\quad \underset{\sim}{B}_o$

($\quad \underset{\sim}{A}_o[\underline{x}_\alpha] \supset B$)

($\quad (E\underline{x}_\alpha)(\underset{\sim}{A}_o[\underline{x}_\alpha]) \supset \underset{\sim}{B}_o$)

n + 1 $\quad \underset{\sim}{B}_o \quad$ (H.2), 1.

The steps in brackets would be omitted, and the right hand side of step n + 1 is put in to show that proposition 1 has been used after the elimination of H.2. A particular case of this form of argument is when H.2 is of the form:

$$\underline{x}_\alpha = \underset{\sim}{M}_\alpha,$$

where $\underline{x}_\alpha$ does not occur free in $\underset{\sim}{M}_\alpha$; the proposition 1 is then trivially provable, and it and its mention in step n + 1 would both be omitted. The introduction of abbreviations can be effected in this way.

The convention of indicating on the right of a hypothesis the variables which it restricts allows one to introduce also variables which are not restricted by it, but which are regarded as being bound by a universal quantifier; thus we may write:

H.1 $\quad \underset{\sim}{A}_o[\underline{x}_\alpha, \underline{y}_\beta] \quad (\underline{x})$,

instead of

H.1 $\quad (\underline{y}_\beta)(\underset{\sim}{A}_o[\underline{x}_\alpha, \underline{y}]) \quad (\underline{x})$.

We use the number of a theorem, or of step, to stand for the appropriate proposition, and we sometimes use 'L.H.S.', 'R.H.S.' to stand for the proposition which is on the left or the right of the principal logical connective in the previous step.

Of course we leave out a great many steps in the proof, especially those which represent well known properties of equality, the quantifiers, and the descriptions operator; a list of some of the most often used theorems of this kind is given in Appendix III. The sign '$\wedge$' on the right hand side means that the rules of conversion have been used; 'P.C.' means that axioms (P) and rule V have been used.

We shall often have occasion to prove the validity of certain inferences; such a proof will also usually be set out line by line, with the above conventions. The premise is marked as a hypothesis; and the fact that, in general, the variables of the premise are not restricted is shown by the absence of a list of variables on the right. We also use '$H \vdash A_o$' to mean 'from the given premise A_o may be inferred' - it being evident from the context what 'the given premise' is.

(G) Development of the system (G).

In this section we prove some theorems and modes of inference in (G), partly because these results are needed for the demonstration of the equivalence of (C) and (G), and partly to show how the system works.

1) (Rule IV') $\quad F_{\alpha\beta} = G_{\alpha\beta} \vdash F_{\alpha\beta}\, A_\beta = G_{\alpha\beta}\, A_\beta$

For $H \vdash (\lambda f_{\alpha\beta} . fA_\beta)F_{\alpha\beta} = (\lambda f_{\alpha\beta} . fA_\beta)G_{\alpha\beta}$ by G.IV.

If necessary, change the bound variables of A_β so that they are distinct from $f_{\alpha\beta}$ and the free variables of $F_{\alpha\beta}$ and $G_{\alpha\beta}$.

Then $H \vdash F_{\alpha\beta}\, A_\beta = G_{\alpha\beta}\, A_\beta$ by II

2) (Substitution). Let M_o be a formula of which the bound variables are distinct both from x_β and from the free variables of A_β, and let M'_o be the result of substituting A_β for the free occurrences of x_β throughout M_o; then $M_o \vdash M'_o$.

For H.1	M_o	
H.2	$M_o = T$	by G.VII
3	$(\lambda x_\beta . M_o) = \lambda x_\beta . T$	by G.VI
4	$(\lambda x_\beta . M_o)A_\beta = (\lambda x_o . T)A_\beta$	by G.IV'
5	$M'_o = T$	by II
6	M'_o	by G.VII

3) (Generalisation). $\underset{\sim}{M}_o \vdash (\underset{\sim}{x}_\beta)(\underset{\sim}{M}_o)$.

for	H.1	$\underset{\sim}{M}_o$	
	2	$\underset{\sim}{M}_o = T$	by G.VII
	3	$\lambda \underset{\sim}{x}_\beta . \underset{\sim}{M}_o = \lambda \underset{\sim}{x}_\beta . T$	by G.VI
	4	$(\underset{\sim}{x}_\beta)(\underset{\sim}{M}_o)$	by definition.

It will be noted that G.VI is used in proving the validity of both substitution and generalisation.

4) $\vdash \quad \underline{x}_\alpha = \underline{x}_\alpha$.

For	$C_o = C_o$	by (P5)
	$(\lambda \underline{p}_o . \underline{x}_\alpha) C_o = (\lambda \underline{p}_o . \underline{x}_\alpha) C_o$	by G.IV
	$\underline{x}_\alpha = \underline{x}_\alpha$	by II

5) $\vdash \quad \underline{p}_o = \underline{q}_o \ . \supset . \ \underline{p}_o \supset \underline{q}_o$.

By substituting $(\lambda \underline{r}_o . \underline{r})$ for $\underline{f}_{oo}$ in (Q), and using II.

6) $\vdash \quad \underline{x}_\alpha = \underline{y}_\alpha \supset \underline{y}_\alpha = \underline{x}_\alpha$

For	$\underline{x}_\alpha = \underline{y}_\alpha \ . \supset . \ (\lambda \underline{z}_\alpha . \underline{z} = \underline{x}_\alpha) \underline{x}_\alpha \supset (\lambda \underline{z}_\alpha . \underline{z} = \underline{x}_\alpha) \underline{y}_\alpha$	by substitution in (Q)
	$\underline{x}_\alpha = \underline{y}_\alpha \supset \underline{y}_\alpha = \underline{x}_\alpha$	by II, P.C., 4).

7) $\vdash \quad \underline{y}_\alpha = \underline{z}_\alpha \ . \supset . \ \underline{x}_\alpha = \underline{y}_\alpha \supset \underline{x}_\alpha = \underline{z}_\alpha$.

By substituting $(\lambda \underline{w}_\alpha . \underline{x}_\alpha = \underline{w})$ for $\underline{f}_{o\alpha}$ in (Q).

8) $\vdash \quad \underline{x}_\beta = \underline{y}_\beta \supset \underline{f}_{\alpha\beta} \underline{x}_\beta = \underline{f}_{\alpha\beta} \underline{y}_\beta$.

By substituting $(\lambda \underline{z}_\beta . \underline{f}_{\alpha\beta} \underline{x}_\beta = \underline{f}_{\alpha\beta} \underline{z})$ for $\underline{f}_{o\beta}$ in (Q), using 4) and P.C.

9) $\vdash \quad \underline{f}_{oo} T \ \& \ \underline{f}_{oo} F \ . \supset . \ \underline{f}_{oo} \underline{p}_o$.

By the same argument as is given to prove (C) in Appendix I.

10) $\vdash \quad (T \supset \underline{p}_o) = \underline{p}_o.$

By P.C. and (T).

11) $\vdash \quad (\underline{x}_\alpha)(F \supset \underline{f}_{o\alpha}\underline{x}) = (F \supset (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x})).$

Both sides are provable by 3) and P.C., and therefore equal by (T).

12) $\vdash \quad (\underline{x}_\alpha)(T \supset \underline{f}_{o\alpha}\underline{x}) = (T \supset (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x})).$

For

$(T \supset \underline{f}_{o\alpha}\underline{x}_\alpha) = \underline{f}_{o\alpha}\underline{x}_\alpha$		by 10)
$\lambda\underline{x}_\alpha.(T \supset \underline{f}_{o\alpha}\underline{x}) = \underline{f}_{o\alpha}$		by G.VI, (E), 7).
$[(\lambda\underline{x}_\alpha.(T \supset \underline{f}_{o\alpha}\underline{x})) = \lambda\underline{x}_\alpha.T] = (\underline{f}_{o\alpha} = \lambda\underline{x}_\alpha.T)$		by substitution in 8).
$(\underline{x}_\alpha)(T \supset \underline{f}_{o\alpha}\underline{x}) = (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x})$		by definition
12)		by 10), 7).

13) $\vdash \quad (\underline{x}_\alpha)(\underline{p}_o \supset \underline{f}_{o\alpha}\underline{x}) = (\underline{p}_o \supset (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x}))$

By suitable substitution in 9), using 11) and 12), and V.

14) (The deduction theorem). If $\underset{\sim}{A}^1_o, \ldots, \underset{\sim}{A}^n_o \vdash \underset{\sim}{B}_o$, by an argument not involving abstraction of the free variables of $\underset{\sim}{A}^n_o$, then $\underset{\sim}{A}^1_o, \ldots, \underset{\sim}{A}^{n-1}_o \vdash \underset{\sim}{A}^n_o \supset \underset{\sim}{B}_o$.

Let $\underset{\sim}{B}^1_o, \ldots, \underset{\sim}{B}^m_o$ be the steps of the argument, each $\underset{\sim}{B}^i_o$ being an $\underset{\sim}{A}^j_o$, or an axiom, or an inference from the preceding $\underset{\sim}{B}^k_o$ by a single application of one of the rules of the system (G). We suppose that $\underset{\sim}{A}^1_o, \ldots, \underset{\sim}{A}^{n-1}_o \vdash \underset{\sim}{A}^n_o \supset \underset{\sim}{B}^k_o$ has been demonstrated for all $k < i$; we show that it will also be true for $k = i$. The theorem then follows by induction over i, since $\underset{\sim}{B}^m_o$ is $\underset{\sim}{B}_o$, and the result is trivial for $i = 1$.

If $\underset{\sim}{B}^i_o$ is an $\underset{\sim}{A}^j_o$ or an axiom this result is trivial. If

B^i_o is inferred by an application of rules I, II, III, V, the result is easily obtained (see Church (1)).

If B^i_o is $F_{\alpha\beta}X_\beta = F_{\alpha\beta}Y_\beta$, and is obtained from B^k_o $(X_\beta = Y_\beta)$ by G.IV, then $\vdash B^k_o \supset B^i_o$ by substitution in 8) (which has no bound variables). Hence $A^n_o \supset B^k_o \vdash A^n_o \supset B^i_o$ by V, and the result follows.

If B^i_o is $\lambda x_\beta . X_\alpha = \lambda x_\beta . Y_\alpha$, and is obtained from B^k_o $(X_\alpha = Y_\alpha)$ by G.VI, where x_β is not a free variable of A^n_o, then:

$A^n_o \supset B^k_o \vdash$	$(x_\beta)(A^n_o \supset B^k_o)$	by generalisation
$H \vdash$	$A^n_o \supset (x_\beta)(X_\alpha = Y_\alpha)$	by substitution in 12) (free variables of A^n_o distinct from x_β), V.
$H \vdash$	$A^n_o \supset (x_\beta)((\lambda x_\beta . X_\alpha)x = (\lambda x_\beta . Y_\alpha)x)$	by III.
$H \vdash$	$A^n_o \supset \lambda x_\beta . X_\alpha = \lambda x_\beta . Y_\alpha$	by substitution in (E), V.

Thus $A^n_o \supset B^k_o \vdash A^n_o \supset B^i_o$, and the result follows.

If B^i_o follows from B^k_o by G.VII, then $\vdash B^k_o \supset B^i_o$;

for	$p_o \supset p_o = T$	by P.C., (T).
and	$p_o = T \supset p_o$	
(from	$T = p_o . \supset . T \supset p_o$	by 5) and P.C.)

The argument is then as before: and this completes the demonstration.

15) (Rule IX - the substitution of equals for equals.) Let A_β be a part of M_α, and let M'_α be like M_α except that the part A_β has been replaced by B_β; and let $c_\gamma, \ldots, d_\delta$, be a complete list of the variables whose occurrences in A_β are

free in A_β and bound in M_α.[1]

Then $\vdash (\lambda c_\gamma \ldots d_\delta . A_\beta = \lambda c_\gamma \ldots d_\delta . B_\beta) \supset M_\alpha = M'_\alpha$.

Let M_α be represented by '$(\ldots A_\beta \ldots)$'; let $f_{\beta\delta\ldots\gamma}$ be a variable that does not occur bound in M_α, and let A'_β be like A_β except that its bound variables have been changed so that $c_\gamma, \ldots, d_\delta$, do not occur bound in A'_β.

1. $M_\alpha = (\ldots((\lambda c_\gamma \ldots d_\delta . A'_\beta) c_\gamma \ldots d_\delta) \ldots)$ by I, III.
2. $M_\alpha = (\lambda f_{\beta\delta\ldots\gamma} . (\ldots (f c_\gamma \ldots d_\delta) \ldots))(\lambda c_\gamma \ldots d_\delta . A_\beta)$ by III, I.

H.3 $(\lambda c_\gamma \ldots d_\delta . A_\beta) = (\lambda c_\gamma \ldots d_\delta . B_\beta)$

4. $M_\alpha = (\lambda f_{\beta\delta\ldots\gamma} . (\ldots (f c_\gamma \ldots d_\delta) \ldots))(\lambda c_\gamma \ldots d_\delta . B_\beta)$ by G.IV, 7).
5. $M_\alpha = M'_\alpha$ by II, I.
6. H.3 $\supset M_\alpha = M'_\alpha$ by 14).

It follows, of course, from the equivalence of (C) and (G), that rule IX is also valid in (C).

(H) Closed Formulae.

A closed formula is one which contains no free occurrences of variables. A closure of a formula A_α, is one of the formulae

$$\lambda b_\beta \ldots c_\gamma . A_\alpha,$$

where $b_\beta, \ldots, c_\gamma$, is a complete list of the free variables of A_α; the only closure of a closed formula is the formula itself.

A combinatorial formula is an abbreviated formula involving the constants of the system ((C) or (G)), and the symbols

---

(1) More precisely: 'the variables whose free occurrences in A_β are bound occurrences in M_α'.

$W_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$ and $K_{\alpha\beta\alpha}$, but not the symbol λ. Thus a combinatorial formula consists of a single symbol, or is of the form $\underset{\sim}{A}_{\alpha\beta}\underset{\sim}{B}_{\beta}$, where $\underset{\sim}{A}_{\alpha\beta}$ and $\underset{\sim}{B}_{\beta}$ are combinatorial formulae. The variables of a combinatorial formula all occur freely in it.

<u>Theorem I</u>

Any formula is provably equal to a combinatorial formula.

<u>Lemma A</u>

$$\vdash I^{\alpha}_{\alpha\alpha} = W_{\alpha\alpha(o\alpha)(\alpha o\alpha)}K_{\alpha o\alpha}(K_{o\alpha o}C_{o})$$

This follows from the definition of W and K and the rules of conversion.

<u>Lemma B</u>

If $\underset{\sim}{A}_{\alpha}$ is a combinatorial formula, then there exists a combinatorial formula $\underset{\sim}{B}_{\alpha\beta}$, such that:

$$\vdash \lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\alpha} = \underset{\sim}{B}_{\alpha\beta}.$$

For if $\underset{\sim}{x}_{\beta}$ is not a free variable of $\underset{\sim}{A}_{\alpha}$, then:

$$\vdash \lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\alpha} = K_{\alpha\beta\alpha}\underset{\sim}{A}_{\alpha}.$$

If $\underset{\sim}{A}_{\alpha}$ is $\underset{\sim}{x}_{\beta}$, then the result follows from lemma A. We suppose therefore that $\underset{\sim}{A}_{\alpha}$ consists of more than one symbol, and that the lemma has been demonstrated for formulae whose length is less than that of $\underset{\sim}{A}_{\alpha}$. But

$$\vdash \underset{\sim}{A}_{\alpha} = \underset{\sim}{X}_{\alpha\gamma}Y_{\gamma},$$

where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are combinatorial formulae. Therefore

$$\begin{aligned}\vdash \lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\alpha} &= \lambda\underset{\sim}{x}_{\beta}.(\lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{X}_{\alpha\gamma})\underset{\sim}{x}((\lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{Y}_{\gamma})\underset{\sim}{x}) \\ &= W_{\alpha\beta(\gamma\beta)(\alpha\gamma\beta)}(\lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{X}_{\alpha\gamma})(\lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{Y}_{\gamma}).\end{aligned}$$

But $(\lambda x_\beta . X_{\alpha\gamma})$, $(\lambda x_\beta . Y_\gamma)$ are provably equal to combinatorial formulae by hypothesis, and hence so is $(\lambda x_\beta . A_\gamma)$; the lemma now follows by induction over the length of the formula.

We suppose now that the theorem has been proved for all formulae in which there are less than n occurrences of the symbol λ, and we suppose that the formula A_α contains just n occurrences of λ. At least one of these occurrences must be an innermost one; i.e. there must be a part $(\lambda x_\gamma . B_\beta)$ of A_α, where λ does not appear in B_β: but this is provably equal to a combinatorial formula $D_{\beta\gamma}$, having the same free variables, and hence, by rule IX, A_α is provably equal to a formula having only n - 1 occurrences of λ; the theorem now follows by induction over n. Q.E.D.

We call a combinatorial formula which is provably equal to A_α, a <u>combinatorial equivalent</u> of A_α.

<u>Corollary to theorem I</u>

If $P_{o\alpha}$ are a set of formulae which satisfy:

$$\vdash \quad P_{o\alpha} X_\alpha ,$$

where X_α is a constant, or a W, or a K; and

$$\vdash \quad P_{o(\alpha\beta)} f_{\alpha\beta} \; \& \; P_{o\beta} x_\beta . \supset . \; P_{o\alpha}(f_{\alpha\beta} x_\beta);$$

then, if A_α is a closed formula,

$$\vdash \quad P_{o\alpha} A_\alpha .$$

For the combinatorial equivalent of a closed formula contains only constants, W's and K's, ~~the corollary follows from the axiom of extensionality.~~

Section 2. Maps and Permutations.

In this section we introduce a number of definitions which will be of use later, and prove some simple properties of the defined objects.

$$\underset{\sim}{F}_{\alpha\beta} : \underset{\sim}{G}_{\beta\gamma} \rightarrow \lambda \underline{x}_{\gamma}.\underset{\sim}{F}_{\alpha\beta}(\underset{\sim}{G}_{\beta\gamma}\underline{x})$$

This defines the product (in the sense of transformation theory) of F and G; we have

2.1) $\vdash \quad (\underset{\sim}{F}_{\alpha\beta} : \underset{\sim}{G}_{\beta\gamma}) : \underset{\sim}{H}_{\gamma\delta} = \lambda\underline{z}_{\delta}.\underset{\sim}{F}_{\alpha\beta}(\underset{\sim}{G}_{\beta\gamma}(\underset{\sim}{H}_{\gamma\delta}\underline{z})) = \underset{\sim}{F}_{\alpha\beta} : (\underset{\sim}{G}_{\beta\gamma} : \underset{\sim}{H}_{\gamma\delta})$,

so that we can omit the brackets from a multiple product.

$$\mathrm{Uni}^{\alpha\beta}_{o(\alpha\beta)} \rightarrow \lambda\underline{f}_{\alpha\beta}.(\underline{x}_{\beta},\underline{y}_{\beta})(\underline{fx} = \underline{fy} \supset \underline{x} = \underline{y})$$

(Here, as we shall often do, we insert an index to indicate the type to which a defined formula refers; this enables us to omit the type suffix, which is often extremely cumbersome; we may also omit the index when this can be done without ambiguity). '$\mathrm{Uni}^{\alpha\beta}\underline{f}$' (or rather the assertion of that formula) means that $\underline{f}$ is a one-to-one map of type β into type α.

$$\mathrm{Ont}^{\alpha\beta}_{o(\alpha\beta)} \rightarrow \lambda\underline{f}_{\alpha\beta}.(\underline{a}_{\alpha})(E!\underline{x}_{\beta})(\underline{fx} = \underline{a}) \,\&\, \underline{f}C_{\beta} = C_{\alpha}$$

'$\mathrm{Ont}^{\alpha\beta}\underline{f}$' means that $\underline{f}$ is a one-to-one map of type β onto type α, and that it maps the nonsense element of one into the nonsense element of the other; this latter restriction is inessential, but very convenient.

$$\mathrm{Per}^{\alpha}_{o(\alpha\alpha)} \rightarrow \mathrm{Ont}^{\alpha\alpha}$$

'$\mathrm{Per}^{\alpha}\underline{f}$' means that $\underline{f}$ is a permutation of the type α which

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$$\underset{\sim}{F}_{\alpha\beta} : \underset{\sim}{G}_{\beta\gamma} \rightarrow \lambda \underline{x}_{\gamma}.\underset{\sim}{F}_{\alpha\beta}(\underset{\sim}{G}_{\beta\gamma}\underline{x})$$

This defines the product (in the sense of transformation theory) of F and G; we have

2.1) $\vdash \quad (\underset{\sim}{F}_{\alpha\beta} : \underset{\sim}{G}_{\beta\gamma}) : \underset{\sim}{H}_{\gamma\delta} = \lambda \underline{z}_{\delta}.\underset{\sim}{F}_{\alpha\beta}(\underset{\sim}{G}_{\beta\gamma}(\underset{\sim}{H}_{\gamma\delta}\underline{z})) = \underset{\sim}{F}_{\alpha\beta} : (\underset{\sim}{G}_{\beta\gamma} : \underset{\sim}{H}_{\gamma\delta})$,

so that we can omit the brackets from a multiple product.

$$\mathrm{Uni}^{\alpha\beta}_{o(\alpha\beta)} \rightarrow \lambda \underline{f}_{\alpha\beta}.(\underline{x}_{\beta},\underline{y}_{\beta})(\underline{fx} = \underline{fy} \supset \underline{x} = \underline{y})$$

(Here, as we shall often do, we insert an index to indicate the type to which a defined formula refers; this enables us to omit the type suffix, which is often extremely cumbersome; we may also omit the index when this can be done without ambiguity). '$\mathrm{Uni}^{\alpha\beta}\,\underline{f}$' (or rather the assertion of that formula) means that $\underline{f}$ is a one-to-one map of type β into type α.

$$\mathrm{Ont}^{\alpha\beta}_{o(\alpha\beta)} \rightarrow \lambda \underline{f}_{\alpha\beta}.(\underline{a}_{\alpha})(E!\underline{x}_{\beta})(\underline{fx} = \underline{a}) \;\&\; \underline{f}C_{\beta} = C_{\alpha}$$

'$\mathrm{Ont}^{\alpha\beta}\,\underline{f}$' means that $\underline{f}$ is a one-to-one map of type β onto type α, and that it maps the nonsense element of one into the nonsense element of the other; this latter restriction is inessential, but very convenient.

$$\mathrm{Per}^{\alpha}_{o(\alpha\alpha)} \rightarrow \mathrm{Ont}^{\alpha\alpha}$$

'$\mathrm{Per}^{\alpha}\underline{f}$' means that $\underline{f}$ is a permutation of the type α which leaves the nonsense element invariant.

2.2) $\vdash \quad Per^{\alpha} I^{\alpha}$

$$Rec^{\alpha\beta}_{\beta\alpha(\alpha\beta)} \rightarrow \lambda \underline{f}_{\alpha\beta} \underline{a}_{\alpha}.(\imath \underline{x}_{\beta})(\underline{fx} = \underline{a})$$

'Rec $\underline{f}$' represents the inverse transformation to $\underline{f}$; it is interesting to note that, due to the nonsense elements, Rec$\underline{f}$ is defined and has useful properties even when $\underline{f}$ is not a one-to-one map onto. For example, we have:

2.3) $\vdash \quad Uni\underline{f}_{\alpha\beta} \;\&\; \underline{f}_{\alpha\beta} C_{\beta} = C_{\alpha} \;.\supset.\; Rec^{\beta\alpha}(Rec\underline{f}_{\alpha\beta}) = \underline{f}_{\alpha\beta}$

2.4) $\vdash \quad Uni\underline{f}_{\alpha\beta} \supset (Rec\underline{f}_{\alpha\beta} : \underline{f}_{\alpha\beta}) = I^{\beta}$

2.5) $\quad Ont\underline{f}_{\alpha\beta} \supset (\underline{f}_{\alpha\beta} : Rec\underline{f}_{\alpha\beta}) = I^{\alpha}$

We prove the first of these.

H.1	$Uni\underline{f}_{\alpha\beta} \;\&\; \underline{f}_{\alpha\beta} C_{\beta} = C_{\alpha}$	($\underline{f}$)
2	$Rec^{\beta\alpha}(Rec\underline{f})\underline{x}_{\beta} = (\imath \underline{a}_{\alpha})[(\imath \underline{y}_{\beta})(\underline{fy} = \underline{a}) = \underline{x}_{\beta}]$	
A.3	$M_{o\alpha\beta} \rightarrow \lambda \underline{x}_{\beta} \underline{a}_{\alpha}.(\imath \underline{y}_{\beta})(\underline{fy} = \underline{a}) = \underline{x}$	
4	$(\imath \underline{y}_{\beta})(\underline{fy} = \underline{fx}_{\beta}) = \underline{x}_{\beta}$	H.1.
5	$M\underline{x}_{\beta}(\underline{fx}_{\beta})$	
6	$\underline{x}_{\beta} \neq C_{\beta} \;\&\; M\underline{x}_{\beta}\underline{a}_{\alpha} \;.\supset.\; \underline{fx}_{\beta} = \underline{a}_{\alpha}$	(D).
7	$\underline{x}_{\beta} \neq C_{\beta} \supset \iota^{\alpha}(M\underline{x}_{\beta}) = \underline{fx}_{\beta}$	5,6,(D).
8	$MC_{\beta}C_{\alpha}$	5,H.1.
9	$\underline{x}_{\beta} = C_{\beta} \supset \iota^{\alpha}(M\underline{x}_{\beta}) = \underline{fx}_{\beta}$	8,H.1.
10	$Rec^{\beta\alpha}(Rec\underline{f}) = \underline{f}$	7,9,(E).
11	2.3)	(H.1).

One of the features of the system of logic we are using is that no individual, except C_{ι}, can be singled out (or named) by a logical formula. This feature is common to some other systems of logic, and to many mathematical systems; for example, one cannot single out a particular point in Euclidean geometry. It is most simply expressed by saying that the system is <u>symmetric</u> in the individuals, just as the points of space occur symmetrically in Euclidean geometry.

Of course one can, by an act of imagination, concentrate one's attention on some particular individual, or point of space; the form of words used in then something like 'let x be an individual', or 'let P and Q be distinct points of space'. But this focussing of the attention is only an accompaniment to the mathematics (one marks two dots on a piece of paper, and labels them P and Q): and further, it is only temporary; for the conclusion of the argument must, on account of the symmetry, be of the form 'for <u>any</u> individual...', 'for <u>any</u> pair of distinct points ...'. If such an argument is presented formally, it must always appear as an instance of the use of the deduction theorem: the premises are hypotheses, and the (general) conclusion is obtained by eliminating them. The temporary names ('x ', 'P', 'Q') that appear in the premises are formally represented by variables which are restricted by hypothesis; and one emphasises to oneself - or to one's audience - the fact that they are so restricted, that they cannot, as long as the argument is in progress, be generalised on or substituted for, by drawing little pictures of the objects and labelling them with the restricted variables which represent them.[1] This giving of temporary names to objects is a matter to which we shall later return.

We wish to be able to express the symmetry of the system within the system itself. To do this it is necessary to define the changes that are undergone by objects of higher type, when a permutation of the individuals is made. We consider the rather more general case of the transformations induced by a map of one type into another.

Let α and β be two given types (complex or basic); to each type γ, we define the transform $\bar{\gamma}$ of γ as follows:

a) If β is not γ, nor a part of γ, then $\bar{\gamma}$ is γ;

b) If γ is β, then $\bar{\gamma}$ is α;

c) If γ is $(\delta\varepsilon)$ and β is a part of γ, then $\bar{\gamma}$ is $(\bar{\delta}\bar{\varepsilon})$.

Due to the definition of 'part' these rules define uniquely the transform of each type. We now define:

$$\mathrm{Tra}^{\gamma}_{\bar{\gamma}\gamma(\alpha\beta)} \rightarrow \lambda \underline{m}_{\alpha\beta}\underline{f}_{\gamma}.\underline{f} \qquad \text{if } \beta \text{ is not } \gamma, \text{ nor a part of } \gamma;$$

$$\mathrm{Tra}^{\beta}_{\alpha\beta(\alpha\beta)} \rightarrow \lambda \underline{m}_{\alpha\beta}.\underline{m}$$

$$\mathrm{Tra}^{\delta\varepsilon}_{\bar{\delta}\bar{\varepsilon}(\delta\varepsilon)(\alpha\beta)} \rightarrow \lambda \underline{m}_{\alpha\beta}\underline{f}_{\delta\varepsilon}\underline{a}_{\bar{\varepsilon}}.\mathrm{Tra}^{\delta}\underline{m}(\underline{f}(\mathrm{Rec}^{\bar{\varepsilon}\varepsilon}(\mathrm{Tra}^{\varepsilon}\underline{m})\underline{a}))$$

$$\text{if } \beta \text{ is a part of } (\delta\varepsilon).$$

'Tra' is short for 'transport'. If $\underline{m}_{\alpha\beta}$ is a one-to-one map of β into α, then $\mathrm{Tra}^{\gamma}\underline{m}$ is a one-to-one map of γ into $\bar{\gamma}$; the map thus defined is analogous to the transformation $P \rightarrow MPM^{-1}$ undergone by an operator in a space when the coordinates

(1) I have sometimes listened to lectures at which the only things that were written on the blackboard were the symbols of restricted variables.

undergo $x \rightarrow Mx$. We have the following theorems:

2.6) $\vdash \quad \mathrm{Uni}\underline{m}_{\alpha\beta} \supset \mathrm{Uni}(\mathrm{Tra}^{\gamma}\underline{m}_{\alpha\beta})$

2.7) $\vdash \quad \mathrm{Uni}\underline{m}_{\alpha\beta} \supset \mathrm{Tra}^{\delta\varepsilon}\underline{m}_{\alpha\beta}\underline{r}_{\delta\varepsilon}(\mathrm{Tra}^{\varepsilon}\underline{m}_{\alpha\beta}\underline{x}_{\varepsilon}) = \mathrm{Tra}^{\delta}\underline{m}_{\alpha\beta}(\underline{r}_{\delta\varepsilon}\underline{x}_{\varepsilon})$

(provided $(\delta\varepsilon)$ is not β).

Both theorems are trivial if γ (= $(\delta\varepsilon)$) does not have β as a part; we suppose that 2.6) has been proved for types δ and ε, and give a proof of both theorems for type $(\delta\varepsilon)$; it follows that they are provable for all types.

H.1	$\mathrm{Uni}\underline{m}_{\alpha\beta}$	($\underline{m}$)
A.2	$M^{\gamma}_{\bar\gamma\gamma} \rightarrow \mathrm{Tra}^{\gamma}\underline{m}$	for all γ.
3	$\mathrm{Uni}M^{\delta}$ & $\mathrm{Uni}M^{\varepsilon}$	2.6) & 2.6).
4	$M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon}(M^{\varepsilon}\underline{y}_{\varepsilon}) = M^{\delta}(\underline{r}_{\delta\varepsilon}(\mathrm{Rec}^{\bar\varepsilon\varepsilon}M^{\varepsilon}(M^{\varepsilon}\underline{y}_{\varepsilon})))$	Definition.
5	$M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon}(M^{\varepsilon}\underline{y}_{\varepsilon}) = M^{\delta}(\underline{r}_{\delta\varepsilon}\underline{y}_{\varepsilon})$	3, 2.4).
P.6	$M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon} = M^{\delta\varepsilon}\underline{s}_{\delta\varepsilon} \supset \underline{r}_{\delta\varepsilon} = \underline{s}_{\delta\varepsilon}$	
H.7	$M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon} = M^{\delta\varepsilon}\underline{s}_{\delta\varepsilon}$	($\underline{r}$,$\underline{s}$)
8	$M^{\delta}(\underline{ry}_{\varepsilon}) = M^{\delta}(\underline{sy}_{\varepsilon})$	H.7,5.
9	$\underline{ry}_{\varepsilon} = \underline{sy}_{\varepsilon}$	8, 3.
10	$\underline{r} = \underline{s}$	(E).
11	P.6	(H.7).
12	$2.6^{\delta\varepsilon}$) & $2.7^{\delta\varepsilon}$)	11,5, (H.1).

In the same way we prove:

2.8) $\vdash \quad \mathrm{Ont}\underline{m}_{\alpha\beta} \supset \mathrm{Ont}(\mathrm{Tra}^{\gamma}\underline{m}_{\alpha\beta})$

H.1	$\mathrm{Ont}\underline{m}_{\alpha\beta}$	($\underline{m}$)
A.2	$M^{\gamma}_{\bar\gamma\gamma} \rightarrow \mathrm{Tra}^{\gamma}\underline{m}$	For all γ.
3	$\mathrm{Uni}M^{\gamma}$	H.1, 2.6).
4	$M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon} = M^{\delta} : \underline{r}_{\delta\varepsilon} : \mathrm{Rec}^{\bar\varepsilon\varepsilon}M^{\varepsilon}$	Definitions.

H.5 $\underline{r}_{\delta\varepsilon} = \mathrm{Rec}^{\bar\delta\delta} M^{\delta} : \underline{a}_{\bar\delta\bar\varepsilon} : M^{\varepsilon}$ ($\underline{r}$, $\underline{a}$)

6 $M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon} = (M^{\delta} : \mathrm{Rec}^{\bar\delta\delta} M^{\delta}) : \underline{a} : (M^{\varepsilon} : \mathrm{Rec}^{\bar\varepsilon\varepsilon} M^{\varepsilon})$

7 $M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon} = I^{\bar\delta} : \underline{a} : I^{\bar\varepsilon}$ 2.8$^{\delta}$), 2.8$^{\varepsilon}$), 2.5).

8 $M^{\delta\varepsilon}\underline{r}_{\delta\varepsilon} = \underline{a}$

9 $(E\underline{r}_{\delta\varepsilon})(M^{\delta\varepsilon}\underline{r} = \underline{a}_{\bar\delta\bar\varepsilon})$ (H.5).

10 $C_{\delta\varepsilon} = \lambda \underline{x}_{\varepsilon}.C_{\delta}$ Definition.

11 $M^{\delta\varepsilon}C_{\delta\varepsilon} = \lambda \underline{a}_{\bar\varepsilon}.M^{\delta}(C_{\delta\varepsilon}(\mathrm{Rec}M^{\varepsilon}\underline{a})$ "

12 $M^{\delta\varepsilon}C_{\delta\varepsilon} = \lambda \underline{a}_{\bar\varepsilon}.M^{\delta}C_{\delta}$ 10.

13 $M^{\delta\varepsilon}C_{\delta\varepsilon} = C_{\bar\delta\bar\varepsilon}$ H.1, 2.8$^{\delta}$).

14 $\mathrm{Ont}M^{\delta\varepsilon}$ 3, 9, 13.

15 2.8$^{\delta\varepsilon}$) (H.1).

When we consider, instead of a map of one type into another, a permutation of a type, the above definitions may be taken over, $\bar\gamma$ being now just γ, and β being replaced by α. We then have further:

2.10) $\vdash$ $\mathrm{Tra}^{\gamma} I^{\alpha} = I^{\gamma}$

2.11) $\vdash$ $\mathrm{Per}\underline{p}_{\alpha\alpha}$ & $\mathrm{Per}\underline{q}_{\alpha\alpha}$ $.\supset.$ $\mathrm{Tra}\ \underline{p}_{\alpha\alpha} : \mathrm{Tra}\ \underline{q}_{\alpha\alpha} = \mathrm{Tra}^{\gamma}(\underline{p}_{\alpha\alpha} : \underline{q}_{\alpha\alpha})$

H.1 L.H.S. ($\underline{p}$.$\underline{q}$).

2 $\mathrm{Tra}^{\delta\varepsilon}(\underline{p} : \underline{q})$

$= \lambda \underline{f}_{\delta\varepsilon}.\mathrm{Tra}^{\delta}(\underline{p} : \underline{q}) : \underline{f} : \mathrm{Rec}^{\bar\varepsilon\varepsilon}(\mathrm{Tra}^{\varepsilon}(\underline{p} : \underline{q}))$ See 4 of 2.8).

3 $= \lambda \underline{f}_{\delta\varepsilon}.\mathrm{Tra}^{\delta}\underline{p} : \mathrm{Tra}^{\delta}\underline{q} : \underline{f} : \mathrm{Rec}(\mathrm{Tra}^{\varepsilon}\underline{p} : \mathrm{tra}^{\varepsilon}\underline{q})$ 2.11$^{\delta}$), 2.11$^{\varepsilon}$).

4 $= \lambda \underline{f}_{\delta\varepsilon}.\mathrm{Tra}^{\delta}\underline{p} : \mathrm{Tra}^{\delta}\underline{q} : \underline{f} : \mathrm{Rec}(\mathrm{Tra}^{\varepsilon}\underline{q}) : \mathrm{Rec}(\mathrm{Tra}^{\varepsilon}\underline{p})$*

5 $= \lambda \underline{f}_{\delta\varepsilon}.\mathrm{Tra}^{\delta\varepsilon}\underline{p}(\mathrm{Tra}^{\delta\varepsilon}\underline{qf})$

6 $= \mathrm{Tra}^{\delta\varepsilon}\underline{p} : \mathrm{Tra}^{\delta\varepsilon}\underline{q}$

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* The theorem justifying this step is easily proved.

7 $\quad 2.11^{\delta\varepsilon})$ (H.1)

2.12) $\vdash$ $\mathrm{Per}\underline{p}_{\iota\iota} \supset \mathrm{Rec}(\mathrm{Tra}^{\gamma}\underline{p}_{\iota\iota}) = \mathrm{Tra}^{\gamma}(\mathrm{Rec}\underline{p}_{\iota\iota})$

H.1 $\quad \mathrm{Per}\underline{p}_{\iota\iota}$ $(\underline{p})$

2 $\quad \mathrm{Tra}^{\gamma}\underline{p} : \mathrm{Tra}^{\gamma}(\mathrm{Rec}\underline{p}) = \mathrm{Tra}^{\gamma}(\underline{p} : \mathrm{Rec}\underline{p})$ 2.11).

3 $\quad = \mathrm{I}^{\gamma}$ H.1, 2.5), 2.10).

4 $\quad \mathrm{Tra}^{\gamma}\underline{p}(\mathrm{Tra}^{\gamma}(\mathrm{Rec}\underline{p})\underline{x}_{\gamma}) = \underline{x}_{\gamma}$

5 $\quad (\imath\underline{z}_{\gamma})(\mathrm{Tra}^{\gamma}\underline{pz} = \underline{x}_{\gamma}) = \mathrm{Tra}^{\gamma}(\mathrm{Rec}\underline{p})\underline{x}_{\gamma}$ 4, H.1.

6 $\quad \mathrm{Rec}(\mathrm{Tra}^{\gamma}\underline{p}) = \mathrm{Tra}^{\gamma}(\mathrm{Rec}\underline{p})$ 5, (E).

7 $\quad$ 2.12). (H.1).

If α is a complex type, the use that can be made of its permutations is rather limited, because 2.8) fails when $(\delta\varepsilon)$ is α. But if α is a basic type, this difficulty does not arise, and we now restrict our discussions to that case. We define:

$$\mathrm{Inv}^{\alpha}_{o\alpha} \rightarrow \lambda\underline{f}_{\alpha}.(\underline{p}_{\iota\iota})(\mathrm{Per}\underline{p} \supset \mathrm{Tra}^{\alpha}\underline{pf} = \underline{f}),$$

$$\mathrm{Cot}^{\alpha}_{o\alpha\alpha} \rightarrow \lambda\underline{f}_{\alpha}\underline{g}_{\alpha}.(\mathrm{E}\underline{p}_{\iota\iota})(\mathrm{Per}\underline{p} \,\&\, \mathrm{Tra}^{\alpha}\underline{pf} = \underline{g}).$$

'Inv' is short for 'invariant'; 'Inv$\underline{f}$' means that $\underline{f}$ is symmetric in the individuals (excluding C_{ι}). 'Cot' is short for 'conjugate'; it is easy to show, using 2.10) - 2.12), that it is an equivalence relation.

2.13) $\vdash$ $\quad \mathrm{Inv}\ \Pi_{o(o\alpha)}$

H.1 $\quad \mathrm{Per}\underline{p}_{\iota\iota}$ $(\underline{p})$

A.2 $\quad \mathrm{P}_{o\alpha(o\alpha)} \rightarrow \mathrm{Tra}^{o\alpha}\underline{p}$

A.3 $\quad \mathrm{P}^{-1}_{o\alpha(o\alpha)} \rightarrow \mathrm{RecP}\ (= \mathrm{Tra}^{o\alpha}(\mathrm{Rec}\underline{p}))$ 2.12).

4 $\quad \mathrm{P}(\lambda\underline{x}_{\alpha}.\mathrm{T}) = \lambda\underline{x}_{\alpha}.\mathrm{T} = \mathrm{P}^{-1}(\lambda\underline{x}_{\alpha}.\mathrm{T})$ Tra, A.3.

5	$\underline{f}_{o\alpha} = \lambda \underline{x}_\alpha.T \supset P^{-1}\underline{f}_{o\alpha} = \lambda\ \underline{x}_\alpha.T$	4.
6	$P^{-1}\underline{f}_{o\alpha} = \lambda\underline{x}_\alpha.T \supset \underline{f}_{o\alpha} = \lambda\underline{x}_\alpha.T$	2.5), 4.
7	$(\lambda\underline{f}_{o\alpha}.\underline{f} = \lambda\underline{x}_\alpha.T) = (\lambda\underline{f}_{o\alpha}.P^{-1}\underline{f} = \lambda\ \underline{x}_\alpha.T)$	5,6,(T),(E).
8	$\mathrm{Inv}^{o(o\alpha)}(\lambda\underline{f}_{o\alpha}.\underline{f} = \lambda\underline{x}_\alpha.T)$	7,Tra,(H.1).
9	$\mathrm{Inv}\ \Pi_{o(o\alpha)}$	8
2.14) $\vdash$	$\mathrm{Inv}\ \iota_{\iota(o\iota)}$	
H.1	$\mathrm{Perp}\underline{p}_{\iota\iota}$	($\underline{p}$)
A.2	$P_{o\iota(o\iota)} \rightarrow \mathrm{Tra}^{o\iota}\underline{p}$	
A.3	$P^{-1} \rightarrow \mathrm{Rec}P,\ \underline{p}^{-1} \rightarrow \mathrm{Rec}\underline{p}$.	
4	$\mathrm{Tra}^{o}\underline{p}^{-1}(\underline{f}_{o\iota}\underline{x}_\iota) = P^{-1}\underline{f}_{o\iota}(\underline{p}^{-1}\underline{x}_\iota) = \underline{f}_{o\iota}\underline{x}_\iota$	2.7), Tra.
5	$J\underline{f}_{o\iota} \supset \iota(P^{-1}\underline{f}_{o\iota}) = \underline{p}^{-1}(\iota\underline{f}_{o\iota})$	4, (D).
6	$\sim J\underline{f}_{o\iota} \supset \sim J(P^{-1}\underline{f}_{o\iota})$	4.
7	$\sim J\underline{f}_{o\iota} \supset \iota(P^{-1}\underline{f}_{o\iota}) = \underline{p}^{-1}(\iota\underline{f}_{o\iota})$	(D), H.1.
8	$\iota(P^{-1}\underline{f}_{o\iota}) = \underline{p}^{-1}(\iota\underline{f}_{o\iota})$	
9	$\mathrm{Tra}^{\iota(o\iota)}\underline{p}\ \iota\ \underline{f}_{o\iota} = \underline{p}(\iota(P^{-1}\underline{f}_{o\iota}))$	Tra.
10	$= \underline{f}_{o\iota}$	8, 2.5).
11	$\mathrm{Inv}\ \iota_{\iota(o\iota)}$	Inv, (E),(H.1).
2.15) $\vdash$	$\mathrm{Inv}K_{\alpha\beta\alpha}$	
H.1	$\mathrm{Perp}\underline{p}_{\iota\iota}$	($\underline{p}$)
A.2	$P^{\gamma} \rightarrow \mathrm{Tra}^{\gamma}\underline{p}$	
A.3	$\overline{P}^{\gamma} \rightarrow \mathrm{Rec}P^{\gamma}$	
4	$P^{\alpha\beta\alpha}K_{\alpha\beta\alpha} = \lambda\underline{x}_\alpha.P^{\alpha\beta}(\lambda\underline{y}_\beta.\overline{P}^{\alpha}\underline{x})$	Tra.
5	$= \lambda\underline{x}_\alpha\underline{y}_\beta.P^{\alpha}(\overline{P}^{\alpha}\underline{x})$	Tra.
6	$= K_{\alpha\beta\alpha}$	H.1, 2.5).
7	2.16)	(H.1).

2.16) $\vdash$ $\mathrm{Inv}W_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$

Proof similar to that of 2.15).

We are now in a position to prove:

Theorem II

If $\underset{\sim}{A}_{\alpha}$ is a closed formula, then $\vdash \mathrm{Inv}\underset{\sim}{A}_{\alpha}$.

For we have:

2.17) $\mathrm{Inv}\underline{f}_{\alpha\beta}$ & $\mathrm{Inv}\underline{z}_{\beta}$.$\supset$. $\mathrm{Inv}(\underline{f}_{\alpha\beta}\underline{z}_{\beta})$;

the proof of this is immediate using 2.7) and the definition of Inv. The theorem now follows from 2.13), 2.14), 2.15), 2.16), and the corollary to theorem I.

We can now express formally the fact that the system is symmetric in the individuals. To say that no individual except C_{ι} can be singled out, is to say that it is not possible to give, in the system, a complete and definite description of any other individual. Since the system contains description operators, this may be more formally expressed by saying that all closed formulae of type ι are provably equal to C_{ι}.

2.18) $\vdash$ $\mathrm{Inv}\underline{x}_{\iota} \supset \underline{x}_{\iota} = C_{\iota}$

H.1	$\underline{x}_{\iota} \neq \underline{y}_{\iota}$ & $\underline{x}_{\iota} \neq C_{\iota}$ & $\underline{y}_{\iota} \neq C_{\iota}$	$(\underline{x}, \underline{y})$.
H.2	$\underline{p}_{\iota\iota} = \lambda\underline{z}_{\iota}.(\urcorner\underline{w}_{\iota})(\underline{z} = \underline{x} \supset \underline{w} = \underline{y} .\&. \underline{z} = \underline{y} \supset \underline{w} = \underline{x}$ & $(\underline{z} \neq \underline{x}$ & $\underline{z} \neq \underline{y}) \supset \underline{w} = \underline{z})$	$(\underline{p})$
3	$\mathrm{Per}\underline{p}$ & $\underline{px} \neq \underline{x}$	Per, (D), H.2.
4	$\sim \mathrm{Inv}\underline{x}$	Inv.
5	H.1 $\supset$ $\sim \mathrm{Inv}\underline{x}_{\iota}$	(H.1, H.2).

6 $\underline{x}_\iota \neq C_\iota \supset (E\underline{y}_\iota)(\underline{y} \neq \underline{x} \ \& \ \underline{y} \neq C)$ (I).

7 $\mathrm{Inv}\underline{x}_\iota \supset \underline{x}_\iota = C_\iota$ 5,6.

Corollary to theorem II. If $\underset{\sim}{A}_\iota$ is a closed formula, then $\vdash \underset{\sim}{A}_\iota = C_\iota$.

Since the translation from (G) into (C) of a closed formula is closed, theorem II is also true of system (G). In some of the lower types one can give closed formulae which represent each of the invariant elements of that type. For example, in types $o\iota$ and $\iota\iota$ the only invariant elements are represented by:

$$\lambda \underline{x}_\iota.T,\ \lambda \underline{x}_\iota.F,\ \lambda \underline{x}_\iota.\underline{x} = C,\ \lambda \underline{x}_\iota.\underline{x} \neq C;$$

and

$$\lambda \underline{x}_\iota.\underline{x},\ \lambda \underline{x}_\iota.C\ ,$$

respectively. But it is easy to see that in the higher types the representation of all invariant elements by closed formulae is not possible. In type $o(o\iota)$ there is a formula corresponding to each natural number: for example

$$\lambda \underline{f}_{o\iota}.(E\underline{x}_\iota)(E\underline{y}_\iota)(\underline{z}_\iota)(\underline{fx} \ \& \ \underline{fy} \ \& \ \underline{x} \neq \underline{y} \ \& \ (\underline{fz} \supset \underline{z} = \underline{x} \vee \underline{z} = \underline{y}))$$

corresponds to 2. Hence any element in type $o(o(o\iota))$ that corresponds to a set of natural numbers is an invariant element; thus the invariant elements of this type are non-denumerable, and therefore they cannot all be represented by closed formulae. A rather similar argument shows that the same is true of type $o(\iota\iota)$.

I do not know at what stage in the development of symbolic logic the invariance of the logical operations first came to be realised; the idea is certainly implicit in Fraenkel's proof of the independence of the selection axiom (Fraenkel (1) 1922). A complete statement, and a discussion of its implications was given by Tarski and Lindenbaum (in (1)) in 1936. Mautner (in (1)) uses the group of permutations of the individuals to discuss and classify logical objects, in the same way that Klein and Weyl used the full linear group and its subgroups to classify geometrical objects; Mautner in fact gives his paper the subtitle 'An extension of the Erlanger programme', and follows as closely as he can the exposition given by Weyl in his 'The classical groups'. But I think that the effort involved in making the parallel a close one is not sufficiently rewarded by any increase in elegance or insight to be worth while; what he expresses in terms of logical tensors and representations in Boolean rings can, I think, be more lucidly and succinctly expressed in terms of the hierarchy of types and the operator 'Tra'.

Section 3. Virtual Types.

One often wishes to concentrate one's attention on certain chosen elements in some type - for example, those elements of the type $\iota\iota(\iota\iota)$ which represent the natural numbers- and on the appropriate elements in higher types which represent functions of the chosen elements, taking chosen elements as values, and so on. The formulae that are required in proving assertions about these elements soon become very unwieldy; but if one introduces a new basic type, whose elements correspond to the chosen elements, this unwieldiness is avoided. The new basic type is called a virtual type; when it is introduced, so must all the associated complex types, and the appropriate constants - Π or Q, ι, C - and the appropriate axioms - (A) or (Q), (D), (E), (but not necessarily (I)). Any expression in this new extended system may be translated into an expression of the old system, which will have the same intuitive meaning; in this way it is possible to show that the new system is consistent if the old one is: and further that one can add to the new system additional constants and axioms - for example, in introducing a virtual type for the natural numbers one might add a constant for the successor function, and Peano's axioms - provided one can give a translation of the constants into definite expressions of the old systems in such a way that the translations of the new axioms are provable propositions of the old system.

Let α be the type to which the chosen elements belong,*
and let $P_{o\alpha}$ represent the set of which they are the only members, and let τ be the symbol adopted for the virtual type. Then evidently the translation of

$$(\underline{t}_{\tau})(A_{o\tau}\underline{t})$$

will be

$$(\underline{x}_{\alpha})(P_{o\alpha}\underline{x} \supset A'_{o\alpha}\ \underline{x}),$$

where $A'_{o\alpha}$ is the translation of $A_{o\tau}$. The range of a variable in the translation has thus to be restricted, and the first thing to be done is to find out what is the proper restriction for each complex type. Evidently the definition of the restrictions and the definition of the translation of a formula must be such that the translation of a closed formula will satisfy the restrictions. There appear to be two methods of ensuring that this will be so; in the first method the definition of the restrictions is simple, but the translation $(\lambda x_{\beta}.A_{\gamma})^T$ of $\lambda x_{\beta}.A_{\gamma}$ is not $\lambda x_{\beta}.(A_{\gamma})^T$; this method will be used in connection with a similar problem in section 4. In the second method $(\lambda x_{\beta}.A_{\gamma})^T$ is $\lambda x_{\beta}.(A_{\gamma})^T$, but the restrictions are more complicated; it is slightly simpler to apply this method to the system (G). Of course, the complications that arise are largely due to the necessity of ensuring that the translation of the axiom of extensionality in one of the added types is a provable proposition of the old system.

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* We suppose that it is not o.

Let β be a type of the system (τ) (i.e. the system in which τ is one of the basic types), and let $\bar{\beta}$ be the corresponding type of the old system (i.e. it is obtained by replacing τ by α throughout β). Then to each β we define an equivalence relation $R^{\beta}_{o\bar{\beta}\bar{\beta}}$, and a restriction $P^{\beta}_{o\bar{\beta}}$, for the type $\bar{\beta}$. It is important to note that these depend on β, for several different β may give rise to the same $\bar{\beta}$. The definitions depend also on $P_{o\alpha}$, and we could abstract with respect to it - as we did with respect to $\underline{m}_{\alpha\beta}$ when defining 'Tra'; but, because of the consequent unwieldiness of the formulae, we do not do so. The definitions of R and P are simultaneously inductive.

$R^{\beta}_{o\bar{\beta}\bar{\beta}} \rightarrow Q_{o\bar{\beta}\bar{\beta}}$ if τ is not ~~, nor~~ a part of β;

$R^{\gamma\delta}_{o(\bar{\gamma}\bar{\delta})(\bar{\gamma}\bar{\delta})} \rightarrow \lambda \underline{f}_{\bar{\gamma}\bar{\delta}} \underline{g}_{\bar{\gamma}\bar{\delta}} . (\underline{x}_{\bar{\delta}})(P^{\delta}_{o\bar{\delta}}\ \underline{x} \supset R^{\gamma}_{o\bar{\gamma}\bar{\gamma}}(\underline{fx})(\underline{gx}))$
if τ is a part of β.

$\underset{\sim}{A}_{\bar{\beta}} \cong \underset{\sim}{B}_{\bar{\beta}} \rightarrow R^{\beta}_{o\bar{\beta}\bar{\beta}}\ \underset{\sim}{A}_{\bar{\beta}} \underset{\sim}{B}_{\bar{\beta}}$.

$P^{\beta}_{o\bar{\beta}} \rightarrow \lambda \underline{x}_{\bar{\beta}} . T$ if τ is not β, nor a part of β;

$P^{\tau}_{o\alpha} \rightarrow P_{o\alpha}$;

$P^{\gamma\delta}_{o(\bar{\gamma}\bar{\delta})} \rightarrow \lambda \underline{f}_{\bar{\gamma}\bar{\delta}} . (\underline{x}_{\bar{\delta}})(\underline{y}_{\bar{\delta}})(P^{\delta}_{o\bar{\delta}}\ \underline{x}\ \&\ P^{\delta}_{o\bar{\delta}}\ \underline{y}\ \&\ \underline{x} \cong \underline{y} . \supset .$
$P^{\gamma\delta}_{o(\bar{\gamma}\bar{\delta})}(\underline{fx})\ \&\ \underline{fx} \cong \underline{fy})$
if τ is a part of $\gamma\delta$.

3.1) $\vdash\ \underline{x}_{\bar{\beta}} \cong \underline{x}_{\bar{\beta}}\ \&\!:\ \underline{x}_{\bar{\beta}} \cong \underline{y}_{\bar{\beta}}\ \&\ \underline{y}_{\bar{\beta}} \cong \underline{z}_{\bar{\beta}} . \supset . \underline{y}_{\bar{\beta}} \cong \underline{x}_{\bar{\beta}}\ \&\ \underline{x}_{\bar{\beta}} \cong \underline{z}_{\bar{\beta}}$.
For the proposition is provable if τ is not a part of β, and

its provability for other types follows from the definition of '$\cong$' by induction over the length of the type symbol.

A clearer idea of the significance of R and P is obtained by expressing them for functions of several arguments; in fact we have:

$$3.2) \vdash \quad f_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon} \cong g_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon} \equiv (\underline{c}_{\bar\gamma}, \ldots, \underline{d}_{\bar\delta}, \underline{e}_{\bar\varepsilon})(P^{\gamma}\underline{c} \,\&\ldots\&\, P^{\delta}\underline{d} \,\&\, P^{\varepsilon}\underline{e} \,.\supset.\, \underline{f}_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon}\underline{ed}\ldots\underline{c} \cong \underline{g}_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon}\underline{ed}\ldots\underline{c})$$

$$3.3) \vdash P^{\beta\gamma\ldots\delta\varepsilon}\underline{f}_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon} \equiv (\underline{c}_{\bar\gamma}, \ldots \underline{d}_{\bar\delta}, \underline{e}_{\bar\varepsilon}, \underline{c}'_{\bar\gamma}, \ldots \underline{d}'_{\bar\delta}, \underline{e}'_{\bar\varepsilon})(P^{\gamma}\underline{c} \,\&\ldots\&\, P^{\delta}\underline{d} \,\&\, P^{\varepsilon}\underline{e} \,\&\, P^{\gamma}\underline{c}' \,\&\ldots\&\, P^{\delta}\underline{d}' \,\&\, P^{\varepsilon}\underline{e}' \,\&\, \underline{c} \cong \underline{c}' \,\&\ldots\&\, \underline{d} \cong \underline{d}' \,\&\, \underline{e} \cong \underline{e}' \,.\supset.\, P^{\beta}(\underline{f}_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon}\underline{ed}\ldots\underline{c}) \,\&\, \underline{f}_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon}\underline{ed}\ldots\underline{c} \cong \underline{f}_{\bar\beta\bar\gamma\ldots\bar\delta\bar\varepsilon}\underline{e}'\underline{d}'\ldots\underline{c}')$$

That these are provable can be shown by induction over the number of arguments. If we translate P and R by the words 'proper' and 'equivalent', then we may say that two functions are equivalent if they take equivalent values for proper arguments, and that a proper function is one that takes proper values for proper arguments, and equivalent values for equivalent arguments.

In defining the translation of a formula of (τ), we have to settle on a translation for C_τ, for C_α may not be one of the chosen elements (i.e. it is not necessary that $\vdash P_{\alpha\alpha}C_\alpha$). For example, in introducing the virtual type of natural numbers it is more convenient to use 0 as the nonsense element than to introduce an element which does not correspond to a natural number.

Finally if we want to introduce in (τ) certain additional constants and axioms, then we must be able to give translations of these constants in an appropriate way - we give a formal statement of the conditions in theorems III and IV below.

We are now able to give an inductive definition of the translation $\underset{\sim}{A}'_{\bar{\gamma}}$ of a formula $\underset{\sim}{A}_{\gamma}$:

a) The constants N_{oo}, A_{ooo}, C_o, C_ι, $\iota_{o(oo)}$, $\iota_{\iota(o\iota)}$ are their own translations.

b) $R^{\gamma}_{\sigma\bar{\gamma}\bar{\gamma}}$ is the translation of $Q_{\sigma\gamma\gamma}$;

c) The translation of a variable $\underset{\sim}{x}_{\beta}$ is the variable $\underset{\sim}{x}_{\bar{\beta}}$;

d) The translations of C_{τ} and any additional constants $X_{\varrho}, \ldots$; are appropriately chosen; we denote them by C'_{α}, $X'_{\bar{\eta}}, \ldots$; one of the implications of 'appropriately chosen' is that $\vdash P^{\tau}C'_{\alpha}$ and $\vdash P^{\varrho}X'_{\bar{\eta}}, \ldots$;

e) The translation of $\iota_{\tau(o\tau)}$ is:

$$\lambda \underline{f}_{o\alpha}.(\imath \underline{x}_{\alpha})((E!\underline{y}_{\alpha})(\underline{f}\underline{y} \,\&\, P^{\tau}\underline{y}) \supset \underline{x} = \iota^{\alpha}\underline{f}$$
$$.\&.\ \sim(E!\underline{y}_{\alpha})(\underline{f}\underline{y} \,\&\, P\,\underline{y}) \supset \underline{x} = C'_{\alpha})\ ;$$

f) The translation of $\underset{\sim}{A}_{\beta\gamma}\underset{\sim}{B}_{\gamma}$ is $\underset{\sim}{A}'_{\bar{\beta}\bar{\gamma}}\underset{\sim}{B}'_{\bar{\gamma}}$;

h) The translation of $\lambda\underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\gamma}$ is $\lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{A}'_{\gamma}$.

3.4) $\vdash P^{o\gamma\gamma}R_{\sigma\bar{\gamma}\bar{\gamma}}$

This follows almost immediately from 3.3) and 3.1).

3.5) $\vdash \; P^{\tau(o\tau)}\iota'_{\lambda(o\lambda)}$

This follows almost immediately from the definitions.

3.6) $\vdash \; P^{\beta\gamma(\delta\gamma)(\beta\delta\gamma)}\,W'_{\bar\beta\bar\gamma(\bar\delta\bar\gamma)(\bar\beta\bar\delta\bar\gamma)}$

For $W'_{\bar\beta\bar\gamma(\bar\delta\bar\gamma)(\bar\beta\bar\delta\bar\gamma)}$ is $W_{\bar\beta\bar\gamma(\bar\delta\bar\gamma)(\bar\beta\bar\delta\bar\gamma)}$, and the result follows by 3.1) and 3.2).

3.7) $\vdash \; P^{\beta\gamma\beta}K'_{\bar\beta\bar\gamma\bar\beta}$

<u>Lemma A</u>. If $\underset{\sim}{A}_\beta$ is a closed formula of (τ), then $\vdash P^\beta \underset{\sim}{A}'_{\bar\beta}$. (Note that in (τ) the additional constants $X_\gamma, \ldots,$ count <u>as</u> constants, not as variables.)

For, by definition, $\vdash P^{\beta\gamma}\underline{f}_{\bar\beta\bar\gamma} \supset P^\beta(\underline{f}_{\bar\beta\bar\gamma}\underline{x}_{\bar\gamma})$. Further if Z_ε is a constant or a W or a K of (τ), then $\vdash P^\varepsilon Z'_{\bar\varepsilon}$. Hence, by the corollary to theorem I the lemma is true.

By the translation of the assertion of a proposition $\underset{\sim}{A}_o$, we shall mean the assertion of $P^\beta \underset{\sim}{b}_{\bar\beta}$ & ... & $P^\gamma \underset{\sim}{c}_{\bar\gamma} \;.\supset.\; \underset{\sim}{A}'_o$, where $\underset{\sim}{b}_\beta, \ldots, \underset{\sim}{c}_\gamma$, is a complete list of the free variables of $\underset{\sim}{A}_o$.

<u>Lemma B</u>. The translations of the assertions of the axioms of (τ) are provable.

For axiom (Q) we have:

P.1. $P^\beta\underline{x}_{\bar\beta}$ & $P^\beta\underline{y}_{\bar\beta}$ & $P^{o\beta}\underline{f}_{o\bar\beta} \;.\supset.\; \underline{x}_{\bar\beta} \cong \underline{y}_{\bar\beta} \supset (\underline{f}_{o\bar\beta}\underline{x}_{\bar\beta} \supset \underline{f}_{o\bar\beta}\underline{y}_{\bar\beta})$.

2 $\quad P^{o\beta}\underline{f}_{o\bar\beta}$ & $\underline{x}_{\bar\beta} \cong \underline{y}_{\bar\beta} \;.\supset.\; \underline{f}_{o\bar\beta}\underline{x}_{\bar\beta} \cong \underline{f}_{o\bar\beta}\underline{y}_{\bar\beta}$

3 $\quad$ P.1. $\qquad$ Q.E.D.

Next we note that the translation of $(\underset{\sim}{x}_\beta)(\underset{\sim}{F}_{o\beta}\underset{\sim}{x})$ is

$$\underset{\sim}{F}'_{o\bar\beta} \cong \lambda\underset{\sim}{x}_{\bar\beta}.T,$$

and hence is provably equivalent to

$$(\underset{\sim}{x}_{\beta})(P^{\beta}\underset{\sim}{x} \supset \underset{\sim}{F}^{1}_{o\bar{\beta}}\, x).$$

Therefore, for axiom (E) we have:

P.1 $P^{\beta\gamma}\underline{f}_{\bar{\beta}\bar{\gamma}}$ & $P^{\beta\gamma}\underline{g}_{\bar{\beta}\bar{\gamma}} \;.\supset.\; (\underline{x}_{\bar{\gamma}})(P^{\gamma}\underline{x} \supset \underline{f}_{\bar{\beta}\bar{\gamma}}\underline{x} \cong \underline{g}_{\bar{\beta}\bar{\gamma}}\underline{x}) \supset \underline{f}_{\bar{\beta}\bar{\gamma}} \cong \underline{g}_{\bar{\beta}\bar{\gamma}}$

But the right hand side is provable by definition of ' $\cong$ '.

For the axioms (D) for the type τ, we have:

P.1 $P^{o\tau}\underline{f}_{o\alpha} \;.\supset.\; (E!\underline{x}_{\alpha})(P^{\tau}\underline{x}\ \&\ \underline{f}_{o\alpha}\underline{x}) \supset \underline{f}_{o\alpha}(\iota'\, \underline{f}_{o\alpha})$,

P.2 $P^{o\tau}\underline{f}_{o\alpha} \;.\supset.\; \sim(E!\underline{x}_{\alpha})(P^{\tau}\underline{x}\ \&\ \underline{f}_{o\alpha}\underline{x}) \supset \iota'\underline{f}_{o\alpha} = C'_{\alpha}$,

where ι', the translation of $\iota_{\tau(o\tau)}$, is defined by the formula on page 43. Using this definition the proof of the above propositions is almost immediate.

The translations of the assertions of the axioms (P) and (T) are evidently provably equivalent to those axioms themselves. It is a condition of the choice of the translations of the additional constants, and of the choice of the additional axioms, that the translations of the assertions of additional axioms should be provable. If axioms of infinity are required for the type τ, they are to be included among the additional axioms.

This completes the demonstration of the lemma.

Lemma C. If a proposition may be inferred from others by the rules of inference, then the translation of its assertion may be inferred from the translations of their assertions, provided that, if $\underset{\sim}{x}_{\beta}$ is variable occurring bound or free in any of the propositions, and if γ is a type symbol such that $\bar{\beta}$ and $\bar{\gamma}$ are the same, then the variable x_{γ} does not occur bound or free in any of the propositions.

The proviso means that if two variables are distinct in the propositions, then the corresponding variables in their translations will also be distinct.

If B_o follows from A_o by an application of rules I, II, III, then evidently $P^{\beta}b_{\bar{\beta}}$ &...& $P^{\gamma}c_{\bar{\gamma}} \supset B_o'$ follows from $P^{\beta}b_{\bar{\beta}}$ &...& $P^{\gamma}c_{\bar{\gamma}} \supset A_o'$ by an application of the same rule.

For rule G.IV we want to show

$$P^{\beta}b_{\bar{\beta}} \;\&\ldots\&\; P^{\gamma}c_{\bar{\gamma}} \supset A'_{\bar{\rho}} \cong B'_{\bar{\rho}} \vdash P^{\beta}b_{\bar{\beta}} \;\&\ldots\&\; P^{\varepsilon}e_{\bar{\varepsilon}}$$
$$\supset F'_{\bar{\sigma}\bar{\rho}}\, A'_{\bar{\rho}} \cong F'_{\bar{\sigma}\bar{\rho}}\, A'_{\bar{\rho}}$$

where $b_{\beta}, \ldots, c_{\gamma};\ d_{\delta}, \ldots, e_{\varepsilon}$, are complete lists of the free variables of A_{ρ} and B_{ρ}; $F_{\sigma\rho}$ respectively (there may be overlapping between the lists).

H.1	$P^{\beta}b_{\bar{\beta}}$ &...& $P^{\gamma}c_{\bar{\gamma}} \supset A'_{\bar{\rho}} \cong B'_{\bar{\rho}}$	
H.2	$P^{\beta}b_{\bar{\beta}}$ &...& $P^{\gamma}c_{\bar{\gamma}}$ & $P^{\delta}d_{\bar{\delta}}$ &...& $P^{\varepsilon}e_{\bar{\varepsilon}}$	
3	$A'_{\bar{\rho}} \cong B'_{\bar{\rho}}$	
4	$F'_{\bar{\sigma}\bar{\rho}} = (\lambda d_{\bar{\delta}} \ldots e_{\bar{\varepsilon}}.F'_{\bar{\sigma}\bar{\rho}})d_{\bar{\delta}} \ldots e_{\bar{\varepsilon}}$	III.
5	$P^{\sigma\rho}F'_{\bar{\sigma}\bar{\rho}}$	Lemma A, H.2, 3.3).
6	$F'_{\bar{\sigma}\bar{\rho}}\, A'_{\bar{\rho}} \cong F'_{\bar{\sigma}\bar{\rho}}\, B'_{\bar{\rho}}$	P.
7	H.2 $\supset$ 6	(H.2).

which is the required inference.

For rule V we want to show that

$$P^{\beta}b_{\bar{\beta}} \supset B_o',\ P^{\beta}b_{\bar{\beta}} \;\&\; P^{\delta}d_{\bar{\delta}} \;.\supset.\; B_o' \supset D_o' \vdash P^{\delta}d_{\bar{\delta}} \supset D_o'$$

where for simplicity we suppose that b_{ρ}, d_{δ}, are the only free variables of B_{σ} and D_{σ} respectively.

H.1 $(P^{\beta} b_{\bar\beta} \supset B'_{\sigma}) \;\&\; (P^{\beta} b_{\bar\beta} \;\&\; P^{\delta} d_{\bar\delta} \;.\supset.\; B'_{\sigma} \supset D'_{\sigma})$

H.2 $P^{\beta} b_{\bar\beta}$ (b)

3 $P^{\delta} d_{\bar\delta} \supset D'_{\sigma}$

4 $P^{\beta} C'_{\bar\beta}$ Lemma A

5 $(E\underline{b}_{\bar\beta})(P^{\beta}\underline{b})$

6 $P^{\delta} d_{\bar\delta} \supset D'_{\sigma}$ (H.2), 5.

which is the required inference.

For rule G.VI we want to show that:

$$P^{\beta} b_{\bar\beta} \;\&\ldots\&\; P^{\gamma} c_{\bar\gamma} \supset A'_{\bar\sigma} \cong B'_{\bar\sigma} \vdash P^{\beta} b_{\bar\beta} \;\&\ldots\&\; P^{\gamma} c_{\bar\gamma} \supset \lambda x_{\bar\rho} . A'_{\bar\sigma} \cong \lambda x_{\bar\rho} . B'_{\bar\sigma}$$

where x_{ρ} may or may not be among the free variables $b_{\rho}, \ldots, c_{\gamma}$, of A_{σ} and B_{σ}.

H.1 $P^{\beta} b_{\bar\beta} \;\&\ldots\&\; P^{\gamma} c_{\bar\gamma} \supset A'_{\bar\sigma} \cong B'_{\bar\sigma}$

H.2 $P^{\beta} b_{\bar\beta} \;\&\ldots\&\; P^{\gamma} c_{\bar\gamma}$ $(b \ldots c)$

3 $A'_{\bar\sigma} \cong B'_{\bar\sigma}$

4 $(\lambda x_{\bar\rho} . A'_{\bar\sigma}) x_{\bar\rho} \cong (\lambda x_{\bar\rho} . B'_{\bar\sigma}) x_{\bar\rho}$ III, may need a change of bound variables.

5 $(x_{\bar\rho})(P^{\rho} x_{\bar\rho} \supset (\lambda x_{\bar\rho} . A'_{\bar\sigma}) x \cong (\lambda x_{\bar\rho} . B'_{\bar\sigma}) x)$ C.VI, P.C.

6 $\lambda x_{\bar\rho} . A'_{\bar\sigma} \cong \lambda x_{\bar\rho} . B'_{\bar\sigma}$ '$\cong$', change bound variables back again.

7 H.2 $\supset$ 6 (H.2)

which is the required inference.

For rule G.VII the lemma is obvious. This concludes the demonstration of Lemma C.

free variables of $\underset{\sim}{B}_{\sigma}$ and $\underset{\sim}{D}_{\sigma}$ respectively.

H.1 $(P^{\beta}\underset{\sim}{b}_{\bar{\beta}} \supset \underset{\sim}{B}'_{\bar{\sigma}}) \& (P^{\beta}\underset{\sim}{b}_{\bar{\beta}} \& P^{\delta}\underset{\sim}{d}_{\bar{\delta}} . \supset . \underset{\sim}{B}'_{\bar{\sigma}} \supset \underset{\sim}{D}'_{\bar{\sigma}})$

H.2 $P^{\beta}\underset{\sim}{b}_{\bar{\beta}}$ (b)

3 $P^{\delta}\underset{\sim}{d}_{\bar{\delta}} \supset \underset{\sim}{D}'_{\bar{\sigma}}$

4 $P^{\beta}C'_{\bar{\beta}}$ Lemma A

5 $(E\underline{b}_{\bar{\beta}})(P^{\beta}\underline{b})$

6 $P^{\delta}\underset{\sim}{d}_{\bar{\delta}} \supset \underset{\sim}{D}'_{\bar{\sigma}}$ (H.2), 5.

which is the required inference.

For rule G.VI we want to show that:

$$P^{\beta}\underset{\sim}{b}_{\bar{\beta}} \& \ldots \& P^{\gamma}\underset{\sim}{c}_{\bar{\gamma}} \supset \underset{\sim}{A}'_{\bar{\sigma}} \cong \underset{\sim}{B}'_{\bar{\sigma}} \vdash P^{\beta}\underset{\sim}{b}_{\bar{\beta}} \& \ldots \& P^{\gamma}\underset{\sim}{c}_{\bar{\gamma}} \supset \lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{A}'_{\bar{\sigma}} \cong \lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{B}'_{\bar{\sigma}}$$

where $\underset{\sim}{x}_{\beta}$ may or may not be among the free variables $\underset{\sim}{b}_{\beta}, \ldots, \underset{\sim}{c}_{\gamma}$, of $\underset{\sim}{A}_{\sigma}$ and $\underset{\sim}{B}_{\sigma}$.

H.1 $P^{\beta}\underset{\sim}{b}_{\bar{\beta}} \& \ldots \& P^{\gamma}\underset{\sim}{c}_{\bar{\gamma}} \supset \underset{\sim}{A}'_{\bar{\sigma}} \cong \underset{\sim}{B}'_{\bar{\sigma}}$

H.2 $P^{\beta}\underset{\sim}{b}_{\bar{\beta}} \& \ldots \& P^{\gamma}\underset{\sim}{c}_{\bar{\gamma}}$ $(\underset{\sim}{b} \ldots \underset{\sim}{c})$

3 $\underset{\sim}{A}'_{\bar{\sigma}} \cong \underset{\sim}{B}'_{\bar{\sigma}}$

4 $(\lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{A}'_{\bar{\sigma}})\underset{\sim}{x}_{\bar{\beta}} \cong (\lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{B}'_{\bar{\sigma}})\underset{\sim}{x}_{\bar{\beta}}$ III, may need a change of bound variables.

5 $(\underset{\sim}{x}_{\bar{\beta}})(P^{\beta}\underset{\sim}{x}_{\bar{\beta}} \supset (\lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{A}'_{\bar{\sigma}})\underset{\sim}{x} \cong (\lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{B}'_{\bar{\sigma}})\underset{\sim}{x})$ C.VI, P.C.

6 $\lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{A}'_{\bar{\sigma}} \cong \lambda\underset{\sim}{x}_{\bar{\beta}}.\underset{\sim}{B}'_{\bar{\sigma}}$ '$\cong$', change bound variables back again.

7 H.2 $\supset$ 6 (H.2)

which is the required inference.

For rule G.VII the lemma is obvious. This concludes the demonstration of Lemma C.

We are now in a position to give formal definitions and theorems about the introduction of virtual types.

Definition A

The type symbols of system (τ) are the same as those of (G) together with the basic type symbol τ, and the consequent complex types.

The constants of (τ) are those of (G), those required by introduction of the new types (viz. $\iota_{\tau(o\tau)}$, C_{τ}, $Q_{o\gamma\gamma}$), and the additional constants $X_{\eta}, \ldots$, (for simplicity we suppose that there is only one of these).

The variables of (τ) are those of (G) together with those required by the introduction of the new types.

The axioms of (τ) are those of (G) together with those required by the introduction of the new types (viz. (D) for type τ, (Q) and (E) for all new types), and the additional axiom $\underset{\sim}{A}_o$ which is to be a closed formula of (τ).

The rules of inference of (τ) are the same as those of (G).

Theorem III

Let α be a type symbol of (G). Let $\underset{\sim}{P}_{o\alpha}$, $\underset{\sim}{C}'_{\alpha}$, $\underset{\sim}{X}'_{\bar{\eta}}$, be closed formulae of (G) - where $\bar{\beta}$ is obtained from β by replacing τ by α throughout β. Let $P^{\gamma}_{o\bar{\gamma}}$ be defined for each type γ as on page 41, and let the translation $\underset{\sim}{B}'_{\bar{\gamma}}$ of a formula $\underset{\sim}{B}_{\gamma}$ of (τ) be defined as on page 43.

Then if:

1) (G) is consistent:

2) $\vdash P^{\tau}\underset{\sim}{C}'_{\alpha}$ & $P^{\eta}\underset{\sim}{X}'_{\bar{\eta}}$;

3) $\vdash \underset{\sim}{A}'_{\delta}$

then:

a) (τ) is consistent;

b) If a proposition is provable in (τ), then the translation of its assertion is provable in (G);

c) If a proposition is provable in (τ), and if it is expressed wholly by means of the symbols which are common to both (τ) and (G), then it is also provable in (G);

d) If a formula is closed in (τ), then its translation is closed in (G);

e) a), b), c), d), remain true, if to (τ) there is adjoined the axiom:

$$\text{(M)} \qquad (E\underline{f}_{\alpha\tau})\Big[(\underline{x}_{\alpha})(P^{\tau}\underline{x} \supset (E!\underline{t}_{\tau})(\underline{f}\underline{t} = \underline{x})) \,\&\, \underline{f}C_{\tau} = \underset{\sim}{C}'_{\alpha} \,\&\, \mathrm{Tra}^{\eta}\underline{f}X_{\eta} = \underset{\sim}{X}'_{\bar{\eta}}\Big].$$

<u>Proof</u>

b) follows from lemmas B and C, for the axioms of (τ) satisfy the proviso of lemma C, and hence the proof of any proposition can be so arranged that the proviso is satisfied for all the steps of the proof. a) follows from b). From the definition of P^{γ} for the types which do not contain τ, it follows almost immediately that the translation of the assertion of a proposition of (τ), which is expressed wholly in terms of the symbols of (G), is provably equivalent, in (G), to the corresponding proposition of (G); hence c) is true. d) is an immediate consequence of the definition of translation. To show that e) is true we have to show that the translation of the axiom is provable in (G): the translation is of the form

$$(E\underline{f}_{o\alpha\alpha})\left[\quad\right],$$

and it can - tediously - be shown that the expression in the square brackets is provable if '$\lambda\underline{z}_\alpha.\underline{z}$' is substituted for '$\underline{f}$'. This completes the proof of the theorem.

Sometimes one may want to introduce a virtual type for which the relevant elements are not represented by closed formulae; for example, one might want to form a virtual type consisting of a certain finite number of individuals. Instead of being represented by a closed formula the defining property will be required to satisfy some condition which is represented by a closed formulae, and the translations of the constants of (τ) may also be required to satisfy certain conditions; we suppose that all the conditions have been rolled into one formula F.

Theorem IV.

Let the system (τ) be defined as on page 48. Let α be a type symbol of (G). Let $P_{o\alpha}$, C'_α, $X'_{\bar{\gamma}}$, be variables of (G). Let $\bar{\beta}$, a type symbol of (G) be obtained from β, a type symbol of (τ) by replacing τ by α throughout β. Let $P^{\gamma}_{o\gamma}$ be defined in terms of $P_{o\alpha}$ as on page 41; and let the translation $\underset{\sim}{B}'_{\bar{\gamma}}$ of a formula $\underset{\sim}{B}_{\gamma}$ of (τ) be defined as on page 43. Let $\underset{\sim}{F}_{o\bar{\eta}\alpha(o\alpha)}$ be a closed formula of (G).

Then if:

1) (G) is consistent;

2) $\vdash (EP_{o\alpha}, C'_{\alpha}, X'_{\bar{\eta}})(\underset{\sim}{F}_{o\bar{\eta}\alpha(o\alpha)} P\, C'\, X')$;

3) $\vdash \underset{\sim}{F}_{o\bar{\eta}\alpha(o\alpha)} P_{o\alpha} C'_{\alpha} X'_{\bar{\eta}} . \supset . P_{o\alpha} C'_{\alpha} \,\&\, P^{\eta}_{o\bar{\eta}} X'_{\bar{\eta}} \,\&\, \underset{\sim}{A}'_{o}$;

then:

a) (τ) is consistent;

b) if $\underset{\sim}{D}_{o}$ is the translation of the assertion of a proposition which is provable in (τ), then

$\underset{\sim}{F}_{o\eta\alpha(o\alpha)} P_{o\alpha} C'_{\alpha} X'_{\bar{\eta}} \supset \underset{\sim}{D}_{o}$;

c) as c) in theorem III;

d) as e) in theorem III.

<u>Proof</u>

We make the hypothesis:

H $\quad \underset{\sim}{F}_{o\bar{\eta}\alpha(o\alpha)} P_{o\alpha} C'_{\alpha} X'_{\bar{\eta}}$,

and then, in virtue of condition 3) of the theorem proceed exactly as in the proof of theorem III, and finally eliminate H, using condition 2) of the theorem.

The first application we make of these theorems is to form the type ν of natural numbers. For $P_{o\alpha}$, the defining property, we take $Num_{o\iota'}$: the additional constants are 0_{ν} and $S_{\nu\nu}$, their translations are $0_{\iota'}$ and $S_{\iota'\iota'}$: the translation of C_{ν} we also take to be 0 (it would be inconvenient to have an element of type ν that did not represent a natural number).

The additional axioms for the type ν are:

(C) $\quad 0_{\nu} = C_{\nu}$;

(O) $\quad S\underline{x}_{\nu} \neq 0_{\nu}$;

(S) $\quad \underline{x}_{\nu} \neq \underline{y}_{\nu} \supset S\underline{x}_{\nu} \neq S\underline{y}_{\nu}$;

(H) $\quad \underline{f}_{o\nu}0 \,\&\, (\underline{y}_{\nu})(\underline{f}_{o\nu}\underline{y} \supset \underline{f}_{o\nu}(S\underline{y})) \,.\supset.\, \underline{f}_{o\nu}\underline{x}_{\nu}$.

It is fairly easy to prove the translations of the assertions of these axioms; for (O) the appropriate theorem is:

$$\mathrm{Num}_{o\iota'}\underline{x}_{\iota'} \supset S_{\iota'\iota'}\underline{x}_{\iota'} \neq 0_{\iota'} \quad ,$$

which is proved in Church (1).

We define 'Nap' ('numerical application') as follows:

$$\mathrm{Nap}_{\alpha'\nu} \rightarrow \lambda\underline{m}_{\nu}.(\imath\underline{x}_{\alpha'})(E!\underline{f}_{\alpha'\nu})(\underline{f}0_{\nu} = 0_{\alpha'} \,\&\, \underline{x} = \underline{f}\underline{m} \,\&$$
$$(\underline{n}_{\nu})(\underline{f}(S_{\nu\nu}\underline{n}) = S_{\alpha'\alpha'}(\underline{f}\underline{n})))$$

This provides an explicit formula for the function whose existence is guaranteed by the axiom (M), and it allows of successive application of a function in any type; for example:

$$\vdash \mathrm{Nap}_{\alpha'\nu}2_{\nu} = \lambda\underline{f}_{\alpha\alpha}\underline{x}_{\alpha}.\underline{f}(\underline{f}\underline{x}) \,.$$

We shall always use 'ν' to denote the type of natural numbers.

Another application of the theory of virtual types is the formation of quotient sets. Let $\underline{r}_{o\alpha\alpha}$ represent an equivalence relation over type α; then we can introduce a virtual type τ by means of the defining property:

$$P^{\tau}_{o(o\alpha)} \rightarrow \lambda\underline{f}_{o\alpha}.(E\underline{x}_{\alpha})(\underline{y}_{\alpha})(\underline{f}\underline{y} \supset \underline{r}_{o\alpha\alpha}\underline{x}\underline{y}) \vee \underline{f} = C_{o\alpha}$$

(The condition '$v\ \underline{f} = C_{o\alpha}$' is inserted to ensure that P C holds; it is not essential, but simplifies the subsequent work.) The elements of τ correspond with the equivalence classes of $\underline{r}$; further, if certain operators - i.e. additional constants - are defined for the type α, and if the equivalence classes of $\underline{r}$ are also congruence classes (in the sense of abstract algebra) with respect to the operators, then it will be possible to introduce corresponding operators for the type τ. Since this process is frequently used both in mathematics and theoretical physics, we investigate it further. First we extend the equivalence relation $\underline{r}$ to higher types ('Eqt$\underline{r}$'). Then we define, for any type, the property of being compatible with the equivalence relation $\underline{r}$ ('Com$\underline{r}$'). To any compatible operator based on the elements of α (i.e. belonging to a type of which α is a part), there corresponds an analogous operator based on the equivalence classes (and so belonging to a type of which $o\alpha$ is a part). This analogous operator - the quotient operator - is obtained from the original operator by means of the function Quo$\underline{r}$. If β is any type symbol we define β' as the symbol obtained by substituting $o\alpha$ for α throughout β, and β_1 as the symbol obtained by substituting τ for α throughout β. Then given a compatible operator X_β, we can introduce a corresponding additional constant U_{β_1} for the system (τ), whose translation will be Quo$\underline{r}$X - of type β'. What is meant by 'analogous' and 'corresponding' in the above rough summary will be more precisely indicated by the theorems which we prove below. In what follows $P^{\tau}_{\sigma\bar{\gamma}}$ and $R^{\tau}_{\sigma\bar{\gamma}\bar{\gamma}}$ (where γ is a type symbol of (τ), and $\bar{\gamma}$ is obtained by substituting $o\alpha$ for τ throughout γ) are defined in terms of $P^{\tau}_{o(o\alpha)}$ - and hence eventually in terms of $\underline{r}$ - as described on page 41. We note that if β is a type symbol of (G) - i.e. if it does not contain τ - then $\bar{\beta}_1$ is the same as β^1.

We define: Eqt^{β}, Com^{β}, Quo^{β}, for all types β of (G) as follows:

$Eqt^{\beta}_{o\beta\beta(o\alpha\alpha)} \rightarrow \lambda \underline{r}_{o\alpha\alpha}\underline{x}_{\beta}\underline{y}_{\beta}.\underline{x} = \underline{y}$ if α is not β nor a part of β;

$Eqt^{\alpha}_{o\alpha\alpha(o\alpha\alpha)} \rightarrow \lambda\underline{r}_{o\alpha\alpha}.\underline{r}$

$Eqt^{\beta\gamma}_{o(\beta\gamma)(\beta\gamma)(o\alpha\alpha)} \rightarrow. \lambda\underline{r}_{o\alpha\alpha}\underline{f}_{\beta\gamma}\underline{g}_{\beta\gamma}.(\underline{x}_{\gamma})(Com^{\gamma}\underline{rx} \supset Eqt^{\beta}\underline{r}(\underline{fx})(\underline{gx}))$

if α is a part of $\beta\gamma$.

$Com^{\beta}_{o\beta(o\alpha\alpha)} \rightarrow \lambda\underline{r}_{o\alpha\alpha}\underline{x}_{\beta}.T$ if α is not a part of β;

$Com^{\beta\gamma}_{o(\beta\gamma)(o\alpha\alpha)} \rightarrow \lambda\underline{r}_{o\alpha\alpha}\underline{f}_{\beta\gamma}.(\underline{x}_{\gamma},\underline{y}_{\gamma})(Com^{\gamma}\underline{rx}$ & $Com^{\gamma}\underline{ry}$ & $Eqt^{\gamma}\underline{rxy}$

$.\supset. Com^{\beta}\underline{r}(\underline{fx})$ & $Eqt^{\beta}(\underline{fx})(\underline{fy}))$

if α is a part of $\beta\gamma$.

It will be noticed that, except for type α, $Eqt^{\beta}\underline{r}$ and $Com^{\beta}\underline{r}$ are defined in the same way as were R^{β} and P^{β}; it follows that if $\underline{r}$ represents an equivalence relation then theorems exactly like 3.1), 3.2), 3.3), are provable. Hence the assertion of $Com^{\beta\ldots\gamma}\ \underline{rf}_{\beta\ldots\gamma}$ means that $\underline{f}$ takes equivalent values for equivalent arguments.

$Quo^{\beta}_{\rho\beta(o\alpha\alpha)} \rightarrow \lambda\underline{r}_{o\alpha\alpha}\underline{f}_{\beta}.\underline{f}$ if α is not ρ, nor a part of β;

$Quo^{\alpha}_{o\alpha\alpha(o\alpha\alpha)} \rightarrow \lambda\underline{r}_{o\alpha\alpha}.\underline{r}$;

$$\mathrm{Quo}^{\beta\gamma}_{\beta'\gamma'(\beta\gamma)(o\alpha\alpha)} \rightarrow \lambda \underline{r}_{o\alpha\alpha}\, \underline{f}_{\beta\gamma}\, \underline{x}_{\gamma'}.(\imath \underline{y}_{\beta'})(E\underline{u}_{\gamma})(\mathrm{Com}\underline{rf}\ \&\ \mathrm{Com}\underline{ru}$$
$$\&\ \underline{x} \cong \mathrm{Quo}\underline{ru}\ \&\ \underline{y} = \mathrm{Quo}\underline{r}(\underline{fu}))$$

if α is a part of $\beta\gamma$.

In this formula '$\underline{x} \cong \mathrm{Quo}\underline{ru}$' stands for '$R^{\gamma_1}_{o\gamma'\gamma'}\underline{x}(\mathrm{Quo}\underline{ru})$'; if we use ordinary equality instead, theorem 3.11) fails. The method by which Quo is defined is analogous to that used for Tra, but is more complicated because $\underline{r}$ is not in general a one to one map of α into $o\alpha$. Quo $\underline{r}$ does in fact define a map of β into β', which is a homomorphism with respect to functional application for all those elements of β which are compatible with $\underline{r}$; the equivalence relation which holds between two such elements if they have the same image under this homomorphism is the same as that represented by Eqt $\underline{r}$.

To make the statement of the theorems below more intelligible, we make the initial hypothesis:

$$V\underline{r}_{o\alpha\alpha}$$

which restricts the variable $\underline{r}_{o\alpha\alpha}$, and then introduce the abbreviations:

$$M^{\beta}_{o\beta} \rightarrow \mathrm{Com}^{\beta}\underline{r} \quad ,$$
$$T^{\beta}_{\beta'\beta} \rightarrow \mathrm{Quo}^{\beta}\underline{r} \quad ,$$
$$\underset{\sim}{A}_{\beta} \doteq \underset{\sim}{B}_{\beta} \rightarrow \mathrm{Eqt}^{\beta}\underline{r}\underset{\sim}{A}\underset{\sim}{B}_{o\beta\beta};$$

so that a complete statement of any of the theorems would be of the form:

$$V\underline{r}_{o\alpha\alpha} \supset \underset{\sim}{D}_{o}$$

3.8) $\vdash V(\mathrm{Eqt}^{\beta}\underline{r})$

3.9) $\vdash P^{\beta'}C_{\beta'}$

For $P^{\beta'}C_{\beta'} \supset P^{\beta'\gamma'}(\lambda\underline{x}_{\gamma'}.C_{\beta'})$.

3.10) $M\underline{f}_{\beta\gamma}$ & $M\underline{u}_{\gamma} \supset T\underline{f}_{\beta\gamma}(T\underline{u}_{\gamma}) = T(\underline{f}_{\beta\gamma}\underline{u}_{\gamma})$ provided $\beta\gamma$ is not α.

3.11) $P^{\beta'}(T\underline{f}_{\beta})$

3.12) $M\underline{f}_{\beta}$ & $M\underline{g}_{\beta}$ & $T\underline{f}_{\beta} \cong T\underline{g}_{\beta} \;.\supset.\; \underline{f}_{\beta} \doteq \underline{g}_{\beta}$

3.13) $M\underline{f}_{\beta}$ & $M\underline{g}_{\beta}$ & $\underline{f}_{\beta} \doteq \underline{g}_{\beta} \;.\supset.\; T\underline{f}_{\beta} = T\underline{g}_{\beta}$.

If α is not a part of $\beta\gamma$ nor of β, these theorems follow immediately from the definitions. We assume that 3.11), 3.12), 3.13), have been proved for types β and γ, and that α is a part of $\beta\gamma$, and we then prove 3.10), and 3.11), 3.12), 3.13), for type $\beta\gamma$.

<u>3.10</u>)

H.1	$M\underline{f}_{\beta\gamma}$ & $M\underline{u}_{\gamma}$	$(\underline{f},\underline{u})$
2	$T\underline{f}(T\underline{u}) = (\imath\underline{y}_{\beta'})(E\underline{v}_{\gamma})(M\underline{v}$ & $T\underline{u} \cong T\underline{v}$ & $\underline{y} = T(\underline{fv}))$	
H.3	$M\underline{v}_{\gamma}$ & $T\underline{v}_{\gamma} \cong T\underline{u}$	$(\underline{v})$
4	$\underline{v} \doteq \underline{u}$	3.12^{γ}).
5	$M(\underline{fu})$ & $M(\underline{fv})$ & $\underline{fu} \doteq \underline{fv}$	H.1, H.3, 4, Com.
6	$T(\underline{fu}) = T(\underline{fv})$	3.13^{β}).
7	$(E\underline{v}_{\gamma})(M\underline{v}$ & $T\underline{u} \cong T\underline{v}$ & $\underline{y}_{\beta'} = T(\underline{fv}))$ $\supset \underline{y}_{\beta'} = T(\underline{fu})$	(H.3).
8	$T\underline{f}(T\underline{u}) = T(\underline{fu})$	2, 7.
9	3.10)	(H.1). Q.E.D.

<u>3.11</u>)

H.1	$M\underline{f}_{\beta\gamma}$	$(\underline{f})$
H.2	$(E\underline{u}_{\gamma})(M\underline{u}$ & $\underline{x}_{\gamma'} \cong Tu)$	$(\underline{x})$
H.3	$M\underline{u}_{\gamma}$ & $\underline{x} \cong T\underline{u}_{\gamma}$	$(\underline{u})$

4	$\underline{x} \cong T\underline{v}_{\gamma} \supset T\underline{u} \cong T\underline{v}_{\gamma}$	.
5	$(\exists\underline{v}_{\gamma})(M\underline{v} \ \& \ \underline{x} \cong T\underline{v} \ \& \ \underline{y}_{\beta'} = T(\underline{fv}))$	
	$\supset \underline{y}_{\beta'} = T(\underline{fu})$	As for 7 above.
6	$T\underline{fx} = T(\underline{fu})$	Quo.
7	$P^{\beta'}(T\underline{fx})$	3.11$^{\beta}$).
H.8	$\underline{y}_{\gamma'} \cong \underline{x}$	($\underline{y}$)
9	$\underline{y} \cong T\underline{u}$	3.1).
10	$T\underline{fy} = T(\underline{fu}) = T\underline{fx}$	As for 6, 6.
11	H.2 .$\supset$. 7 & (H.8 $\supset$ 10)	(H.2, H.3, H.8).
H.12	$\sim$H.2	($\underline{x}$)
13	$T\underline{fx} = C_{\beta'}$	Quo.
14	$P^{\beta'}(T\underline{fx})$	3.9)
H.15	H.8	($\underline{y}$)
16	$\sim(\exists\underline{u}_{\gamma})(M\underline{u} \ \& \ \underline{y} = T\underline{u})$	H.12.
17	$T\underline{fx} = C_{\beta'} = T\underline{fy}$	13.
18	$P^{\beta'}(T\underline{fx}_{\gamma'}) \ \& \ (\underline{x}_{\gamma'} \cong \underline{y}_{\gamma'} \supset T\underline{fx}_{\gamma'} = T\underline{fy}_{\gamma'})$	(H.12, H.15), 11.
19	H.1 $\supset$ 18	(H.1)
20	$\sim M\underline{f}_{\beta\gamma} \supset T\underline{f}_{\beta\gamma} = C_{\beta'\gamma'}$	Quo.
21	3.11$^{\beta\gamma}$)	19, 20, 3.9)
3.12)		
H.1	$M\underline{f}_{\beta\gamma} \ \& \ M\underline{g}_{\beta\gamma} \ \& \ T\underline{f}_{\beta\gamma} \cong T\underline{g}_{\beta\gamma}$	($\underline{f}$,$\underline{g}$)
H.2	$M\underline{u}_{\gamma}$	($\underline{u}$)
3	$P^{\gamma'}(T\underline{u})$	3.11$^{\gamma}$).
4	$T\underline{f}(T\underline{u}) \cong T\underline{g}(T\underline{u})$	$\cong$.
5	$T(\underline{fu}) \cong T(\underline{gu})$	3.10$^{\beta\gamma}$).

6	$\underline{fu} \doteq \underline{gu}$	3.12^{β}).
7	H.2 $\supset$ 6	(H.2).
8	$\underline{f} \doteq \underline{g}$	Eqt.
9	$3.12^{\beta\gamma}$)	(H.1).

<u>3.13</u>)

H.1	$M\underline{f}_{\beta\gamma} \ \& \ M\underline{g}_{\beta\gamma} \ \& \ \underline{f}_{\beta\gamma} \doteq \underline{g}_{\beta\gamma}$	($\underline{f}$, $\underline{g}$)
H.2	$(E\underline{u}_{\gamma})(M\underline{u} \ \& \ \underline{x}_{\gamma'} \cong T\underline{u})$	($\underline{x}$)
H.3	$M\underline{u}_{\gamma} \ \& \ \underline{x} \cong T\underline{u}_{\gamma}$	($\underline{u}$)
4	$\underline{fu} \doteq \underline{gu}$	Eqt.
5	$T(\underline{fu}) = T(\underline{gu})$	3.13^{β}).
6	$T\underline{fx} = T\underline{gx}$	As in proof of $3.10^{\beta\gamma}$).
7	H.2 $\supset$ 6	(H.2, H.3).
H.8	$\sim$H.2	($\underline{x}$)
9	$T\underline{fx} = C_{\beta'} = T\underline{gx}$	Quo
10	$T\underline{f} = T\underline{g}$	(H.8), 7, (E).
11	$3.13^{\beta\gamma}$)	(H.1).

We can now construct the virtual type τ which represents the quotient set of the type α with respect to a given equivalence relation $\underline{r}$, and which has additional constants corresponding to certain constants which have been specified in connection with type α. (For instance, α may itself be a virtual type, and the specified constants may be just the additional constants for that type.) The interpretation of any type symbol is obtained by substituting $\alpha\alpha$ for τ ; the defining property $P^{\tau}_{o(o\lambda)}$ has already been specified. Let $X_{\delta}, \ldots, Y_{\varepsilon}$, be the constants specified in connection with the type α; (they will be closed formulae of the system as it stands before τ is introduced, but may of course involve additional constants - e.g. S_{w} - belonging to virtual types which have been introduced previously). Let $U_{\delta_1}, \ldots, V_{\varepsilon_1}$, be the corresponding additional constants belonging to the type τ. We define their translations:

$$U'_{\delta'} \text{ is } T^{\delta} X_{\delta},$$

$$\cdots\cdots\cdots\cdots$$

$$V'_{\varepsilon'} \text{ is } T^{\varepsilon} Y_{\varepsilon},$$

$$C'_{o\lambda} \text{ is } C_{o\lambda}.$$

Then provided that:

(C) $\qquad M^{\delta} X_{\delta} \,\&\ldots\&\, M^{\varepsilon} Y_{\varepsilon}$,

the system (τ) will have the properties specified in theorems III and IV; for from (C) we may infer:

$$\cdot P^{\delta_1} U'_{\delta'} \,\&\ldots\&\, P^{\varepsilon_1} V'_{\varepsilon'}$$

by 3.11). If for the equivalence relation $\underline{r}$ we choose a closed formula then (C) must be a provable proposition, and theorem III applies. If not, then we can regard (C) as an hypothesis which restricts the variable $\underline{r}$, and theorem IV applies.

The constants $X_{\delta}, \ldots, Y_{\varepsilon}$, will satisfy certain propositions or axioms, and it is natural to ask whether these axioms can be taken over into (τ); in general the answer must be 'no', but there will be many particular cases in which it is 'yes'. For example, suppose we have as an axiom:

$$(A)\quad \underline{x}_\alpha \neq C \ \& \ \underline{y}_\alpha \neq C \ \& \ \underline{z}_\alpha \neq C \ .\supset.\ X\underline{x}_\alpha(X\underline{y}_\alpha\underline{z}_\alpha) = X(X\underline{x}_\alpha\underline{y}_\alpha)\underline{z}_\alpha\ ,$$

where X is of type $\alpha\alpha\alpha$, so that, except for C_α, the type α is a semigroup with multiplication defined by X. To take (A) over into type τ means that we replace the suffix α by τ, and $X_{\alpha\alpha\alpha}$, by $U_{\tau\tau\tau}$; the question is then whether the translation of the assertion of (A^τ) is provable - we assume that a suitable equivalence relation has been specified. In fact it is provable, for we have:

$$(P) \vdash \quad P^\tau \underline{x}_{o\alpha} \ \& \ \underline{x}_{o\alpha} \neq C_{o\alpha} \ .\supset.\ (E\underline{u}_\alpha)(M\underline{u} \ \& \ \underline{x}_{o\alpha} = T\underline{u}),$$

and the result then follows by repeated applications of 3.10). Hence the axiom (A^τ) may be adopted as an additional axiom for the type τ ; this is of course a well known result. Now it is not difficult to prove a result similar to (P) for higher types, and it follows that any axiom, which, like (A), consists of a simple equality and does not contain bound variables, can be taken over in the same way. This is the situation that occurs in abstract algebra, where considerable use is made of the notion of a quotient set. (See, for example, lectures given by P.Hall in Cambridge, 1947 - 9). It would be interesting to investigate other kinds of axioms that can be taken over into the type τ, but we shall not pursue the matter here. It may seem at first sight that the employment of a system of symbolic logic in such investigations is quite unnecessary; but without some form of type notation, the definition of concepts like Com and Quo for objects of arbitrarily high type would be more unwieldy and less clear.

We shall later have occasion to use the type ρ of real numbers. Of course there are a large number of ways in which this can be introduced: perhaps the simplest is to start with the type $o\nu$, which can be interpreted as the set of all binary decimals; we then define the equivalence relation which holds between two elements if the corresponding decimals represent the same real number (in the ordinary sense), and which also holds between $\lambda \underline{x}_\nu . T$ and $\lambda \underline{x}_\nu . F$. The quotient of the type $o\nu$ by this relation we call type μ; it may be interpreted as the set of real numbers modulo an integer. Then we pick out from the type $o\nu\mu$ all those elements which take the value T for just one set of arguments, thus forming the type ρ. Of course it is possible to introduce additional constants in this type corresponding to the usual arithmetic and topological concepts, and to provide translations of these constants in such a way that the translations of the assertions of the usual axioms are provable propositions. We shall not carry out this programme, but we shall suppose it has been done.

So far as I know, the idea of introducing virtual types is due to A.M. Turing; (see footnote in Newman and Turing (1)).

He has not published his version, and I do not know to what extent the version given here is in agreement with his. The method really combines two processes, both of which have been current for some time. The first is simply the restriction of the ranges of variables - and is thus almost as old as algebra: it only becomes complicated when applied to all the types simultaneously. The second is the translation of the formulae of one system into those of another; it has been extensively used in the study of axiomatic systems, and goes back at least to Bolyai and Lobachevsky. Its applications in symbolic logic are especially due to the Polish school; we shall have more to say about it in the next section.

Section 4. Models.

Let us suppose that we have a set of closed formulae $\mathrm{Bas}^{\alpha}_{o\alpha}$, one for each type α, which satisfy the following conditions:

(B) i) If $\underset{\sim}{A}_{\alpha}$ is a closed formula, then $\vdash \mathrm{Bas}_{o\alpha}\underset{\sim}{A}_{\alpha}$;

ii) $\vdash \mathrm{Bas}_{o(\alpha\beta)}\,\underline{f}_{\alpha\beta}\ \&\ \mathrm{Bas}_{o\beta}\,\underline{x}_{\beta}\ .\supset .\ \mathrm{Bas}_{o\alpha}(\underline{f}_{\alpha\beta}\underline{x}_{\beta})$;

iii) $\vdash \mathrm{Bas}_{oo}\,\underline{p}_{o}\ \&\ \mathrm{Bas}_{o\iota}\,\underline{x}_{\iota}$.

Then, very roughly speaking, we are going to show that if all variables, both bound and free, are restricted to the ranges indicated by Bas, we obtain a true model of the system (G) - i.e. one in which the axioms are provable and the rules valid. The procedure is similar to that used in the last section; we provide a translation for every formula of (G) in such a way that the translations lie within the model, and the translation of provable propositions are again provable.

As before, it is the axiom of extensionality that gives trouble: there we translated ' = ' by ' $\stackrel{\frown}{=}$ ', so that distinct elements became identified; here it is the translation of ' λ ' which is important; it is such that the translation of any function takes the nonsense value for all irrelevant arguments. I believe that both methods are applicable to both cases; the one used in this section is, I think, a little easier to visualise, and a little more tiresome formally.

First we define a rather narrower restriction than that described by Bas. ('Bas' stands for 'basis', 'Mod' for 'model').

$\mathrm{Mod}_{oo} \to \lambda \underline{p}_o T$;

$\mathrm{Mod}_{o\iota} \to \lambda \underline{x}_\iota T$;

$\mathrm{Mod}_{o(\beta\gamma)} \to \lambda \underline{f}_{\beta\gamma} . \mathrm{Bas}\underline{f} \,\&\, (\underline{x}_\gamma)\, (\mathrm{Mod}\underline{x} \supset \mathrm{Mod}(\underline{fx})$

$.\&. \sim \mathrm{Mod}\underline{x} \supset \underline{fx} = C_\beta)$.

Next we define the translation A'_α of any formula A_α:

i) All constants, except $Q_{o\alpha\alpha}$ where α is a complex type, and all variables, are their own translations;

ii) $Q'_{o\alpha\alpha}$ is $\lambda \underline{x}_\alpha \underline{y}_\alpha . (\imath \underline{p}_o)(\mathrm{Mod}\underline{x} \,\&\, \mathrm{Mod}\underline{y} \,\&\, \underline{p} = Q\underline{xy})$ if α is a complex type;

iii) $(A_{\alpha\beta} B_\beta)'$ is $A'_{\alpha\beta} B'_\beta$;

iv) $(\lambda x_\beta . A_\alpha)'$ is $\lambda x_\beta . (\imath y_\alpha)(\mathrm{Mod}x \,\&\, y = A'_\alpha)$, where y_α is a variable that does not occur free in A_α.

By the translation of the assertion of a proposition P_o, we shall mean, as before, the proposition

$$\mathrm{Mod}\underline{a}_\alpha \,\&\ldots\&\, \mathrm{Mod}\underline{b}_\beta . \supset . P'_o ,$$

where $\underline{a}_\alpha, \ldots, \underline{b}_\beta$, is a complete list of the free variables of P_o.

Now we prove a series of lemmas.

Lemma A

If $\underline{b}_\beta, \ldots, \underline{c}_\gamma$, is a complete list of the free variables of A_α, then $\vdash \mathrm{Mod}\underline{b}_\beta \,\&\ldots\&\, \mathrm{Mod}\underline{c}_\gamma . \supset . \mathrm{Mod}A'_\alpha$.

First we note that if X_η is a constant of (G), then $\mathrm{Bas}^\eta X_\eta$ by (B.i); hence we have:

$\vdash \mathrm{ModN}'_{oo}$ & ModA'_{ooo} & $\mathrm{Mod}\,\iota'_{o(oo)}$ & ModC'_{o} & ModC'_{ι} & ModQ'_{ooo}
& $\mathrm{ModQ}'_{o\iota\iota}$.

Since $Q'_{o\alpha\alpha}$ is a closed formula, $\vdash \mathrm{BasQ}'_{o\alpha\alpha}$, and so, evidently,

$\vdash \mathrm{ModQ}'_{o\alpha\alpha}$

Now $\vdash (E!\underline{x}_{\iota})(\underline{f}_{o\iota}\underline{x}) \supset (E\underline{x}_{\iota})(\underline{f}_{o\iota} = \lambda \underline{y}_{\iota}.\underline{y} = \underline{x})$,

so $\vdash (E!\underline{x}_{\iota})(\underline{f}_{o\iota}\underline{x}) \supset \mathrm{Bas}\underline{f}_{o\iota}$, by (B.i), (B.ii), and (B.iii);

and since, evidently,

$\vdash \mathrm{Bas}\underline{f}_{o\iota} \supset \mathrm{Mod}\underline{f}_{o\iota}$,

we have

$\vdash \mathrm{Mod}\,\iota'_{\iota(o\iota)}$.

Thus the lemma is true if $\underset{\sim}{A}_{\alpha}$ consists of a single symbol.

$\vdash \mathrm{Mod}\underset{\sim}{A}'_{\alpha\beta}$ & $\mathrm{Mod}\underset{\sim}{B}'_{\beta} \supset \mathrm{Mod}(\underset{\sim}{A}_{\alpha\beta}\,\underset{\sim}{B}_{\beta})'$ is obvious.

$\vdash \mathrm{Bas}\underset{\sim}{b}_{\beta}$ &...& $\mathrm{Bas}\underset{\sim}{c}_{\gamma} \supset \mathrm{Bas}\underset{\sim}{A}_{\alpha}$, since

$$\underset{\sim}{A}_{\alpha} = (\lambda \underset{\sim}{b}_{\beta}\ldots\underset{\sim}{c}_{\gamma}.\underset{\sim}{A}_{\alpha})\underset{\sim}{b}_{\beta}\ldots\underset{\sim}{c}_{\gamma} \;;$$

and so

$\vdash \mathrm{Mod}\underset{\sim}{b}_{\beta}$ &...& $\mathrm{Mod}\underset{\sim}{c}_{\gamma} \supset \mathrm{Bas}\underset{\sim}{A}'_{\alpha}$, for <u>any</u> formula $\underset{\sim}{A}_{\alpha}$, the only free variables of which are $\underset{\sim}{b}_{\beta}, \ldots, \underset{\sim}{c}_{\gamma}$.

Further

$\vdash \mathrm{Mod}\underset{\sim}{x}_{\beta} \supset (\lambda \underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\alpha})'\underset{\sim}{x}_{\beta} = \underset{\sim}{A}'_{\alpha}$.&. $\sim\mathrm{Mod}\underset{\sim}{x}_{\beta} \supset (\lambda \underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\alpha})'\underset{\sim}{x}_{\beta} = C_{\beta}$,

hence

$\vdash \mathrm{Mod}\underset{\sim}{A}'_{\alpha} \supset \mathrm{Mod}(\lambda \underset{\sim}{x}_{\beta}.\underset{\sim}{A}_{\alpha})'$.

The truth of the lemma now follows by induction over the length of $\underset{\sim}{A}_{\alpha}$.

Lemma B

1) $\vdash \text{Mod}A'_{\alpha}$ & $\text{Mod}B'_{\alpha}$ $.\supset.\ (A_{\alpha} = B_{\alpha})' \equiv A'_{\alpha} = B'_{\alpha}$; obvious.

2) $\vdash \text{Mod}A'_{\alpha}$ & $\text{Mod}B'_{\alpha}$ $.\supset.\ ((\lambda x_{\beta}.A_{\alpha})' = (\lambda x_{\beta}.B_{\alpha}))'$

$$\equiv (x_{\beta})(\text{Mod}x \supset A'_{\alpha} = B'_{\alpha});$$

for $\vdash (\lambda x_{\beta}.A'_{\alpha}) = (\lambda x_{\beta}.B'_{\alpha})$

$$\equiv (\lambda x_{\beta}.(\imath y_{\alpha})(\text{Mod}x \ \&\ y = A'_{\alpha})) = (\lambda x_{\beta}.(\imath y_{\alpha})(\text{Mod}x \ \&\ y = B'_{\alpha}))$$

$$\equiv (x_{\beta})(\text{Mod}x \supset A'_{\alpha} = B'_{\alpha} \ .\&.\ \sim\text{Mod}x \supset C_{\alpha} = C_{\alpha})$$

$$\equiv (x_{\beta})(\text{Mod}x \supset A'_{\alpha} = B'_{\alpha})$$

and so 2) follows from 1).

3) $\vdash \text{Mod}b_{\beta}$ &...& $\text{Mod}c_{\gamma}$ $.\supset.\ ((x_{\delta})(A_{o}))' \equiv (x_{\delta})(\text{Mod}x \supset A'_{o})$

For $(x_{\delta})(A_{o})$ is an abbreviation for $\lambda x_{\delta}.A_{o} = \lambda x_{\delta}.T$, and so 3) follows from lemma A and 2).

Lemma C

The translations of the assertions of the axioms of (G) are provable in (G).

This is obvious for axioms (P) and (T). For (Q), we have

P.1 $\quad \text{Mod}\underline{x}_{\alpha}$ & $\text{Mod}\underline{y}_{\alpha}$ & $\text{Mod}\underline{f}_{o\alpha}$

$$.\supset.\ (\underline{x}_{\alpha} = \underline{y}_{\alpha})' \supset (\underline{f}_{o\alpha}\underline{x}_{\alpha} \supset \underline{f}_{o\alpha}\underline{y}_{\alpha})$$

which is provable by 1) of lemma B and (Q).

For axioms (D) for the type o, the lemma is obvious; for the type ι, we have:

P.1 $\quad \text{Mod}\underline{f}_{o\iota}\ .\supset.((E!\underline{x}_{\iota})(\underline{f}_{o\iota}\underline{x}))' \supset \underline{f}_{o\iota}(\iota\underline{f}_{o\iota})$;

P.2 $\quad \text{Mod}\underline{f}_{o\iota}\ .\supset.\ \sim((E!\underline{x}_{\iota})(\underline{f}_{o\iota}\underline{x}))' \supset \iota\underline{f}_{o\iota} = C_{\iota}$.

H.3 $\quad \text{Mod}\underline{f}_{o\iota}$ $\qquad$ ($\underline{f}$)

4 $((E!\underline{x}_\iota)(\underline{fx}))' \equiv (E!\underline{x}_\iota)(\mathrm{Mod}\underline{x} \ \& \ \underline{fx})$ Lemma B, 3).

$\equiv (E!\underline{x}_\iota)(\underline{fx})$

5 P.1 & P.2 (H.3), (D).

For (E) we have:

P.1 $\mathrm{Mod}\underline{f}_{\alpha\beta} \ \& \ \mathrm{Mod}\underline{g}_{\alpha\beta} \ . \supset . \ ((\underline{x}_\beta)(\underline{f}_{\alpha\beta}\underline{x} = \underline{g}_{\alpha\beta}\underline{x}))'$

$\supset (\underline{f}_{\alpha\beta} = \underline{g}_{\alpha\beta})'$

H.1 $\mathrm{Mod}\underline{f}_{\alpha\beta} \ \& \ \mathrm{Mod}\underline{g}_{\alpha\beta}$ $(\underline{f}, \underline{g})$

H.2 $((\underline{x}_\beta)(\underline{fx} = \underline{gx}))'$

3 $(\underline{x}_\beta)(\mathrm{Mod}\underline{x} \supset (\underline{fx} = \underline{gx})')$ Lemma B, 3).

4 $\mathrm{Mod}\underline{x}_\beta \supset \underline{fx}_\beta = \underline{gx}_\beta$ H.1, Mod, Lemma B 1).

5 $\sim\mathrm{Mod}\underline{x}_\beta \supset \underline{fx}_\beta = \underline{gx}_\beta = \mathfrak{C}_\beta$ H.1, Mod.

6 $\underline{f} = \underline{g}$ (E).

7 $(\underline{f} = \underline{g})'$ Lemma B 1).

8 P.1 (H.1, H.2).

For the axiom

$(E\underline{x}_\iota)(E\underline{y}_\iota)(\underline{x} \neq \underline{y})$,

the lemma is immediate; the other part of the axiom (I) is harder to deal with, and we shall first prove some subsidiary results. We note that

$$\vdash (\lambda\underline{x}_\iota . \underset{\sim}{A}_\alpha)' = \lambda\underline{x}_\iota . \underset{\sim}{A}'_\alpha ,$$

and that

$$\vdash \mathrm{Mod}\underset{\sim}{A}_\alpha[\underline{x}_\iota] \supset \mathrm{Mod}\underset{\sim}{A}_\alpha[\underset{\sim}{B}_\iota] ,$$

if $\underline{x}_\iota$ occurs free in $\underset{\sim}{A}_\alpha[\underline{x}_\iota]$, and the free variables of $\underset{\sim}{B}_\iota$ are distinct from the bound variables of $\underset{\sim}{A}_\alpha[\underline{x}_\iota]$.

We shall be concerned with the natural numbers in the model; we have:

$\mathrm{Num}'_{o\iota'}$ is $(\lambda \underline{j}_{\iota'}.(\imath \underline{p}_o)[\mathrm{Mod}\underline{j} \,\&\, \underline{p} \equiv (\underline{f}_{o\iota'})\mathrm{Mod}\underline{f} \,\&\, \underline{f}0'$
$\&\, (\underline{k}_{\iota'})(\mathrm{Mod}\underline{k} \,\&\, \underline{fk} \supset \underline{f}(S'\underline{k})).\supset.\ \underline{fj})]$.

From this there follows a rule of induction:

$\underset{\sim}{F}_{o\iota'}0'$, $\mathrm{Mod}\underline{k}_{\iota'} \,\&\, \underset{\sim}{F}_{o\iota'}\underline{k}_{\iota'} \supset \underset{\sim}{F}_{o\iota'}(S'\underline{k}_{\iota'}) \vdash \mathrm{Num}'\underline{j}_{\iota'}$
$\&\, \mathrm{Mod}\underline{j}_{\iota'} \supset \underset{\sim}{F}_{o\iota'}\underline{j}_{\iota'}$.

(To any $\underset{\sim}{F}$ satisfying the premises there corresponds a $\underset{\sim}{G}$ which also satisfies them and for which $\vdash \mathrm{Mod}\underset{\sim}{G}$).

4.1) $\vdash \mathrm{Mod}\underline{g}_{\iota\iota} \supset 0_{\iota'}\underline{g}_{\iota\iota} = 0'_{\iota'}\underline{g}_{\iota\iota}$;

4.2) $\vdash \mathrm{Mod}\underline{g}_{\iota\iota} \,\&\, \mathrm{Mod}\underline{j}_{\iota'} \,\&\, \underline{m}_{\iota'}\underline{g}_{\iota\iota} = \underline{j}_{\iota'}\underline{g}_{\iota\iota}$
$.\supset.\ S\underline{m}_{\iota'}\underline{g}_{\iota\iota} = S'\underline{j}_{\iota'}\underline{g}_{\iota\iota}$;

4.3) $\vdash \mathrm{Mod}\underline{j}_{\iota'} \,\&\, \mathrm{Num}'\underline{j}_{\iota'}.\supset.\ (E\underline{m}_{\iota'})(\mathrm{Num}\underline{m} \,\&\, (\underline{g}_{\iota\iota})(\mathrm{Mod}\underline{g} \supset \underline{mg} = \underline{j}_{\iota'}\underline{g}))$.

The proofs of the above are all straightforward; we omit them.

4.4) $\vdash \mathrm{Num}\underline{m}_{\iota'} \supset (E\underline{f}_{\iota\iota})(\mathrm{Mod}\underline{f} \,\&\, (\underline{n}_{\iota'})(\underline{n} \leqslant \underline{m} \supset \underline{nfx}_{\iota} = \underline{ng}_{\iota\iota}\,\underline{x}_{\iota}))$

This is evidently true if for $\underline{m}_{\iota'}$ we substitute $0_{\iota'}$;

H.1 $\mathrm{Mod}\underline{f}_{\iota\iota} \,\&\, (\underline{n}_{\iota'})(\underline{n} \leqslant \underline{m}_{\iota'} \supset \underline{nfx}_{\iota} = \underline{ng}_{\iota\iota}\underline{x}_{\iota})$ $(\underline{f},\underline{g},\underline{m},\underline{x})$

H.2 $\underline{h}_{\iota\iota} = \lambda\underline{z}_{\iota}.(\imath y_{\iota})\Big[(E\underline{n}_{\iota'})(\underline{n} \leqslant \underline{m} \,\&\, \underline{nfx} = \underline{z}) \supset \underline{y} = \underline{fx}$.&.
$\underline{z} = S\underline{mgx} \supset \underline{y} = g(S\underline{mgx})$.&. $(\underline{z} \neq S\underline{mgx}\ \&$
$\sim(E\underline{n}_{\iota'})(\underline{n} \leqslant \underline{m} \,\&\, \underline{nfx} = \underline{z}) \supset \underline{y} = \underline{z}\Big]$

If in the expression for $\underline{h}$ we substitute free variables, say $\underline{u}_{\iota}$ and $\underline{v}_{\iota}$, for '$S\underline{mgx}$' and '$\underline{g}(S\underline{mgx})$', we obtain a formula of type $\iota\iota$, whose only other free variable is $\underline{f}$, and whose bound variables are distinct from $\underline{m}$, $\underline{g}$, $\underline{x}$; hence, from the remark on the previous page, we have:

3	$\mathrm{Mod}\underline{h}$	
4	$(\underline{n}_{\iota'})(\underline{n} \leqslant S\underline{m} \supset \underline{nhx} = \underline{ngx})$	

The theorem now follows by the rule of induction.

The translation of the assertion of the second part of (I) is:

P.1	$\mathrm{Mod}\underline{j}_{\iota'}\ \&\ \mathrm{Mod}\underline{k}_{\iota'}\ \&\ \mathrm{Num}'\underline{j}_{\iota'}\ \&\ \mathrm{Num}'\underline{k}_{\iota'}\ \&\ \underline{j}_{\iota'} \neq \underline{k}_{\iota'}$ $.\supset.\ S'\underline{j}_{\iota'} \neq S'\underline{k}_{\iota'}$	
H.2	L.H.S. of P.1	$(\underline{j}, \underline{k})$
H.3	$\mathrm{Num}\underline{m}_{\iota'}\ \&\ (\underline{g}_{\iota\iota})(\mathrm{Mod}\underline{g} \supset \underline{m}_{\iota'}\underline{g} = \underline{jg})$	$(\underline{m})$
H.4	$\mathrm{Num}\underline{n}_{\iota'}\ \&\ (\underline{g}_{\iota\iota})(\mathrm{Mod}\underline{g} \supset \underline{n}_{\iota'}\underline{g} = \underline{kg})$	$(\underline{n})$
H.5	$\mathrm{Mod}\underline{g}_{\iota\iota}\ \&\ \underline{jg}_{\iota\iota} \neq \underline{kg}_{\iota\iota}$	$(\underline{g})$
6	$\underline{m} \neq \underline{n}$	(E)
7	$S\underline{m} \neq S\underline{n}$	(I)
H.8	$S\underline{mf}_{\iota\iota}\,\underline{x}_{\iota} \neq S\underline{nf}_{\iota\iota}\,\underline{x}_{\iota}$	$(\underline{f}, \underline{x})$
H.9	$\mathrm{Mod}\underline{h}_{\iota\iota}\ \&\ (\underline{p}_{\iota'})(\underline{p} \leqslant \mathrm{Max}(S\underline{m}, S\underline{n}) \supset \underline{ph}_{\iota\iota}\underline{x} = \underline{pfx})$	$(\underline{h})$
10	$S'\underline{jh} = S\underline{mh}\ \&\ S'\underline{kh} = S\underline{nh}$	H.3, H.4, 4.2).
11	$S'\underline{jhx} \neq S'\underline{khx}$	H.8.
12	$S'\underline{j} \neq S'\underline{k}$	(E).
13	P.1	(H.9), 4.4); (H.5, H.8), (E); (H.3, H.4), 4.3); (H.1).

This completes the proof of Lemma C.

Lemma D

If $\underset{\sim}{Q}_0$ can be inferred from $\underset{\sim}{P}_0$ by a single application of one of the rules of inference, then the translation of the assertion of $\underset{\sim}{Q}_0$ can be inferred from the translation of the assertion of $\underset{\sim}{P}_0$.

For rule I this follows immediately from the definition of translation. In order to deal with rules II and III we show that rule IX may be applied in the model. Let A_β be a part of M_α, and let N_α be the result of substituting the formula B_β for A_β in M_α; let $c_\gamma, d_\delta, \ldots, e_\varepsilon$, be a complete list of the variables which occur free in A_β and bound in M_α.

IX' $\quad (c_\gamma \ldots e_\varepsilon)(\mathrm{Mod}\,c \,\&\ldots\&\, \mathrm{Mod}\,e \supset A'_\beta = B'_\beta) \vdash M'_\alpha = N'_\alpha$

Of course A'_β is a part of M'_α, and further, N'_α is obtained from M'_α by substituting B'_β for A'_β; and $c_\gamma, \ldots, e_\varepsilon$, is a complete list of the free variables of A'_β which are bound in M'_α; we demonstrate IX' by induction over the length of this list.

If none of the free variables of A_β occur bound in M_α, then the above inference is simply an application of rule IX, and hence is valid; we suppose that its validity has been established whenever the length of the list is less than the length of the list $c_\gamma, d_\delta, \ldots, e_\varepsilon$. Let c_γ be the variable of this list for which the binding occurrence in M_α occurs furthest to the left, so that M_α contains a part $X_{\rho\gamma}$ of the form

$$(\lambda c_\gamma . R_\rho) ,$$

where A_β is a part of R_ρ, and all the variables of the list except c_γ occur bound in R_ρ. Let $Y_{\rho\gamma}$ and S_ρ be obtained from $X_{\rho\gamma}$ and R_ρ by substituting B_β for A_β; then

$$\underset{\sim}{X}'_{\rho\gamma} \text{ is } (\lambda \underset{\sim}{c}_\gamma . (\imath \underset{\sim}{y}_\rho)(\mathrm{Mod}\underset{\sim}{y} \,\&\, \underset{\sim}{y} = \underset{\sim}{R}'_\rho)) ,$$

$$\text{and } \underset{\sim}{Y}'_{\rho\gamma} \text{ is } (\lambda \underset{\sim}{c}_\gamma . (\imath \underset{\sim}{y}_\rho)(\mathrm{Mod}\underset{\sim}{y} \,\&\, \underset{\sim}{y} = \underset{\sim}{S}'_\rho)) ,$$

and $\underset{\sim}{N}'_\alpha$ is obtained from $\underset{\sim}{M}'_\alpha$ by substituting $\underset{\sim}{Y}'_{\rho\gamma}$ for $\underset{\sim}{X}'_{\rho\gamma}$. We start from the premise:

H.1 $(\underset{\sim}{c}_\gamma, \underset{\sim}{d}_\delta, \ldots, \underset{\sim}{e}_\varepsilon)(\mathrm{Mod}\underset{\sim}{c} \,\&\ldots\&\, \mathrm{Mod}\underset{\sim}{e} \supset \underset{\sim}{A}'_\beta = \underset{\sim}{B}'_\beta)$

2 $\underset{\sim}{R}'_\rho = \underset{\sim}{S}'_\rho$ by the induction hypothesis.

3 $(\underset{\sim}{c}_\gamma)(\mathrm{Mod}\underset{\sim}{c} \supset \underset{\sim}{X}'_{\rho\gamma}\underset{\sim}{c} = \underset{\sim}{Y}'_{\rho\gamma}\underset{\sim}{c} \;.\&.\; \sim\mathrm{Mod}\underset{\sim}{c} \supset \underset{\sim}{X}'_{\rho\gamma}\underset{\sim}{c} = \underset{\sim}{Y}'_{\rho\gamma}\underset{\sim}{c})$;

4 $\underset{\sim}{X}'_{\rho\gamma} = \underset{\sim}{Y}'_{\rho\gamma}$ by (E).

5 $\underset{\sim}{M}'_\alpha = \underset{\sim}{N}'_\alpha$ by (E) and rule IX.

(Note that $\underset{\sim}{X}_{\rho\gamma}$ may contain free variables, other than those of $\underset{\sim}{A}_\beta$, which are bound in $\underset{\sim}{M}_\alpha$; but these variables will also appear free in $\underset{\sim}{Y}_{\rho\gamma}$ and hence also free in 4; so that rule IX may correctly be applied.) It follows now that inference IX' is valid.

Now consider rules II and III; let a part $\underset{\sim}{X}_\alpha$ of the formula $\underset{\sim}{P}_o$ be:

$$((\lambda \underset{\sim}{x}_\beta . \underset{\sim}{A}_\alpha)\underset{\sim}{N}_\beta),$$

and let $\underset{\sim}{B}_\alpha$ be obtained by substituting $\underset{\sim}{N}_\beta$ for $\underset{\sim}{x}_\beta$ throughout $\underset{\sim}{A}_\alpha$; we suppose that the bound variables of $\underset{\sim}{A}_\alpha$ are distinct both from the free variables of $\underset{\sim}{N}_\beta$, and from $\underset{\sim}{x}_\beta$. Let $\underset{\sim}{Q}_o$ be obtained from $\underset{\sim}{P}_o$ by substituting $\underset{\sim}{B}_\alpha$ for $\underset{\sim}{X}_\alpha$. Let $\underset{\sim}{s}_\sigma, \ldots, t_\tau$, be a complete list of the free variables of $\underset{\sim}{P}_o$, and let $\underset{\sim}{c}_\gamma, \ldots, \underset{\sim}{d}_\delta$, be a complete list of those variables which occur free in $\underset{\sim}{N}_\beta$ and bound in $\underset{\sim}{P}_o$. We want to show that

$$\mathrm{Mod}\underset{\sim}{s}_\sigma \,\&\ldots\&\, \mathrm{Mod}\underset{\sim}{t}_\tau \;.\supset.\; \underset{\sim}{Q}'_o$$

can be inferred from

$$\text{Mod}\underset{\sim}{s}_\sigma \ \& \ldots \& \ \text{Mod}\underset{\sim}{t}_\tau \ . \supset . \ \underset{\sim}{P}'_o \ ,$$

and vice-versa. $\underset{\sim}{X}'_\alpha$ is

$$((\lambda \underset{\sim}{x}_\beta.(\imath \underset{\sim}{y}_\alpha)(\text{Mod}\underset{\sim}{x} \ \& \ \underset{\sim}{y} = \underset{\sim}{A}'_\alpha)\underset{\sim}{N}'_\beta),$$

and $\underset{\sim}{B}'_\alpha$ is the result of substituting $\underset{\sim}{N}'_\beta$ for $\underset{\sim}{x}_\beta$ throughout $\underset{\sim}{A}'_\alpha$.

H.1	$\text{Mod}\underset{\sim}{s}_\sigma \ \& \ldots \& \ \text{Mod}\underset{\sim}{t}_\tau$	$(\underset{\sim}{s}, \ldots, \underset{\sim}{t})$
H.2	$\text{Mod}\underset{\sim}{c}_\gamma \ \& \ldots \& \ \text{Mod}\underset{\sim}{d}_\delta$	$(\underset{\sim}{c}, \ldots, \underset{\sim}{d})$
3	$\text{Mod}\underset{\sim}{N}'_\beta$	Lemma A
4	$\underset{\sim}{X}'_\alpha = \underset{\sim}{B}'_\alpha$	Rule II, (D).
5	$(\underset{\sim}{c}_\gamma, \ldots, \underset{\sim}{d}_\delta)(\text{Mod}\underset{\sim}{c} \ \& \ldots \& \ \text{Mod}\underset{\sim}{d} \supset \underset{\sim}{X}'_\alpha = \underset{\sim}{B}'_\alpha)$	(H.2).
6	$\underset{\sim}{P}'_o = \underset{\sim}{Q}'_o$	IX'.
7	$\text{Mod}\underset{\sim}{s}_\sigma \ \& \ldots \& \ \text{Mod}\underset{\sim}{t}_\tau \ . \supset . \ \underset{\sim}{P}'_o \equiv \underset{\sim}{Q}'_o$	(H.1).

The required inferences are now obviously valid.

Let $\underset{\sim}{c}_\gamma, \ldots, \underset{\sim}{d}_\delta$; $\underset{\sim}{s}_\sigma, \ldots, \underset{\sim}{t}_\tau$; be lists of the free variables of $\underset{\sim}{A}_\beta$ and $\underset{\sim}{B}_\beta$, and $\underset{\sim}{F}_{\alpha\beta}$, respectively. Then for rule G.IV we want to show that:

$$\text{Mod}\underset{\sim}{c}_\gamma \ \& \ldots \& \ \text{Mod}\underset{\sim}{d}_\delta \supset \underset{\sim}{A}'_\beta = \underset{\sim}{B}'_\beta \vdash \text{Mod}\underset{\sim}{c}_\gamma \ \& \ldots \& \ \text{Mod}\underset{\sim}{s}_\sigma$$
$$\& \ldots \& \ \text{Mod}\underset{\sim}{t}_\tau \supset \underset{\sim}{F}'_{\alpha\beta}\underset{\sim}{A}'_\beta = \underset{\sim}{F}'_{\alpha\beta}\underset{\sim}{B}'_\beta.$$

But this follows immediately from G.IV.

For rule V the argument is the same as was used in proving Lemma C in the section on virtual types.

For rule VI, we wish to show that from

$$\text{Mod}\underset{\sim}{c}_\gamma \ \& \ldots \& \ \text{Mod}\underset{\sim}{d}_\delta \supset \underset{\sim}{A}'_\alpha = \underset{\sim}{B}'_\alpha$$

we can infer

$$\text{Mod}\underset{\sim}{c}_\gamma \ \& \ldots \& \ \text{Mod}\underset{\sim}{d}_\delta \supset \lambda \underset{\sim}{x}_\beta.(\imath \underset{\sim}{y}_\alpha)(\text{Mod}\underset{\sim}{x} \ \& \ \underset{\sim}{y} = \underset{\sim}{A}'_\alpha)$$
$$= \lambda \underset{\sim}{x}_\beta.(\imath \underset{\sim}{y}_\alpha)(\text{Mod}\underset{\sim}{x} \ \& \ \underset{\sim}{y} = \underset{\sim}{B}'_\alpha).$$

If $\underset{\sim}{x}_\beta$ is not one of $\underset{\sim}{c}_\gamma, \ldots \underset{\sim}{d}_\delta$, (the free variables of $\underset{\sim}{A}'_\alpha$ and $\underset{\sim}{B}'_\alpha$) the inference can be obtained by using the deduction theorem. If $\underset{\sim}{x}_\beta$ is one of that list, then the inference follows from (D) and (E).

This completes the demonstration of lemma D.

Theorem V (The model theorem).

Let there be given a set of closed formulae $\mathrm{Bas}_{o\alpha}$ which satisfy the conditions (B), and let the formulae $\mathrm{Mod}_{o\alpha}$, and the translation $\underset{\sim}{A}'_\alpha$ of any formula $\underset{\sim}{A}_\alpha$, be defined as above; then if $\underset{\sim}{P}_o$ is a provable proposition of (G), the translation of the assertion of $\underset{\sim}{P}_o$ is also a provable proposition of (G).

This theorem follows immediately from lemmas C and D. Before we discuss its implications, we show by an example that non-trivial sets of formulae satisfying (B) do exist.

We define:

$$\mathrm{Fin}_{o(o\alpha)} \rightarrow \lambda \underline{f}_{o\alpha}.(E\underline{n}_\gamma)(E\underline{h}_{\alpha\gamma})(\underline{x}_\alpha)(\underline{fx} \supset (E!\underline{m}_\gamma)(\underline{m} \leqslant \underline{n} \;\&\; \underline{hm} = \underline{x}))$$

$$\mathrm{Con}_{o\alpha} \rightarrow \lambda \underline{z}_\alpha.(E\underline{f}_{o\iota})\Big[\mathrm{Fin}\underline{f} \;\&\; (\underline{t}_{\iota\iota})(\mathrm{Per}\underline{t} \;\&\; (\underline{x}_\iota)(\underline{fx} \supset \underline{tx} = \underline{x}) \;.\!\supset.\; \mathrm{Tra}\underline{tz} = \underline{z})\Big]$$

'Fin' stands for 'finite', 'Con' for 'constructive'; a function is 'constructive' if there exists a finite set of individuals such that all the permutations of the individuals which leave that set invariant, also - when transported to the appropriate type - leave the function invariant. In the types $o\iota$ and $\iota\iota$, the 'constructive' functions are just those which may be explicitly described, using the names of a finite number of individuals; that is, to be more precise, those functions which are represented by formulae whose only free variables are of type ι. In the higher types there are 'constructive' functions which cannot be explicitly described in this way; that this is so follows from the existence of invariant functions which cannot be represented by closed formulae.

Now it is easy to show that $\mathrm{Con}_{o\alpha}$ satisfies the conditions (B).

$4.5)\ \vdash \mathrm{Inv}\underline{z}_{\alpha} \supset \mathrm{Con}\underline{z}_{\alpha}$ obvious.

But if $\underset{\sim}{A}_{\alpha}$, is a closed formula, $\vdash \mathrm{Inv}\underset{\sim}{A}_{\alpha}$ by theorem II; thus (B.i) is satisfied.

$4.6)\ \vdash \mathrm{Fin}\underline{f}_{o\alpha} \,\&\, \mathrm{Fin}\underline{g}_{o\alpha} \supset \mathrm{Fin}(\lambda \underline{x}_{\alpha}.\underline{f}_{o\alpha}\underline{x} \vee \underline{g}_{o\alpha}\underline{x})$

The proof of this is straightforward.

$4.7)\ \vdash \mathrm{Con}\underline{f}_{\alpha\beta} \,\&\, \mathrm{Con}\underline{z}_{\beta} \,.\supset.\, \mathrm{Con}(\underline{f}_{\alpha\beta}\underline{z}_{\beta})$

H.1	L.H.S.	$(\underline{f}, \underline{z})$
H.2	$\mathrm{Fin}\underline{u}_{o\iota} \,\&\, (\underline{t}_{\iota\iota})(\mathrm{Per}\underline{t} \,\&\, (\underline{x}_{\iota})(\underline{u}_{o\iota}\underline{x} \supset \underline{tx} = \underline{x})$ $.\supset.\, \mathrm{Tra}\underline{tf} = \underline{f})$	$(\underline{u})$
H.3	$\mathrm{Fin}\underline{v}_{o\iota} \,\&\, (\underline{t}_{\iota\iota})(\mathrm{Per}\underline{t} \,\&\, (\underline{x}_{\iota})(\underline{v}_{o\iota}\underline{x} \supset \underline{tx} = \underline{x})$ $.\supset.\, \mathrm{Tra}\underline{tz} = \underline{z})$	$(\underline{v})$
H.4	$\underline{w}_{o\iota} = \lambda \underline{x}_{\iota}.\underline{ux} \vee \underline{vx}$	$(\underline{w})$
5	$\mathrm{Fin}\underline{w}$	4.6).
H.6	$\mathrm{Per}\underline{t}_{\iota\iota} \,\&\, (\underline{x}_{\iota})(\underline{wx} \supset \underline{t}_{\iota\iota}\underline{x} = \underline{x})$	$(\underline{t})$
7	$\mathrm{Tra}\underline{tf} = \underline{f} \,\&\, \mathrm{Tra}\underline{tz} = \underline{z}$	H.2, H.3, H.4, H.6.

8	$\mathrm{Tra}\underline{t}(\underline{fz}) = \mathrm{Tra}\underline{tf}(\mathrm{Tra}\underline{tz})$	2.8).
9	$\mathrm{H.6} \supset \mathrm{Tra}\underline{t}(\underline{fz}) = \underline{fz}$	(H.6).
10	$\mathrm{Con}(\underline{fz})$	(H.2, H.3, H.4).
11	4.7)	(H.1).

Thus Con satisfies (B.ii); and evidently it satisfies (B.iii). Thus our assertion is justified. We investigate some properties of the model of which Con is the basis.

4.8) $\vdash \mathrm{Con}\underline{f}_{o\iota} .\supset. \mathrm{Fin}\underline{f}_{o\iota} \vee \mathrm{Fin}(\lambda \underline{x}_\iota \sim \underline{f}_{o\iota}\underline{x})$

The proof is straightforward.

4.9) $\vdash \mathrm{Fin}(\lambda \underline{x}_\alpha.T) \supset (E\underline{m}_{\alpha'}, \underline{n}_{\alpha'})(\mathrm{Num}\underline{m} \;\&\; \mathrm{Num}\underline{n} \;\&\; \underline{m} \neq \underline{n} \;\&\; S\underline{m} = S\underline{n})$

The proof of this is a trifle tedious: if N is the finite cardinal of the type α, then appropriate values to take for m and n in the above are 0 and N! + 1.

4.10) $\vdash \mathrm{Fin}\underline{f}_{o\iota} \supset (E\underline{x}_\iota, \underline{y}_\iota)(\sim \underline{f}_{o\iota}\underline{x} \;\&\; \sim\underline{f}_{o\iota}\underline{y} \;\&\; \underline{x} \neq \underline{y} \;\&\; \underline{x} \neq C \;\&\; \underline{y} \neq C)$.

4.11) $\vdash \sim\mathrm{Fin}(\lambda \underline{x}_\iota.T) \equiv (I)$.

The proof of this is, of course, conducted without using (I) as an axiom; the implication from right to left is an immediate consequence of 4.9); the reverse implication is easily proved, by introducing an $\underline{h}_{\iota\iota}$ similar to that used in 4.4). This theorem gives an intuitive interpretation of Church's axiom of infinity.

4.12) $\vdash (\underline{f}_{o\iota})(\mathrm{Con}\underline{f} \;\&\; \Sigma\underline{f} \supset \underline{f}(\underline{j}_{\iota(o\iota)}\underline{f})) \supset \sim\mathrm{Con}\underline{j}_{\iota(o\iota)}$

H.1	L.H.S.	($\underline{j}$)
H.2	$\mathrm{Fin}\underline{g}_{o\iota}$	($\underline{g}$)

H.3 $\sim \underline{gx}_{\iota}$ & $\sim \underline{gy}_{\iota}$ & $\underline{x}_{\iota} \neq C$ & $\underline{y}_{\iota} \neq C$ & $\underline{x}_{\iota} \neq \underline{y}_{\iota}$ $(\underline{x}, \underline{y})$

H.4 $\underline{t}_{\iota\iota}\underline{x} = \underline{y}$ & $\underline{t}_{\iota\iota}\ \underline{y} = \underline{x}$ & $(\underline{z}_{\iota})(\underline{z} \neq \underline{x}$ & $\underline{z} \neq \underline{y}$

$. \supset .\ \underline{t}_{\iota\iota}\ \underline{z} = \underline{z})$ $(\underline{t})$

H.5 $\underline{h}_{o\iota} = \lambda \underline{z}_{\iota} . \underline{z} = \underline{x} \vee \underline{z} = \underline{y}$ $(\underline{h})$

6 $\underline{jh} = \underline{x} \vee \underline{jh} = \underline{y}$ H.1.

7 $\mathrm{Tra}\underline{th} = \underline{h}$ Tra, H.4, H.5.

8 $\underline{jh} \neq \mathrm{Trat}(\underline{jh}) = \mathrm{Trat}\underline{jh}$ 6, H.4, 2.8), 7.

9 $\underline{j} \neq \mathrm{Trat}\underline{j}$

10 $\mathrm{Fing}_{o\iota} \supset (\mathrm{E}\underline{t}_{\iota\iota})(\mathrm{Per}\underline{t}$ & $(\underline{x}_{\iota})(\underline{g}_{o\iota}\underline{x} \supset \underline{tx} = \underline{x})$

& $\mathrm{Trat}\underline{j} \neq \underline{j})$ (H.2, H.3, H.4, H.5), 4.10).

11 $\sim \mathrm{Con}\underline{j}$

12 4.12) (H.1).

4.13) $\vdash$ $\mathrm{Mod}\underline{j}_{\iota(o\iota)} \supset \mathrm{Con}\underline{j}_{\iota(o\iota)}$ & $\mathrm{Mod}_{o(o\iota)} = \mathrm{Con}_{o(o\iota)}$

Obvious. Now the translation of the selection axiom for the type ι is:

(S)' $(\mathrm{E}\underline{j}_{\iota(o\iota)})(\mathrm{Mod}\underline{j}$ & $(\underline{f}_{o\iota})(\mathrm{Mod}\underline{f}$ & $\sum \underline{f} \supset \underline{f}(\underline{jf}))$

But, from 4.12) and 4.13), (S) is provably false.

Theorem VI

If (G) is consistent, then the following propositions are not consequences of the axioms:

i). (S) for the type ;

ii). $(\mathrm{E}\underline{f}_{o\iota})(\sim \mathrm{Fin}\underline{f}$ & $\sim \mathrm{Fin}(\lambda \underline{x}_{\iota} . \sim \underline{fx}))$;

iii). $(\mathrm{E}\underline{f}_{\iota\iota})(\mathrm{E}\underline{x}_{\iota})(\mathrm{Uni}\underline{f}$ & $(\underline{y}_{\iota})(\underline{x} \neq \underline{fy}))$.

i) follows from the provable falsity of (S)' and theorem V.ii) and iii) may be shown in an entirely analogous fashion.

Similar theorems have been proved by Fraenkel concerning various forms of the selection axiom (Fraenkel (1) and (2), see also Mostowski and Lindenbaum (2)); theorems showing the progressive independence of six axioms of infinity have been proved by Mostowski and Lindenbaum (Mostowski (1), Mostowski and Lindenbaum (1)); and using similar methods Mostowski (in (2)) has shown the independence of the selection axiom from an axiom of simple ordering. Except for Mostowski (1) and Mostowski and Lindenbaum (1), these investigations refer to systems of the set theory kind.

All the studies of the selection axiom depend on showing that those elements of the system whose existence is guaranteed by the axioms have a property similar to that defined by 'Con'; in fact 'Con' is a special case of Mostowski's 'G-M ausgezeichnet'; (a definition of this term is obtained by substituting an arbitrary subgroup G, and an arbitrary ring of sets M, for the <u>complete</u> permutation group, and the ring of <u>finite</u> sets, in the definition of 'Con'). Fraenkel's proofs lie almost entirely <u>outside</u> the system he is considering, and use the ordinary methods of mathematical argument. Mostowski (in (2)) proceeds by constructing a model of one system of set theory inside <u>another</u> system of set theory; that is he uses an <u>outer</u> model, in the same sort of way that we have used an <u>inner</u> model. It appears that the facilities of definition ('ι' and 'λ') afforded by the system (G), the combinatorial character of its formulae, and the fact that it is a type theory, combine together to make our proof of theorem VI a good deal more compact than any in the investigations considered[1].

In the statement of theorem VI we used the phrase 'are not consequences of the axioms' instead 'are not provable', because we wished to suggest that the lack of provability involved is of a rather different sort than that established in Godel's theorem. For example, I think it clear that one could not hope to prove (S) merely by adjoining an axiom of the form:

$$(E\underline{m}_{\nu})(\mathrm{Proof}_{o\nu\nu}\,\underline{mn}_{\nu}) \supset \underset{\sim}{N}_{o}$$

where $\mathrm{Proof}_{o\nu\nu}\,\underline{mn}$ represents the statement that $\underline{m}$ is the Godel number of a proof of the proposition $\underset{\sim}{N}_{o}$ whose Godel number is $\underline{n}_{\nu}$; while it is known that the adjoining of such an axiom does render Godel's proposition provable (see Turing (2)).

It may be possible to distinguish between 'consequence of the axioms' and 'provable' by setting up a certain class of models for (G), and then defining 'consequence of the axioms' as 'valid in all models of the given class': but the

(1) I may add that I discovered the above proof in ignorance of the references cited.

results of Henkin (in (1)) make it clear that this would not be quite so straightforward as it might, at first sight, appear. He defines a standard model as a universe which contains two representatives for the type o, an infinity of individuals for type ι, and all the functions of higher types; together with the natural interpretation of the constants of the system (G) in this universe, and a typically correct, though otherwise arbitrary, interpretation of the variables of (G). Thus the only difference between two standard models is in the interpretation of the variables. Of course the rules governing the interpretation of the constants are such that the interpretation of a provable proposition in any standard model is the element of the universe corresponding to truth; provable propositions are valid in all standard models.

A general model is like a standard model except that only some of the functions of higher types are present in the universe of the model, with the proviso that sufficiently many functions of each type are included to ensure that every provable proposition is valid in the model. Now Henkin shows that a proposition is provable only if it is valid in every general model. It follows that there exist general models in which Godel's proposition is interpreted as truth, and ones in which it is interpreted as falsehood. Thus the class of general models is too large for our purpose, while the class of standard models is too small.

The use of models to define terms like 'consequence' and 'true' is due, I believe, to Tarski (see (2)), and has since become a major preoccupation of the semanticists. But I think it is a mistake to suppose that the method will provide new and satisfactory formal definitions of semantical concepts: if, for example, one defines a 'true' proposition as one whose interpretation is valid in all standard models, one has merely, as it were, 'passed the buck' from the original system to some other system in which the universe of the model must be described; and one can only be quite clear about what is and what is not the case, if the universe of the model is finite - but for a system which admits only finite models 'true' can be identified with 'provable'. On the other hand models are certainly very useful on the intuitive level: by choosing an appropriate model one can see 'why' such and such a proposition is not provable; (indeed, if the model is an inner one, one can show that it is not provable). One can do this because most mathematicians feel more at home in classical set theory than in some particular logical system. (Another way of putting it: most mathematicians believe that some adequate system of set theory is consistent.) Thus should I try to communicate to the reader the distinction I feel there to be between the non-provability of Godel's proposition, and the non-provability of (S), by reference to a class of models, the communication will be successful if the reader's notion of set theory is like mine; but if his notion is very different, if, say, he is an intuitionist in thought as well as word, then communication will fail; and I doubt that an increase of formality on my part - by, for example, a re-statement of my definitions in the notation of Godel's set theory - will avail to restore it. I do not wish to assert that there are no formal uses to which models may be put: they may certainly be used to establish questions concerning relative consistency and independence; but I do wish to emphasise that some of their uses are <u>essentially</u> informal, and stand in no need therefore of excessive formal elaboration.

We return now to a consideration of theorem V. We ask whether there is a set of functions $\mathrm{Int}^{\alpha}_{\alpha\alpha}$ of the system which represent the (metalogical) process of translation; that is, which satisfy

$$\vdash \mathrm{Int}^{\alpha}_{\alpha\alpha}\, \underset{\sim}{A}_{\alpha} = \underset{\sim}{A}'_{\alpha}$$

for any closed formula $\underset{\sim}{A}_{\alpha}$. It is not hard to see that there can be no such functions, because the process of translation is not purely extensional. Let us suppose that we have to do with a strictly inner model, so that

(X) $\qquad (E\underline{x}_{\alpha})(\sim \mathrm{Mod}\underline{x})$

may, for some particular type α, be consistently adjoined to the axioms of (G). We define:

$$A_{o\alpha} \rightarrow \lambda \underline{x}_{\alpha}. \sim \mathrm{Mod}\underline{x}$$

$$B_{o\alpha} \rightarrow \lambda \underline{x}_{\alpha}. F$$

$$M_{o(o\alpha)} \rightarrow \lambda \underline{f}_{o\alpha} . \underline{f} = A \ ;$$
$$N_{o(o\alpha)} \rightarrow \lambda \underline{f}_{o\alpha} . \underline{f} = A \ \& \ \underline{f} \neq B \ .$$

Then we have:

$$\vdash (X) \supset M = N$$
$$\vdash A' = (\lambda \underline{x}_{\alpha} . (\imath \underline{p}_o)(\mathrm{Mod}\underline{x} \ \& \ \underline{p} = F)) = B' \quad *$$
$$\vdash M' = \lambda \underline{f}_{o\alpha} . (\imath \underline{p}_o)(\mathrm{Mod}\underline{f} \ \& \ \underline{p} \equiv \underline{f} = A')$$
$$\vdash N' = \lambda \underline{f}_{o\alpha} . (\imath \underline{p}_o)(\mathrm{Mod}\underline{f} \ \& \ \underline{p} \equiv (\underline{f} = A' \ \& \ \underline{f} \neq B'))$$
$$= \lambda \underline{f}_{o\alpha} . (\imath \underline{p}_o)(\mathrm{Mod}\underline{f} \ \& \ \underline{p} = F)$$
$$\vdash M' \neq N'$$

But, since (X) is consistent with the axioms, this shows that we could not have

$$M' = \mathrm{Int}^{o(o\alpha)} M \ \& \ N' = \mathrm{Int}^{o(o\alpha)} N.$$

I do not know if one could redefine the process of translation in such a way that it became purely extensional, and at the same time preserved the validity of theorem V.

Suppose that $\underset{\sim}{G}_{o\alpha}$ is a formula for which

(Y) $\vdash \Sigma \underset{\sim}{G}_{o\alpha}$,

then

$$\vdash (E\underline{x}_{\alpha})(\mathrm{Mod}\underline{x} \ \& \ \underset{\sim}{G}'_{o\alpha} \underline{x}) \ ,$$

by theorem V; but can we say anything about the proposition

(Z) $(E\underline{x}_{\alpha})(\mathrm{Mod}\underline{x} \ \& \ \underset{\sim}{G}_{o\alpha}\underline{x})$?

If

(M') $\vdash \mathrm{Mod}\underline{x}_{\alpha} \supset \mathrm{Mod}'\underline{x}_{\alpha}$,

we shall say that the model in question is a <u>final</u> model. It

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* Provided that the model on question is a final one; see below.

is easy to verify that the model based on 'Con' is a final model. For such models the translation of the assertion of

(M) $\qquad (\underline{x}_\alpha)(\mathrm{Mod}\underline{x})$,

is provable; and hence (M) is consistent with the axioms of (G). It follows that if (Y) is provable then (Z) may consistently be adjoined to the axioms. I do not know if non-final models exist, or for what sorts of models (Z) (or rather, (Z) with 'Mod' replaced by 'Bas') may be actually provable whenever (Y) is provable.

The next point that we consider is the application of the model theorem to a system which includes some virtual types; we illustrate the procedure to be adopted by discussing the case of the virtual type $\vee$. To the conditions (B) we add:

$$(\mathrm{iv}) \vdash \mathrm{Bas}_{o\vee}\underline{x}_\vee \quad ;$$

$$(\mathrm{v}) \vdash \mathrm{Bas}_{o(\vee\vee)}S_{\vee\vee} \quad ;$$

and we define the translations $O_\vee'$ and $S_\vee'$ to be $O_\vee$ and $S_{\vee\vee}$. It is then easy to see that the translations of the assertions of the axioms for type $\vee$ are provable, and hence that theorem V (mutatis mutandis) is again true. Of course for some virtual types more severe restrictions on Bas may be necessary if the translations of the assertions of the additional axioms are to be provable, but provided these restrictions are made, the appropriate form of theorem V will continue to be true.

Finally we ask if the complexity of the definitions and the proofs leading up to theorem V was really necessary. The simple way of defining a model is to use system (C) and define the translation of $\Pi_{o(o\alpha)}$ to be:

$$\lambda \underline{f}_{o\alpha}.(\underline{x}_{\alpha})(\mathrm{Bas}\underline{x} \supset \underline{fx}),$$

and let everything else be its own translation; but if one does this one has no guarantee that the axiom of extensionality will hold in the model, although given some particular formulae for Bas one may well find that it does in fact hold, or can be made to hold by a slight modification of the formulae for Bas. What, in effect, our method does, is to show that such a modification can always be made, provided that the original formulae satisfy the conditions (B): it is of course possible that this <u>general</u> demonstration can also be carried out more simply.

Section 5. Closed formulae.

Our first object in this section will be to show that it is possible to define <u>within the system</u> the property of being representable by a closed formula, the bound variables of which are <u>not</u> of arbitrarily high type. We define the <u>length</u>, $l(\alpha)$, of a type α, to be the total number of o's and ι's occurring in the type symbol 'α'. We define the type symbols 'ι_n' by:

$$\iota_1 \text{ is } \iota$$

$$\iota_n \text{ is } \iota\iota_{n-1}$$

We show that it is possible to map all elements of types of length less than or equal to n into the type ι_{n+1}, the map being one-one. We first single out elements T_ι, F_ι, X_ι, which are all distinct from each other, and from C_ι. We define:

$$\iota'_{\iota(o\iota)} \rightarrow \lambda \underline{f}_{o\iota}.(\imath\underline{x}_\iota)(J\underline{f} \supset \underline{x} = \iota\underline{f} .\&. \sim J\underline{f} \supset \underline{x} = T_\iota);$$

$$\iota''_{\iota(o\iota)} \rightarrow \lambda \underline{f}_{o\iota}.(\imath\underline{x}_\iota)(J\underline{f} \supset \underline{x} = \iota\underline{f} .\&. \sim J\underline{f} \supset \underline{x} = F_\iota);$$

$$X_{\iota_n} \rightarrow \lambda \underline{f}_{\iota_{n-1}}.X_\iota .$$

$$\mathrm{Pai}^n_{\iota_n\iota_{n-1}\iota_{n-1}} \Rightarrow \lambda \underline{g}_{\iota_{n-1}}\underline{h}_{\iota_{n-1}}\underline{u}_{\iota_{n-1}}(\imath\underline{x}_\iota)(\underline{u} = \underline{g} \;\&\; \underline{x} = T_\iota .v. \underline{u} = \underline{h} \;\&\; \underline{x} = F_\iota)$$

ι' and ι'' are descriptions operators with T_ι and F_ι as their respective nonsense elements; we use $(\imath'\underline{x}_\iota)(\underset{\sim}{P}_o)$, $(\imath''\underline{x}_\iota)(\underset{\sim}{P}_o)$ in the obvious way. The properties of Pai are given by:

$\vdash \mathrm{Pai}^{n+1}\underline{g}_{\iota_n}\underline{h}_{\iota_n}\underline{g}_{\iota_n} = T_\iota \ \& \ \mathrm{Pai}^{n+1}\underline{g}_{\iota_n}\underline{h}_{\iota_n}\underline{h}_{\iota_n} = F_\iota;$

$\vdash \underline{u}_{\iota_n} \neq \underline{g}_{\iota_n} \ \& \ \underline{u}_{\iota_n} \neq \underline{h}_{\iota_n} .\supset. \ \mathrm{Pai}^{n+1}\underline{g}_{\iota_n}\underline{h}_{\iota_n}\underline{u}_{\iota_n} = C_\iota.$

Now we define the required maps inductively:

$$\mathrm{Map}^0_{\iota o} \rightarrow \lambda \underline{p}_o(\neg \underline{x}_\iota)(\underline{p} \supset \underline{x} = T_\iota \ .\&.\ \sim\underline{p} \supset \underline{x} = F_\iota)$$

$$\mathrm{Map}^1_{\iota\iota\iota} \rightarrow \lambda \underline{t}_\iota \underline{u}_\iota.(\neg'' \underline{x}_\iota)(\underline{u} = T_\iota \ \& \ \underline{x} = \underline{t});$$

$$\mathrm{Map}^1_{\iota\iota o} \rightarrow \lambda \underline{p}_o \underline{u}_\iota.(\neg \underline{x}_\iota)(\underline{u} = C_\iota \ \& \ \underline{x} = \mathrm{Map}^0\underline{p})$$

Now suppose that

$$l(\alpha) \leqslant n \ .$$

Then

a) α is ι_n ;

or b) α is $o\iota_{n-1}$;

or c) α is $\beta\gamma$, and $l(\beta) \leqslant n-1$, and $l(\gamma) \leqslant n-2$;

or d) α is ι ;

or e) α is o .

We define $\mathrm{Map}^n\underline{f}$ according to which of these cases holds; throughout what follows it is assumed that α,β,γ, satisfy the conditions given above.

a) $\mathrm{Map}^n_{\iota_{n+1}\iota_n} \rightarrow \lambda \underline{f}_{\iota_n}\underline{u}_{\iota_n}.(\neg'' \underline{x}_\iota)(E\underline{g}_{\iota_{n-1}})(\underline{u} = \mathrm{Map}^{n-1}\underline{g} \ \& \ \underline{x} = \underline{fg});$

b) $\mathrm{Map}^n_{\iota_{n+1}(o\iota_{n-1})} \rightarrow \lambda \underline{f}_{o\iota_{n-1}}\underline{u}_{\iota_n}(\neg \underline{x}_\iota)(E\underline{g}_{\iota_{n-1}})(\underline{u} = \mathrm{Map}^{n-1}\underline{g}$
$\& \ \underline{x} = \mathrm{Map}^0(\underline{fg}));$

c) $\mathrm{Map}^n_{\iota_{n+1}(\beta\gamma)} \rightarrow \lambda \underline{f}_{\beta\gamma}\underline{u}_{\iota_n}.(\neg' \underline{x}_\iota)(E\underline{g}_{\iota_{n-1}})(E\underline{k}_\gamma)(\underline{u} = \mathrm{Pai}^n(\mathrm{Map}^{n-2}\underline{k})\underline{g}$
$\& \ \underline{x} = \mathrm{Map}^{n-1}(\underline{fk})\underline{g});$

d) $\mathrm{Map}^n_{\iota_{n+1}\iota} \rightarrow \lambda \underline{t}_\iota\underline{u}_{\iota_n}.(\neg \underline{x}_\iota)(\underline{u} = (\lambda \underline{h}_{\iota_{n-1}}.T_\iota) \ \& \ \underline{x} = \underline{t});$

e) $\mathrm{Map}^n_{\iota_{n+1}o} \rightarrow \lambda \underline{p}_o\underline{u}_{\iota_n}.(\neg \underline{x}_\iota)(\underline{u} = (\lambda \underline{h}_{\iota_{n-1}}.C_\iota) \ \& \ \underline{x} = \mathrm{Map}^0\underline{p}).$

Theorem VII

Let $l(\alpha) \leqslant n$, and $l(\delta) \leqslant n$; then from

$$\mathrm{Map}^n \underline{f}_\alpha = \mathrm{Map}^n \underline{h}_\delta$$

we may infer that δ is α, and

$$\underline{f}_\alpha = \underline{h}_\delta .$$

We give an outline of the proof; we shall state a number of formal lemmas, the proofs of which proceed by induction over n, and are straightforward enough to be omitted. The theorem is trivial for n = 1.

5.1) $\vdash \ (\underline{f}_\alpha)(\mathrm{Map}^n \underline{f} \neq X_{\iota_{n+1}})$. $(n = 1, 2 \ldots)$

5.2) From $\mathrm{Map}^n \underline{f}_\alpha X_{\iota_n} = \begin{cases} T_\iota \\ C_\iota \\ F_\iota \end{cases}$ we can infer that α is $\begin{cases} \beta\gamma \\ o\iota_{n-1} \\ \iota_n \end{cases}$, or o, or ι.

$(n = 2, 3, \ldots)$

5.3) $\vdash \ (E!\underline{u}_{\iota_n})(\mathrm{Map}^n \underline{f}_{o\iota_{n-1}} \ \underline{u} \neq C_\iota)$ $(n = 2, 3, \ldots)$

5.4) $\vdash \ \mathrm{Map}^n \underline{p}_o(\lambda \underline{h}_{\iota_{n-1}} . C_\iota) \neq C_\iota$ $(n = 2, 3, \ldots)$

5.2), 5.3), 5.4), show that for $n \geqslant 2$ we cannot have

$$\mathrm{Map}^n \underline{f}_\alpha = \mathrm{Map}^n \underline{g}_\delta$$

unless α and δ come under the same case; to show that α and δ must be the same type we have now only to deal with case c); we note that this case only arises if $n \geqslant 3$.

5.5) $(E\underline{u}_{\iota_n})(\mathrm{Map}^n \underline{f}_\alpha \underline{u} \neq T_\iota)$ $(n = 1, 2, \ldots)$

This is immediate for all except case b), and also for n = 1. For case b) it follows by induction over n.

5.6) If γ' is not γ, then $\vdash \mathrm{Map}^n\underline{f}_{\beta\gamma} \neq \mathrm{Map}^n\underline{h}_{\beta'\gamma'}$.

Short proof:

H.1	$\mathrm{Map}^{n-1}(\underline{f}_{\beta\gamma}\,\underline{k}_{\gamma})\underline{g}_{\iota_{n-1}} \neq T_\iota$	$(\underline{f}, \underline{k}, \underline{g})$
H.2	$\underline{u}_{\iota_n} = \mathrm{Pai}^n(\mathrm{Map}^{n-2}\underline{k})\underline{g}$	$(\underline{u})$
3	$\mathrm{Map}^n\underline{fu} \neq T_\iota$	Map.
4	$\underline{u} \neq \mathrm{Pai}^n(\mathrm{Map}^{n-2}\underline{j}_{\gamma'})\underline{m}_{\iota_{n-1}}$	Pai, induction hypothesis
5	$\mathrm{Map}^n\underline{h}_{\beta'\gamma'}\underline{u} = T_\iota$	Map.
6	5.6)	(H.2, H.1), 5.5).

5.7) If β' is not β, then $\vdash \mathrm{Map}^n\underline{f}_{\beta\gamma} \neq \mathrm{Map}^n\underline{h}_{\beta'\gamma}$.

This concludes the demonstration that the maps of elements of distinct types are distinct.

5.8) $\vdash \mathrm{Map}^n\underline{f}_\alpha = \mathrm{Map}^n\underline{g}_\alpha \supset \underline{f}_\alpha = \underline{g}_\alpha$

The proof of this has to be taken case by case; it is straightforward.

This conclude the demonstration of theorem VII.

$$\mathrm{App}^n_{\iota_{n+1}\iota_{n+1}\iota_{n+1}} \rightarrow \lambda\underline{r}_{\iota_{n+1}}\underline{s}_{\iota_{n+1}}.(\iota\underline{t}_{\iota_{n+1}})$$

$$\left[\sum_{l(\alpha\delta)\leq n}(E\underline{f}_{\alpha\delta})(E\underline{x}_\delta)(\underline{s} = \mathrm{Map}^n\underline{f} \;\&\; \underline{r} = \mathrm{Map}^n\underline{x} \;\&\; \underline{t} = \mathrm{Map}^n(\underline{fx})\right]$$

where the $\sum$ means the logical disjunction of all the propositions of the given form.

5.9) If $l(\alpha\delta) \leq n$, and $n \geq 2$, then

$$\vdash \mathrm{App}^n(\mathrm{Map}^n\underline{f}_{\alpha\delta})(\mathrm{Map}^n\underline{x}_\delta) = \mathrm{Map}^n(\underline{f}_{\alpha\delta}\,\underline{x}_\delta).$$

Thus we can map all those formulae of the system which have no parts of type of length greater than n, into the type and the map preserves the logical relations between formulae. (We have not actually dealt with formulae containing free variables, nor with abstraction, but evidently it would be possible to do so.)

We note that

$$\mathrm{Map}^n C_\iota = \lambda \underline{u}_{\iota_n} . C_\iota ;$$

it is however convenient to have a nonsense element in which is not the image of any element under Map; accordingly we define:

$$\bar{\iota}_{\iota_{n+1}(o\iota_{n+1})} \rightarrow \lambda \underline{f}_{o\iota_{n+1}} . (\imath \underline{x}_{\iota_{n+1}})(J\underline{f} \supset \underline{x} = \iota \underline{f} \;.\&.\; \sim J\underline{f} \supset \underline{x} = X_{\iota_{n+1}})$$

$(n = 1, 2, \ldots)$.

We also introduce η as an abbreviation for ι_{n+1}.

$$\alpha\text{-Sub}_{o\eta} \rightarrow \lambda \underline{f}_\eta . (E\underline{g}_\alpha)(\underline{f} = \mathrm{Map}^n \underline{g}), \qquad n = 1, 2, \ldots$$

$$\mathrm{Typ}_{o(o\eta)} \rightarrow \lambda \underline{r}_{o\eta} \left[\sum_{l(\alpha) \leq n} (\underline{r} = \alpha\text{-Sub}) \right].$$

$$\mathrm{Tot}_{o\eta} \rightarrow \lambda \underline{f}_\eta . (E\underline{r}_{o\eta})(\mathrm{Typ}\underline{r} \;\&\; \underline{r}\underline{f}).$$

The above definitions define the image sets of the various types of length not greater than n, the set of all such image sets, and the union of all such image sets, respectively.

Consider now a closed formula the bound variables of which are all of type less than n. (We say 'of type less than n', instead of 'of type of length less than n', for brevity). The formula has a combinatorial equivalent; but unfortunately the more bound variables there are in the formula, the higher the type of the W's and K's in that equivalent. If there were an upper bound N to the length of the type of the W's and K's involved, one could map all types not greater than N into a higher type, and therein describe the combinatorial process, and so obtain a formula representing the class of all closed formulae of the sort considered. But since there is not such an upper bound, we proceed rather differently, following the method proposed by Tarski in Tarski (2).

We define the <u>argument parts</u>, and the <u>value part</u>, of any type:

a) A type whose type symbol consists of a single symbol is its own value part, and has no argument part;

b) The value part of $\alpha\beta$ is the value part of α ; the argument parts of $\alpha\beta$ are β and the argument parts of α.

Thus an element of any complex type may be considered as a function of several arguments ranging over the various argument parts, and taking its values in the appropriate value part, which is always o or ι . This way of looking at the structure of a complex type is of course reflected in the conventions concerning the omission of brackets in a type symbol. For later use we define:

α-Nar_v is 1_v, if α is o or ι ;

$\alpha\beta$-Nar_v is α-$\mathrm{Nar}_v + 1_v$.

α-Nar is the number of value and argument parts of the type α.

Consider now a function of type $\kappa\alpha_1\ldots\alpha_m$ where κ is o or ι; instead of thinking of it as a function of several arguments, we may think of it as a function of a sequence, the elements of the sequence lying in the types $\alpha_1,\ldots,\alpha_m$, and the function taking its values in κ. This is important because if the lengths of the types $\alpha_1,\ldots,\alpha_m$ are less than n, then we can represent any such sequence in the type $(\eta\nu)$. Thus any function, none of whose argument parts are of length greater than n, can be distinctly represented in the type $\iota_2(\eta\nu)$; and it turns out that such functions suffice for the making of a definition of the class of closed formulae with bound variables of length not greater than n.

In what follows we use a number of conventions:

α,β,γ are of length less than n; and n $\geqslant$ 2;

κ is o or ι;

$\bar{\nu}$ is the type of <u>positive</u> integers; in connection with it we use the usual arithmetical symbols;

X is an abbreviation for X_η;

Y is an abbreviation for X_{ι_2};

σ is an abbreviation for $\eta\bar{\nu}$;

[<u>f</u>] denotes the function of sequences corresponding to the element $\underline{f}_{\kappa\beta\ldots\gamma}$; for a sequence ($\underline{z},\ldots,\underline{x}$)

[<u>f</u>] ($\underline{z},\ldots,\underline{x}$) is <u>fx</u>...z,

if <u>x</u> is in type γ, ..., <u>z</u> is in type β, otherwise it is nonsense; (this usage is only required informally).

We now introduce a number of definitions.

$$\mathrm{Seq}_{o\sigma} \rightarrow \lambda \underline{s}_{\sigma}.(E\underline{m}_{\bar{\nu}})(\underline{p}_{\bar{\nu}})(\underline{p} < \underline{m} \supset \mathrm{Tot}(\underline{sp}) \ .\&.\ \underline{p} \geqslant \underline{m} \supset \underline{sp} = X).$$

Seq$\underline{s}$ means that for the first so many integers - possibly none - $\underline{s}$ takes values in η representing elements of type not greater than n, and thereafter takes a nonsense value.

$$\mathrm{Alo}_{o\eta\bar{\nu}\sigma} \rightarrow \lambda \underline{s}_{\sigma}\underline{m}_{\bar{\nu}}\underline{u}_{\eta}.\ (E\underline{r}_{o\eta})(\mathrm{Typ}\underline{r} \ \&\ \underline{r}(\underline{sm}) \ \&\ \underline{ru})$$
$$.v.\ \underline{sm} = X \ \ \&\ \underline{u} = X\ .$$

Alo$\underline{sm}$ gives the typical range of the mth member of $\underline{s}$.

$$\mathrm{Cut}_{\sigma\sigma\bar{\nu}} \rightarrow \lambda \underline{m}_{\bar{\nu}}\underline{s}_{\sigma}\underline{p}_{\bar{\nu}}.(\bar{\imath}\underline{u}_{\eta})(\underline{s}(\underline{p} + \underline{m}) \neq X \ \&\ \underline{u} = \underline{sp})$$

Cut$\underline{ms}$ is the sequence which is like $\underline{s}$ but with the last m terms deleted.

$$\mathrm{Cub}_{\sigma\sigma\bar{\nu}} \rightarrow \lambda \underline{m}_{\bar{\nu}}\underline{s}_{\sigma}\underline{p}_{\bar{\nu}}.\underline{s}(\underline{p} + \underline{m})$$

Cub$\underline{ms}$ is like $\underline{s}$ but with the first m terms deleted.

$$\mathrm{Fir}_{\sigma\sigma\bar{\nu}} \rightarrow \lambda \underline{m}_{\bar{\nu}}\underline{s}_{\sigma}\underline{p}_{\bar{\nu}}.(\bar{\imath}\underline{u}_{\eta})(\underline{p} \leq \underline{m} \ \&\ \underline{u} = \underline{sp})$$

Fir$\underline{ms}$ has the same first m elements as $\underline{s}$.

$$\kappa\text{-}\mathrm{Cla}_{o\sigma} \rightarrow \lambda \underline{s}_{\sigma}.\underline{s} = \lambda \underline{m}_{\bar{\nu}}.X$$

$$\alpha\beta\text{-}\mathrm{Cla}_{o\sigma} \rightarrow \lambda \underline{s}_{\sigma}.(E\underline{t}_{\sigma})(\underline{m}_{\bar{\nu}})(\alpha\text{-}\mathrm{Cla}\underline{t}$$
$$.\&.\ \underline{m} < \alpha\text{-}\mathrm{Nar} \supset \mathrm{Alo}\underline{sm} = \mathrm{Alo}\underline{tm}$$
$$.\&.\ \underline{m} = \alpha\text{-}\mathrm{Nar} \supset \mathrm{Alo}\underline{sm} = \beta\text{-}\mathrm{Sub}$$
$$.\&.\ \underline{m} > \alpha\text{-}\mathrm{Nar} \supset \underline{sm} = X).$$

If α is $\kappa\alpha_1 \ldots \alpha_p$ then α-Cla$\underline{s}$ means that $\underline{s}$ is a sequence whose mth element lies in the image (in η) of the type α_m.

We now turn to functions of sequences, taking their values in ι_2 ; ι_2 of course contains image sets (under Map[1]) of the types σ and ι.

$$\mathrm{Pro}_{o(\iota_2\sigma)} \rightarrow \lambda \underline{f}_{\iota_2\sigma}.(E\underline{s}_\sigma)(E\underline{a}_{o\iota_2})(\underline{t}_\sigma)\Big\{\mathrm{Seq}\underline{s}\ \&\ \mathrm{Typ}\underline{a}\ \&\ \underline{a}(\underline{fs})$$
$$\&\ (\underline{ft} \neq Y\ .\supset.\ \mathrm{Alo}\underline{s} = \mathrm{Alo}\underline{t}\ \&\ \underline{a}(\underline{ft}))\Big\}$$

Pro$\underline{f}$ ('$\underline{f}$ is proper') means that there is a type κ and a sequence of types $\alpha_1, \ldots \alpha_m$, such that $\underline{fs}$ lies in the image (in ι_2) of κ, if the elements of $\underline{s}$ lie in the images (in η) of the types $\alpha_1, \ldots, \alpha_m$, respectively, and $\underline{fs}$ is nonsense otherwise.

$$\kappa\text{-Fus}_{\iota_2\sigma\eta} \rightarrow \lambda \underline{u}_\eta\ \underline{s}_\sigma.(\bar{\imath}\underline{x}_{\iota_2})(E\underline{q}_\kappa)(\underline{s} = \lambda \underline{m}_\nu.X\ \&\ \underline{u} = \mathrm{Map}^n\underline{q}$$
$$\&\ \underline{x} = \mathrm{Map}^1\underline{q}).$$

$$\alpha\beta\text{-Fus}_{\iota_2\sigma\eta} \rightarrow \lambda \underline{u}_\eta\ \underline{s}_\sigma.(\bar{\imath}\underline{x}_{\iota_2})\Big\{\alpha\beta\text{-Sub}\underline{u}\ \&\ \alpha\beta\text{-Cla}\underline{s}$$
$$\&\ \underline{x} = \alpha\text{-Fus}[\mathrm{App}\underline{u}(\underline{s}(\alpha\beta\text{-Nar}))]\ (\mathrm{Cut}1_\nu\underline{s})\Big\}.$$

$$\mathrm{Fus}_{\iota_2\sigma\eta} \rightarrow \lambda \underline{u}_\eta\ \underline{s}_\sigma.(\bar{\imath}\underline{x}_{\iota_2})\Big\{\sum_{\ell(\alpha)\leq n}(\alpha\text{-Sub}\underline{u}\ \&\ \underline{x} = \alpha\text{-Fus}\underline{us}) \quad .$$

If $\underline{u}$ is the image in η of an element $\underline{g}_\alpha$, then Fus$\underline{u}$ is $[\underline{g}]$. ('fus' stands for 'FUnction of Sequences'). If $\underline{g}$ is in o or ι, then Fus$\underline{us}$ is the image of $\underline{g}$ in ι_2 if $\underline{s}$ is the empty sequence, and is nonsense otherwise.

$$\mathrm{Las}_{\eta\sigma} \rightarrow \lambda \underline{s}_\sigma.(\bar{\imath}\underline{u}_\eta)(E\underline{m}_\nu)(\underline{sm} \neq X\ \underline{s}(S\underline{m}) = X\ \&\ \underline{u} = \underline{sm}).$$

Las$\underline{s}$ is the last element of the sequence $\underline{s}$.

$$\mathrm{Mix}_{\iota_2\sigma\eta(\iota_2\sigma)} \rightarrow \lambda \underline{f}_{\iota_2\sigma}\ \underline{u}_\eta\ \underline{s}_\sigma.(\bar{\imath}\underline{x}_{\iota_2})(E\underline{t}_\sigma)(\mathrm{Seq}\underline{t}\ \&\ \underline{s} = \mathrm{Cut}1_\nu\underline{t}$$
$$\&\ \mathrm{Las}\underline{t} = \underline{u}\ \&\ \underline{x} = \underline{ft}).$$

$$\mathrm{Sap}_{\iota_2\sigma(\iota_2\sigma)(\iota_2\sigma)} \rightarrow \lambda \underline{f}_{\iota_2\sigma}\ \underline{g}_{\iota_2\sigma}(\bar{\imath}\underline{h}_{\iota_2\sigma})(E\underline{u}_\eta)(\underline{g} = \mathrm{Fus}\underline{u}\ \&\ \underline{h} = \mathrm{Mix}\underline{fu}).$$

$$\mathrm{Dam}_{\eta\sigma(\iota_2\sigma)} \rightarrow \lambda \underline{f}_{\iota_2\sigma}.\underline{s}_\sigma.(\bar{\imath}\underline{u}_\eta)(E\underline{t}_\sigma)(E\underline{m}_\nu)(\mathrm{Pro}\underline{f}\ \&\ \underline{ft} \neq Y$$
$$\&\ \underline{s} = \mathrm{Cub}\underline{mt}\ \&\ \underline{ft} = \mathrm{Fus}\underline{u}(\mathrm{Fir}\underline{mt})).$$

For brevity we now stop distinguishing (in these informal remarks) between an element and its image under Map, and we ignore the typical chaos that results. Mix $[\underline{f}_{\delta\beta}]\underline{z}_{\beta}$ is $[\underline{f}_{\delta\beta}\underline{z}_{\beta}]$ (where δ can be greater than n). Sap $[\underline{f}_{\delta\beta}][\underline{g}_{\beta}]$ is $[\underline{f}_{\delta\beta}\underline{g}_{\beta}]$. $\mathrm{Dam}[\underline{f}_{\kappa\alpha\ldots\beta\gamma\ldots\delta}](\underline{x}_{\gamma},\ldots,\underline{y}_{\delta})$ is $\underline{f}_{\kappa\alpha\ldots\beta\gamma\ldots\delta}\,\underline{y}_{\delta}\ldots\underline{x}_{\gamma}$. ('Sap' stands for 'functional APplication in terms of Sequences; 'Mix' for 'the mixed case'; 'Dam' expresses the author's, and probably the reader's feelings).

$$\begin{aligned}\mathrm{Dou}_{\iota_2\sigma(\iota_2\sigma)(\iota_2\sigma)\bar{\nu}} \rightarrow \lambda\underline{m}_{\bar{\nu}}\,\underline{f}_{\iota_2\sigma}\underline{g}_{\iota_2\sigma}\,\underline{s}_{\sigma}.(\bar{\imath}\underline{x}_{\iota_2})(\mathrm{E}\underline{t}_{\sigma})(\mathrm{Seq}\underline{t}\ \&\ \underline{x} = \underline{ft}\\ \&\ \underline{t}(\mathrm{S}\underline{m}) = \mathrm{Dam}\underline{g}(\mathrm{Cub}\underline{ms})\\ \&\ (\underline{p}_{\bar{\nu}})(\underline{p} \leqslant \underline{m} \supset \underline{sp} = \underline{tp}\ .\&.\ \underline{p} > \underline{m} \supset \underline{sp} = \underline{t}(\mathrm{S}\underline{p}))).\end{aligned}$$

$\mathrm{Dou}\underline{m}[\underline{f}_{\kappa\alpha\ldots\beta\gamma\delta\ldots\varepsilon}][g_{\gamma\delta\ldots\varepsilon}]$ is

$$[\lambda\underline{x}_{\varepsilon}\ldots\underline{y}_{\delta}.\underline{f}_{\kappa\alpha\ldots\beta\gamma\delta\ldots\varepsilon}\,\underline{x}_{\varepsilon}\ldots\underline{y}_{\delta}(\underline{g}_{\gamma\delta\ldots\varepsilon}\,\underline{x}_{\varepsilon}\ldots\underline{y}_{\delta})]$$

where $\alpha,\ldots,\beta,\gamma,\delta,\ldots,\varepsilon$, are all of length not greater than n, and the list $\alpha,\ldots,\beta$, is of length m; if m is greater than the total number of argument parts of $\underline{f}$, then $\mathrm{Dou}\underline{m}[\underline{f}][\underline{g}]$ is just $[\underline{f}]$. ('Dou' stands for 'Double U', for Dou represents a glorified version of W).

$$\begin{aligned}\mathrm{All}_{\iota_2\sigma(\iota_2\sigma)} \rightarrow\ & \lambda\underline{f}_{\iota_2\sigma}\,\underline{s}_{\sigma}.(\bar{\imath}\underline{x}_{\iota_2})(\mathrm{E}\underline{t}_{\sigma})\Big[\mathrm{Seq}\underline{t}\ \&\ \underline{s} = \mathrm{Cub1}_{\bar{\nu}}\underline{t}\\ & \&\ \underline{x} = \mathrm{Map}^1\Big\{(\underline{r}_{\sigma})(\mathrm{Seq}\underline{r}\ \&\ \underline{s} = \mathrm{Cub1}_{\bar{\nu}}\underline{r}\ \&\ \underline{fr} \neq \mathrm{Y}\\ & .\supset.\ \underline{fr} = \mathrm{Map}^1\mathrm{T})\Big\}\Big]].\end{aligned}$$

All $\underline{f}_{o\alpha\beta\ldots\gamma}$ is $\lambda\underline{x}_{\gamma}\ldots\underline{y}_{\beta}.(\underline{z}_{\alpha})(\underline{fx}\ldots\underline{yz})$.

$$\begin{aligned}\mathrm{Pic}_{o(\iota_2\sigma)\bar{\nu}} \rightarrow \lambda\underline{m}_{\bar{\nu}}\,\underline{f}_{\iota_2\sigma}.(\mathrm{E}\underline{p}_{\bar{\nu}})(\mathrm{Pro}\underline{f}\ \&\ (\underline{s}_{\sigma})(\mathrm{Seq}\underline{s}\ \&\ \underline{fs} \neq \mathrm{Y}\\ .\supset.\ \mathrm{Fus}(\underline{s}(\underline{m}+\underline{p}))(\mathrm{Fir}\underline{ps}) = \underline{fs})).\end{aligned}$$

$\underline{f}$ satisfies Pic$\underline{m}$[$\underline{f}$], if $\underline{f}$ is of the form

$$\lambda \underline{x}_\gamma \ldots \underline{y}_\beta \ldots \underline{z}_\alpha . \underline{y}_\beta ,$$

where $\underline{y}$ is the mth from the right in the list $\underline{x}_\gamma, \ldots, \underline{y}_\beta, \ldots \underline{z}_\alpha$.

$$\mathrm{Sei}_{\iota_2 o \alpha} \rightarrow \lambda \underline{z}_\alpha . \mathrm{Fus}(\mathrm{Map}^n \underline{z})$$

Sei$\underline{z}$ is [$\underline{z}$].

$$\text{n-Clo}^\alpha_{o\alpha} \rightarrow \lambda \underline{z}_\alpha . (\underline{h}_{o(\iota_2 o)}) \Big[\prod_{B_\beta \in \mathcal{C}} \{ \underline{h}(\mathrm{Sei}B_\beta) \} \ \& \ (\underline{m}_\gamma)(\underline{h}(\mathrm{Pic}\underline{m}))$$

$$\& \ \big\{ (\underline{p}_\gamma)(\underline{f}_{\iota_2 o})(\underline{g}_{\iota_2 o})(\underline{hf} \ \& \ \underline{hg} \ . \supset . \ \underline{h}(\mathrm{All}\underline{f}) \ \& \ \underline{h}(\mathrm{Doup}\underline{fg})$$

$$\& \ \underline{h}(\mathrm{Sap}\underline{fg})) \big\} \ . \supset . \ \underline{h}(\mathrm{Sei}\underline{z}) \Big] \qquad (n \geqslant 3)$$

where $\prod$ stands for the conjunction of the given propositions, and $B_\beta \in \mathcal{C}$ means that B_β is one of the constants:

$$N_{oo}, \ A_{ooo}, \ C_o, \ \iota_{o(oo)}, \ C_\iota, \ \iota_{\iota(o\iota)} .$$

This is the definition that we set out to find. By a <u>proper</u> closed formula of system (C) we mean a closed formula in which $\prod_{o(o\alpha)}$ only occurs in parts of the form $(\prod_{o(o\alpha)}(\lambda \underset{\sim}{x}_\alpha . \underset{\sim}{A}_o))$.

<u>Theorem VIII</u>

If $\underset{\sim}{A}_\alpha$ is a proper closed formula of type not greater than n, and all the bound variables of $\underset{\sim}{A}_\alpha$ are of type not greater than n, and $n \geqslant 3$ then

$$\vdash \text{n-Clo}\underset{\sim}{A}_\alpha .$$

We shalll not give a complete proof of this theorem, but shall content ourselves with demonstrating the following lemma:

If $\underset{\sim}{A}_\alpha$ is as above, then $[\underset{\sim}{A}_\alpha]$ is obtained from a finite number of the functions of sequences

$$\frac{[B_\beta]}{[\lambda x_\alpha \ldots y_\beta \ldots z_\gamma . y]} \qquad B_\beta \in \mathcal{C}$$

by a finite number of applications of the operations which are represented by Dou, All, and Sap.

Firstly we note that every (well formed) formula of (C) has a normal form - i.e. for any formula D_δ there exists a formal E_δ such that $\vdash D_\delta = E_\delta$, and no application of rule II to E_δ is possible. Because of the axiom of extensionality it is sufficient to prove the theorem and the lemma for any formula which is in normal form.

We suppose now that the lemma has been demonstrated for any formula which is shorter than A_α; it is obvious if A_α consists of a single symbol.

Case 1. A_α is of the form $D_{\delta\varepsilon}\, E_\varepsilon$. Then $D_{\delta\varepsilon}$ must be $N_{oo}\ \iota_{o(oo)},\ \iota_{o(o\iota)},\ \Pi_{o(o\beta)}$, or of the form $(A_{ooo}\ P_o)$; for it cannot be of the form $(\lambda x_\varepsilon . M_\delta)$, nor can its first proper symbol be a free variable. If $D_{\delta\varepsilon}$ is $\Pi_{o(o\beta)}$ then E_ε is of the form $(\lambda x_\beta . Q_o)$, where $l(\beta) \leq n$, and so a single application of All to $[E_\varepsilon]$ gives $[A_\alpha]$. In the other cases a single application of Sap to a constant and a closed part of A_α gives $[A_\alpha]$.

Case 2. A is of the form

$$\lambda x_\beta \ldots y_\gamma . D_\delta ,$$

where D_δ consists of a single symbol.
If D_δ is a variable then $[A_\alpha]$ is one of the original list.
If D_δ is a constant $B_\delta (\in \mathcal{C})$, then $[A_\alpha]$ is obtained by an

application of Sap to $[\lambda\, \underline{b}_{\delta}. x_{\beta} \ldots y_{\gamma} . \underline{b}\ ]$ and $[B_{\delta}]$

Case 3. A_{α} is of the form

$$\lambda x_{\beta} \ldots y_{\gamma} . D_{\delta\varepsilon} E_{\varepsilon} .$$

If $D_{\delta\varepsilon} E_{\varepsilon}$ is of the form $\Pi_{o(o\zeta)}(\lambda z_{\zeta} . P_{o})$, then $[A_{\alpha}]$ is obtained, by an application of All, from the function of sequences

$$[\lambda x_{\beta} \ldots y_{\gamma} . z_{\zeta} . P_{o}],$$

which corresponds to a closed formula of length less than the length of A_{α}; hence the result. If $D_{\delta\varepsilon}$ is not $\Pi_{o(o\zeta)}$, then $D_{\delta\varepsilon} E_{\varepsilon}$ is of one of the forms

$$B_{\delta\varepsilon} E_{\varepsilon}, \qquad B \text{ is } N_{oo}, \iota_{o(oo)}, \text{ or } \iota_{\varepsilon(o\iota)} ;$$

$$(A_{ooo} P_{o}) E_{o} ;$$

$$(z_{\delta\varepsilon\rho\ldots\sigma} M_{\sigma} \ldots N_{\rho}) E_{\varepsilon}, \text{ where } z_{\delta\varepsilon\rho\ldots\sigma} \text{ is one of}$$

$x_{\beta}, \ldots y_{\gamma}$. But in each of these cases the type of E cannot be greater than n; so that $[A_{\alpha}]$ can be obtained by an application of Dou$\underline{m}$ (with appropriate $\underline{m}$) to:

$$[\lambda x_{\beta} \ldots y_{\gamma} . D_{\delta\varepsilon}\ ],$$

and

$$[\lambda x_{\beta} \ldots y_{\gamma} . E_{\varepsilon}\ ] .$$

But these functions of sequences correspond to closed formulae of length less than A_{α}; hence the result. This concludes the demonstration of the lemma, for due to the requirement that A_{α} be in normal form, no cases other than those considered can arise.

To pass from this lemma to a proof of theorem VIII, we should have to prove a large number of formal lemmas which would show that the formulae we have introduced do in fact have the properties we have claimed for them, we shall not do this.

We now discuss some of the implications of theorem VIII. Firstly we remark that there is no essential difficulty in extending it to the case where there are other non-complex types besides o and ι; in particular, if these types are o and ν, or o, ι, and ν, and there is an axiom which allows the mapping of the integers one-to-one into the individuals, then we can make the extension without altering the types of the variables that occur in the formula 'n-Clo' - except that, in the first case, ι will be everywhere replaced by ν.

Secondly we note that it is possible to enumerate all the functions of sequences which correspond to n-closed formulae, and that it is possible to define such an enumeration within the system, and so produce a series of formulae 'n-$\text{Enu}_{\alpha\nu}$' which enumerate all the 'n-clo' elements of type α. We shall say that an element of type α which can be described by a closed formula with no bound variables of type greater than n, and cannot be described by a closed formula with variables of type less than n, is of <u>order</u> n. The term was first used in this sense by Tarski (in (2)); but our meaning of the term is slightly different from his, since his system does not contain λ or ι, and only contains the types ν, $o\nu$, $o(o\nu)$, ... This means that the actual order of a given quantity (say, for example, a class of integers) will depend on which definition is adopted, but whether or not the quantity has a finite order will be independent of the exact definition. We here remark again on the economy which is achieved by using Church's system: in (2) Tarski gives in English (or rather, in German) but not formally, a definition of 'of order 1'; this does not take up very much less space than our formal definition (including all the concomitants) of 'n-Clo'. The term 'order' suggests, and is meant to suggest, the orders of the ramified theory of types, for our 'order' also serves to prevent situations, which are analogous to those that occur in the 'linguistic' paradoxes, from arising; indeed - assuming that system (C) is consistent - positive information may be obtained from the attempt to set up such a situation. For example, it is perfectly possible to set up in (C) a theory of all ordinals less than some given ω_n; this is best done by introducing a special virtual type with certain additional constants. Then a suitable definition of 'n-Clo' for the extended system can be made, and one has only to consider the expression 'the least ordinal which is not n-Clo' - an analogue of Grelling's paradox - to see that n-Clo in the extended system is certainly of order greater than n. By showing that it is possible to set up an explicit well ordering of some of the elements of types $o(o\iota)$, $o(o(o\iota))$,..., Tarski (in (4)) has shown that the formula of 'of order n' for these types cannot itself be of order less than (n + 1); I think it evident that his argument could be taken over into our system, so we have:

$$\vdash \sim \text{n-Clo}(\text{n-Clo}^{o(ov)});$$

on the other hand, for the system based only on the types o and ν, by simply substituting ν for ι, and using theorem VIII, we have:

$$\vdash (n + 5)\text{-Clo}(\text{n-Clo}^{\alpha}) \qquad (1(\alpha) \leqslant n,\ n \geqslant 3).$$

The 5 in this proposition could certainly be replaced by a smaller integer; in Tarski's system the value in the equivalent proposition is 1. The question whether or not the proposition

(X) $\qquad \sim \text{n-Clo}(\text{n-Clo}^{o\nu})$

is provable remains open; it would be very surprising if the negation of (X) were provable. But by using a version of Cantor's theorem we evidently have:

$$\vdash \sim \text{n-Clo}(\text{n-Enu}^{o\nu}).$$

We consider now an extension of the system (C); we introduce a set of new symbols '$\text{Clo}^{\alpha}_{o\alpha}$', and a new set of axioms:

(N) $\qquad \text{n-Clo}^{\alpha}\ \underline{z}_{\alpha} \supset \text{Clo}^{\alpha}\ \underline{z}_{\alpha}; \qquad$ (n 3)

and a new rule:

Rule N. From $\text{Clo}^{\alpha}\ \underset{\sim}{A}_{\alpha}$ to infer $\text{n-Clo}^{\alpha}\ \underset{\sim}{A}_{\alpha}$ for some integer n. Thus Clo^{α} represents the set of all closed formulae of type α. It is possible to make a model of the simple theory of types within Godel's system of set theory (see Rosser and Wang (1)), and hence it is evident that, assuming the consistency of set theory, one could prove the consistency of the above additions. From the theorems mentioned above we can deduce:

$$\vdash \sim \mathrm{Clo}(\mathrm{Clo}^{o(ov)})$$

$$\vdash (\underline{f}_{\alpha\nu})\left\{(\underline{r}_{\alpha})(\mathrm{Clo}\,\underline{r} \supset (\mathrm{E}\underline{m}_{\nu})(\underline{fm} = \underline{r})) \supset \sim \mathrm{Clo}\,\underline{f}\right\}$$

The latter theorem is an 'explanation' of Richard's paradox. And, as for (X), we do not know whether the proposition

$$\sim \mathrm{Clo}(\mathrm{Clo}^{ov})$$

is provable or not.

If we confine ourselves to a system in which the only basic types are o and ν, then Clo evidently satisfies the conditions (B) of section 4; this suggests that it should be possible to construct a model based on Clo. But lemma A, and hence theorem V depends essentially on the formulae Bas being closed, and therefore satisfying

$$\vdash \mathrm{Bas}^{o\alpha}\mathrm{Bas}^{\alpha} .$$

Thus the method used for theorem V is not available; but nevertheless it seems to me plausible that there could be constructed a model based on Clo. (The chief ground for this belief is my inability to see how one could possibly prove the existence of an unclosed element in any type without using either an enumeration of the closed elements or the selection axiom.) If such a model could be constructed it would be evidently a minimum model, for the existence of any element represented by a closed formula is assured. Secondly the existence of such an inner model would ensure that by taking just the elements representable by closed formulae in every type one could construct (outer) general models in Henkin's sense; and 'valid in every such model' might be the definition of 'is a consequence of the axioms' for which we were looking in the last section.

Finally I wish to stress that the properties Clo and n-Clo are not merely of logical interest, but have real mathematical significance. For definiteness, let us cpnsider the type $o\nu$ - that is the real numbers between 0 and 1 considered as binary decimals with the possibility of dual representation. In a sense every mathematically definable real number between 0 and 1 is representable by a closed formula, and Tarski (in (2) and (4)) uses the word definable in this sense. But by enumerating all the closed formulae of type $o\nu$ (or their combinatorial equivalents), and applying Cantor's diagonal process one does define - metamathematically - a number which is not representable by any closed formula. Other possible methods of defining such a number are: the number which corresponds to such and such an ordinal in a well-ordering of the real numbers; the number whose binary digits are determined by an infinite succession of tosses of a specified coin. But the first of these is not a proper definition unless a well-ordering is explicitly given; and if it is given within the system it will be represented by a closed formula, while if it is given outside the system, the definition is again metamathematical. And the second proposed definition is really absurd; for it refers to a physical process which is physically impossible. Thus if we rule out metamathematical definitions, we can conclude that all definable real numbers are representable by closed formulae. Now whenever a real number is mentioned in a mathematical argument it must be referred to either by means of a description, or by means of a variable which has been restricted by hypothesis ('let x be a number such that...'); and similarly for objects of higher type. If we are right in supposing that a model may be based on Clo, it follows that any mathematical argument, which does not use metamathematical considerations, can be interpreted as referring <u>entirely</u> to elements representable by closed formulae. (An exception might have to be made for arguments which used the selection axiom, for it seems to me likely that the axiom would not hold for type ω in a model based on Clo.).

It is not usual for mathematics, or mathematical physics to concern themselves with objects of very high type, so that the order of defined quantities is in practice very low. For example, I reckon that the order of any computable binary decimal is less than eight. Certainly the mental effort required in handling a concept increases rapidly with the length of its type, and progress depends on inventing techniques and analogies which will lessen that effort. For instance, the analogy (and the accompanying techniques) between the application of a linear functional to a function and the scalar product in a finite-dimensional vector space has made possible an elaborate theory of functionals; a theory which would seem incredibly abstract and hard to grasp to anyone unfamiliar with the analogy. One of the reasons why modern quantum field theory is so difficult is that it deals with objects of rather high type - functionals of functions defined on arbitrary spacelike surfaces and so on; but it does not provide a convincing analogy with objects of lower type, nor does it use an adequate notation. Indeed, if we order the types in such a way that a lesser type can always be mapped one-to-one into a greater type, then we might well take the greatest type in common use as an index of mathematical progress!

CHAPTER II.

Section 1. The deduction theorem.

The function of symbolic logic and of foundational studies is not, in my opinion, to dictate to a subject how it should conduct its arguments, but to elucidate the way in which it does conduct them: the right words of appreciation for a successful attempt are 'I see', not 'I hear and obey'. To achieve this aim the logician must fix his subject at a particular stage of its development, and then must codify, and classify, and make more precise, the methods of argument which it uses, and the nature and interrelations of its concepts. And when he has done this, he should leave the subject free to go its own way. He should be like a surgeon who performs an operation to examine the condition of his patient, and to display it to his students; and who, when the operation is finished, says to the patient 'run along now, we'll take another look at you later'. He should not be like an anatomist who first kills his subject, or a Frankenstein who makes monsters. By this last remark I do not mean to say that a logician should never indulge his fancy; to do so is a privilege which belongs to all mathematicians. But I do mean that his first duty is to make a logical picture of the subject he is studying as he finds it. I do not think it is possible to give strict criteria for deciding what is a satisfactory logical picture; in particular it is not necessary that the picture should be, as it were, a photographic likeness. (Poincaré believed that it should be; hence his disputes with the logicians). But if the picture is violently non-representational it cannot, evidently, fulfil its purpose - to elucidate and explain. And I believe that the majority of modern logicians are guilty of just this fault - the picture they present is too hopelessly unlike life to be of any use. For they assert that all propositions are either analytic, or contradictory, or synthetic; while I claim that many of the propositions of mathematics, and almost all of the propositions of theoretical physics are none of these things.

Of course, it all depends what you mean by 'proposition'. Let us consider some of the possible interpretations of

$$\underline{x} > 3$$

a) '$\underline{x} > 3$' means the same as '$\lambda \underline{x}_v . \underline{x} > 3$';

b) '$\underline{x} > 3$' means the same as '$(\underline{x}_v)(\underline{x} > 3)$';

c) $\underline{x} > 3$ is not a proposition, but a propositional function (in the old sense of the term), or a <u>matrix</u>; it becomes a proposition when the symbol for an integer is substituted for $\underline{x}$;

d) $\underline{x} > 3$ is a proposition whose truth value depends on $\underline{x}$.

Interpretation a) can be ruled out straight away, for it leads to hopeless confusion: according to it, the presence of the free variable $\underline{x}$ indicates that the expression is a function of an integral argument. Consider the proposition

(X) $\quad (\lambda \underline{x}_v . \underline{x} > 3) = (\lambda \underline{x}_v . \underline{x} > 3)$;

by a)

$$(\underline{x} > 3) = (\underline{x} > 3)$$

will mean the same as (X). But it contains the free variable $\underline{x}$, and so is also a function of an integral argument; which is absurd. We may note that the interpretation a) is based on the seventeenth century convention of writing 'f($\underline{x}$)' to mean 'the function f', and the consequent (or precedent ?) failure to distinguish between a function and its values. Fallacies based on confusions similar to the one we have expounded do still occur in papers on theoretical physics[1]; of course they can only arise when functions of functions are being considered.

In system (C) a proposition which has been proved (or an axiom) bears the interpretation b). But a proposition which has not been proved does not: for although

$$\underline{x}_v > 3 \vdash (\underline{x}_v)(\underline{x} > 3)$$

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(1) See for example Eddington (1), pp.26-27; H^{Δ} is regarded both as a function of the occupation function j, and of the state parameters X_i, which are the arguments of j, and a detailed analysis of the argument shows that this does really represent a confusion of the kind considered.

is a valid inference (rule C.VI), it will never in fact be used, because it does not lead towards the proof of any proposition; an uncertain proposition, in the sense of interpretation d), occurs only in contexts which involve - sooner or later - an application of the deduction theorem. This is one reason why the deduction theorem is important; it allows contexts in which expressions can bear the interpretation d). And it may be noted in passing that the objections usually raised against material implication fail when applied to a system which allows the interpretation d); for if $\underset{\sim}{A}_o$ and $\underset{\sim}{B}_o$ are two uncertain propositions, and if

$$\vdash \underset{\sim}{A}_o \supset \underset{\sim}{B}_o$$

then $\underset{\sim}{B}_o$ <u>really</u> <u>is</u> a consequence of $\underset{\sim}{A}_o$ - that is, everyone would agree that $\underset{\sim}{B}_o$ is a consequence of $\underset{\sim}{A}_o$.

Of course it is possible to set up a satisfactory system based on the interpretation b); the first objection to such a system is that it is unbearably cumbersome to use. For consider a step in a proof in system (C) - a proposition $\underset{\sim}{A}_o$, say; the corresponding step in the system considered will consist of an implication sign, on the right of which will stand $\underset{\sim}{A}_o$, and on the left of which will stand the conjunction of all the uneliminated hypotheses which have been used in the derivation of $\underset{\sim}{A}_o$. A glance at some of the proofs in sections 3, 4, and 5, will show why such a system simply is not practical. A second objection against systems of the kind considered is that they depart from normal mathematical usage; for '$x > 3$' on the page of a mathematical work would never bear the interpretation b). A third objection is that it does not allow free variables to be used as names; this will be discussed shortly.

We now discuss interpretation c); be it noted that we can force system (C) to bear this interpretation simply by asserting that formulae containing free variables are never propositions, and that free variables do not represent elements of the appropriate type, but are just symbols which - if not restricted by hypothesis - may be replaced by the name of an element of the appropriate type, or may be generalised on. We are going to show that this interpretation is not suitable for the elucidation of the concepts and the methods of argument of modern mathematics.

Before doing this we make more precise the notion of a name. By a name we mean an unabbreviated closed formula of system (C) (possibly extended by the introduction of a number of virtual types). Two closed formulae are (provably) names of the same object if they are (provably) equal. A short name is an abbreviation for a closed formula (e.g. 'Tra ' is a short name). A nickname is a variable restricted by hypothesis; (if the reader considers this too light a word for a learned work he may use the term 'improper name' instead). Now certainly names (excluding nicknames) in our sense are names in the accepted sense - accepted, that is, by those who would not reject system (C); for example, the integers all have names - viz. the formulae $\lambda \underline{f}_{\iota\iota}\ \underline{x}_{\iota}.\underline{f}(\ldots(\underline{fx})\ldots)$ of type ι', or the formulae $S(\ldots(SO_{\nu})\ldots)$ of type ν. Many people however - for instance the authors of Principia Mathematica - would claim that our definition was too narrow. They would urge that the individuals, and some of the elements of type o_{ι} (corresponding to atomic propositions), for example, do have names. I agree that one may wish to introduce a type of individuals with names - to represent, say, a series of events. But such names will not be purely logical, and are therefore best represented by introducing a series of additional constants, A_{ι}, B_{ι}, D_{ι},...; personally I believe that only a finite number of such additional constants are necessary - that an infinity of names always involves a rule of generation from a finite number of symbols, as in the case of the names of the integers. However that may be, there certainly are occasions when one wants to deal with a type of individuals which do not have names - the points of space, or the elements of an abstract set, in the sense in which that term is used in abstract algebra or general topology (for examples see Boubarki (1)). And in classical mathematics too, there are elements - the non-definable real numbers, for instance - which do not have names. These elements constitute, as it were, a sort of underworld; for while the 'respectable' elements have proper names, the inhabitants of the underworld can only be known by nicknames; and although, admittedly, one cannot identify someone by a mere nickname, one can at least make some sort of reference to them. But under interpretation c) neither free nor restricted variables are names at all, but only symbolic devices; and hence the elements of the underworld become unmentionable - except in the mass.

Let us consider some examples. First, supposing to (C) there be adjoined the axiom and rule (N) which govern the use of Clo, and also the axiom:

(U) $(E\underline{r}_{o\vee})(\sim \mathrm{Clo}\underline{r})$,

and consider the expression

(R) $\sim \mathrm{Clo}\underline{r}_{o\vee}$.

Under interpretation c) this is a matrix which becomes a false proposition if we substitute, say,

$$\lambda \underline{m}_{\vee}.\mathrm{T}$$

for $\underline{r}$; but there is no substitution which makes it a true proposition. On the other hand, because of (U), the generalisation of its negation,

$$(\underline{r}_{o\vee})(\mathrm{Clo}\underline{r})$$

is provably false; a curious state of affairs! But (R) is certainly an expression which might occur in a mathematical work - as 'let $\underline{r}$ be an undefinable binary decimal'. Before settling finally against interpretation c), however, let us consider some of the ways in which the situation might be met by the proponents of c).

1). They might reject the axiom (U); this position is not unreasonable, especially when it is remembered that it may be possible to base a model on Clo. We will call it a 'definitist' approach; it amounts to denying the existence of the underworld[1].

2). The proponents of c) might claim that all elements really had names, (those of the underworld being known, I suppose, to the prince of darkness); but that the names of elements not representable by closed formulae were secret, and beyond the ken of our limited reason. I believe this opinion would have been advanced - or at least defended - by Ramsey when he was writing (1). But it seems to me that one who holds this view is as much a fraud as the man mentioned by Wittgenstein, who promised to instal a telephone in every house in Cambridge, and who, when shown a house without one, said 'Ah well, you see, I've given them an <u>invisible</u> telephone'.

Consider now another example; let the type ι represent

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(1) More refined positions are also possible! The constructivist will only admit the existence of those highly respectable members of society - computable elements - whose names guarantee that a search in the library of the college of heralds will eventually yield further information about them; while the social world of the finitist is limited to members of the royal family - the integers, which, as Kroenecker remarked, are there of divine right - and their closest relations.

an abstract set - that is let C_ι be the only named individual. We define:

$$\mathrm{Cas}_{o(\iota\iota\iota)} \rightarrow \lambda \underline{p}_{\iota\iota\iota}.(\underline{x}_\iota,\underline{y}_\iota,\underline{z}_\iota)(\underline{x} \neq C \;\&\; \underline{y} \neq C \;\&\; \underline{z} \neq C \,.\supset.\; \underline{pxy} \neq C \;\&\; \underline{px}(\underline{pyz}) = \underline{p}(\underline{pxy})\underline{z} \;\&\; \underline{pxy} = \underline{pyx}).$$

Thus Cas$\underline{p}$ means that $\underline{p}$ is an associative commutative product defined on the set consisting of all the individuals except C. It is easy to prove

$$\vdash \mathrm{Cas}\underline{p}_{\iota\iota\iota} \supset \sim \mathrm{Inv}\underline{p}_{\iota\iota\iota},$$

and hence

$$\vdash \sim \mathrm{Cas}\underline{p}_{\iota\iota\iota}$$

for any closed formula $\underline{p}_{\iota\iota\iota}$. On the other hand, I cannot believe that anyone would assert the proposition

$$(\underline{p}_{\iota\iota\iota})(\sim \mathrm{Cas}\underline{p});$$

in this case the underworld is, as it were, too respectable to be denied, and so the defence 1) of interpretation c), is no longer possible. Defence 2) this time is more reasonable, for one can produce examples of named commutative and associative products (on the integers or the real numbers). But I think it misrepresents the case: for, according to it, when the hypothesis

$$\mathrm{Cas}\underline{p}_{\iota\iota\iota}$$

is made; when, that is, the algebraist says 'let $\underline{p}$ be a commutative and associative product be given on an abstract set', what is really meant is 'consider, say, the ordinary multiplication of integers'. But I think that the algebraist means what he says; he is not concerned with how the product is given, nor with the nature of the elements of the set, nor with its cardinal number: and it is the task of the logician to express this meaning in logical terms, not to tell him he means something quite different.

We may sum up in this way: we imagined elements which were either too random or too abstract to be representable by closed formulae. The proponents of interpretation c) denied our right to imagine elements of the first kind, and assured us that when we mentioned an element of the second kind we were really only mentioning some particular - though unspecified - concrete instance. Proponents of interpretation b) would say that elements of the kinds considered can only be referred to *en masse*, so that when we think we are mentioning a single such element we are merely writing or uttering the symbol of a bound variable. We said at the start that it all depends on what is meant by a proposition; those who wish to assert that many of the proposition-like expressions of mathematics and physics are in fact matrices, and that symbols which appear to refer to mathematical and physical quantities are in fact only bound variables, are free to do so. I hope that by the end of this dissertation I shall have said enough to show that such people should be thought mildly eccentric.

There is a way of referring to members of the underworld what is really meant is 'consider, say, the ordinary multiplication of integers'. But I think that the algebraist means what he says; he is not concerned with how the product is given, nor with the nature of the elements of the set, nor with its cardinal number: and it is the task of the logician to express this meaning in logical terms, not to tell him he means something quite different.

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There is a way of referring to members of the underworld which has not yet been mentioned, and that is by introducing a constant selection operator in every type, which, as it were, hauls out a hostage; but this method involves the very strong assumption that axiom (S) holds in every type, and also destroys the symmetry of an abstract set (see theorem VI).

The interpretation d) is capable of a semantic formulation as follows: a class of models (e.g. the class of standard models) is chosen. A formula $\underset{\sim}{A}_o$ which is interpreted as truth in some of the models and falsehood in others is an uncertain proposition; either it or its negation may be taken as hypothesis. The range of a free variable $\underline{x}_\alpha$ of $\underset{\sim}{A}_o$, when that variable is restricted by the hypothesis $\underset{\sim}{A}_o$, is the set of all the interpretations of $\underline{x}_\alpha$ in all those models of the given class for which the interpretation of $\underset{\sim}{A}_o$ is truth. A proposition $\underset{\sim}{B}_o$ is true on the assumption $\underset{\sim}{A}_o$ if the interpretation of $\underset{\sim}{B}_o$ is truth for every model of the given class for which the interpretation of $\underset{\sim}{A}_o$ is truth. As before, we regard this formulation as providing a reasonably precise but intuitive meaning to the various terms, not as providing formal definitions.

Finally we recall once more the english rendering of A considered as a hypothesis: 'let us imagine that $\underset{\sim}{A}_o$ is true for some elements, and let such an element be denoted by (the nickname) $\underline{x}_\alpha$'. And herein lies the philosophic importance of the deduction theorem and the interpretation d): they show the logical status of those acts of imagination which are so essential a part of mathematics.

Section 2. Mathematical Structure.

The elements of type ι other than C_ι form an abstract or <u>structureless</u> set; no particular element can be singled out, the only binary relation that is singled out is the identical relation $Q_{o\iota\iota}$, and so on. For definiteness we will assume the axiom (I), so that the set considered is not a finite one. If now we are given an element X_α in some type α, we say that X_α determines a <u>structure</u> on the set; X_α may single out some particular individual, or a set of individuals; or it may be a successor-like function $S_{\iota\iota}$, so that every individual may be expressed in terms of it, through an expression

$$\mathrm{Nap}\underline{n}_\nu S_{\iota\iota}\ (\imath \underline{x}_\iota)(\underline{y}_\iota)(S_{\iota\iota}\underline{y} \neq \underline{x});$$

or X_α may be an invariant element, so that it does not in fact determine any structure at all. For the sake of uniformity we say in this last case that X determines the <u>logical</u> or the <u>symmetric</u> structure on the set.

When shall we say that two elements X_α and Y_β determine the same structure? We give two answers to this question; the first is provided by the formula:

$$\mathrm{Sam}^{\alpha\beta}_{o\alpha\beta} \rightarrow \lambda \underline{y}_\beta \underline{x}_\alpha.(\underline{p}_{\iota\iota})(\mathrm{Per}\underline{p}\ .\supset.$$
$$\mathrm{Tra}\underline{px} = \underline{x} \equiv \mathrm{Tra}\underline{py} = \underline{y}).$$

X_α and Y_β define the same structure in this sense (the extensional sense, as we shall call it) if the subgroups of the permutation group on the abstract set which leave X_α and Y_β invariant are the same subgroup. $Sam^{\alpha\beta} Y_\beta$ defines the set of all elements in type α which define the same structure as Y_β. With this definition we can actually define the structure determined by X_α as the subgroup of permutations which leave it invariant:

$$Exu_{o(\iota\iota)\alpha} \longrightarrow \lambda \underline{x}_\alpha \underline{p}_{\iota\iota} . Per\underline{p} \;\&\; Tra\underline{p}\underline{x} = \underline{x} .$$

('Exu' stands for 'EXtensional strUcture').

2,1) $\vdash Sam\underline{x}_\alpha \underline{y}_\beta \equiv Exu\underline{x}_\alpha = Exu\underline{y}_\beta$.

Next we define:

$$Wea^{\alpha\beta}_{o\alpha\beta} \rightarrow \lambda \underline{y}_\beta \underline{x}_\alpha.(\underline{p}_{\iota\iota})(Per\underline{p} \;\&\; Tra\underline{p}\underline{y} = \underline{y} \;.\supset.\; Tra\underline{p}\underline{x} = \underline{x}).$$

$$Mon^{\alpha}_{o\alpha} \rightarrow \lambda \underline{x}_\alpha.(\underline{p}_{\iota\iota})(Tra\underline{p}\underline{x} = \underline{x} \supset \underline{p} = I^{\iota}).$$

The structure determined by X_α is weaker than the structure determined by Y_β if the subgroup of X_α(i.e. $ExuX_\alpha$) includes the subgroup of Y_β. The weakest possible structure is the logical one. $MonX_\alpha$ means that X_α determines the strongest possible structure; that is, no permutation other than the identical one leaves X_α invariant; we say then that X_α is monomorphic.

2.2) $\vdash$ $Sam\underline{y}_\beta \, \underline{x}_\alpha \equiv (E\underline{f}_{\alpha\beta})(Inv\underline{f} \;\&\; \underline{x} = \underline{f}\underline{y})$

H.1 $\quad Sam\underline{y}_\beta \, \underline{x}_\alpha$ $\qquad\qquad$ $(\underline{x}, \underline{y})$

H.2 $\underline{f}_{\alpha\beta} = \lambda \underline{v}_\beta . (\imath \underline{u}_\alpha)(E\underline{p}_{\iota\iota})(\mathrm{Per}\underline{p}\ \&\ \mathrm{Tra}\underline{pv} = \underline{y}$

$\&\ \mathrm{Tra}\underline{pu} = \underline{x})$ $(\underline{f})$

3 $\mathrm{Inv}\underline{f}\ \&\ \underline{x} = \underline{fy}$

4 H.1 $\supset$ L.H.S. of 2.2) (H.1, H.2).

5 2.2)

The omitted steps in the above proof are straightforward; we note that the map $\underline{f}$ defined by H.3 is a one-to-one map of the set of all the conjugates of $\underline{y}$ onto the set of all the conjugates of $\underline{x}$ (see section 2 of Chapter I). If $\underline{x}$ and $\underline{y}$ are invariant, they are themselves their only conjugates.

Given an element X_β, there will be a range of elements which can be defined explicitly in terms of it, by expressions $\underset{\sim}{F}_{\alpha\beta} X_\beta$, where $\underset{\sim}{F}_{\alpha\beta}$ is a closed formula; by theorem II all these elements will determine the same extensional structure as X_β. But the converse is not true, and this leads us to make an intensional definition of 'determining the same structure' as follows:

$\mathrm{Mud}^{\alpha\beta}_{o\alpha\beta} \rightarrow \lambda \underline{y}_\beta \underline{x}_\alpha . (E\underline{f}_{\alpha\beta}, \underline{g}_{\alpha\beta})(\mathrm{Clo}\underline{f}\ \&\ \mathrm{Clo}\underline{g}\ \&\ \underline{x} = \underline{fy}\ \&\ \underline{y} = \underline{gx})$.

Mud$\underline{yx}$ ('$\underline{x}$ and $\underline{y}$ are mutually interdefinable') means that $\underline{x}$ may be represented by a formula whose only free variable is $\underline{y}$, and vice-versa. It is perhaps worthwhile to give an example showing the difference between Sam and Mud. Let there be given a set of individuals H_n having the cardinal n, and an individual x_n, for each positive integer n; and let the sets H_n all be distinct, and let no x_n belong to an H_m, and let every individual be an x_n or belong to an H_n.

We define functions (of type $\iota\iota$) f and g as follows:

a) if y belongs to H_n, then $fy = x_n$, and $gy = x_{n'}$, where $n \rightarrow n'$ is a permutation of the positive integers;

b) $fx_n = x_n = gx_n$.

Then it is fairly obvious that f and g determine the same structure in the extensional sense, but they will only be expressible explicitly in terms of each other if the permutation $n \rightarrow n'$ is explicitly given; if we assume that there exist random permutations, not representable by closed formulae, then there will exist functions f and g which are not mutually interdefinable.

Of more fundamental importance than the concept of the sameness of two structures, is the concept of <u>isomorphism</u>. Let ι and κ be two basic types, and if α is a type symbol not involving κ, let $\bar{\alpha}$ be the type symbol obtained from α by substituting κ for ι throughout α. Then an element of type α, and one of type $\bar{\alpha}$, are said to be isomorphic if they satisfy $\text{Iso}\underline{y}_{\bar{\alpha}}\,\underline{x}_{\alpha}$, where:

$$\text{Iso}_{o\alpha\bar{\alpha}} \rightarrow \lambda \underline{y}_{\bar{\alpha}}.\underline{x}_{\alpha}.(\text{E}\underline{f}_{\iota\kappa})(\text{Ont}\underline{f} \,\&\, \text{Tra}\underline{f}\underline{y} = \underline{x}).$$

If in this definition we substitute κ for ι, that is if we consider two elements determining structures on the same abstract set, then we simply get the definition of 'Cot'. The above definition is that usually given for the isomorphism of two <u>structures</u> (see, for example, Bourbaki (2), and also the discussion below). But it is obviously better to regard it as only defining isomorphism between two <u>elements</u>. For example, let the function f be defined as above, and let g be defined by:

a) if y belongs to H_n, then $gy = x_n$;

b) $gx_n = x_{n'}$, where $n \to n'$ is an explicitly given permutation of the positive integers.

Then f and g determine the same structure in both the extensional and the intensional sense, but they are not isomorphic. We therefore introduce new definitions for the isomorphism of structures.

Let α, β, be type symbols in which, respectively, κ, ι, do not occur.

$$\mathrm{Sis}_{o\alpha\beta} \rightarrow \lambda \underline{y}_{\beta}\, \underline{x}_{\alpha}.(\mathrm{E}\underline{f}_{\iota\kappa})(\mathrm{Ont}\underline{f} \;\&\; \mathrm{Mud}(\mathrm{Tra}\underline{f}\underline{y})\underline{x});$$

$$\mathrm{Eso}_{o\alpha\beta} \rightarrow \lambda \underline{y}_{\beta}\, \underline{x}_{\alpha}.(\mathrm{E}\underline{f}_{\iota\kappa})(\mathrm{Ont}\underline{f} \;\&\; \mathrm{Sam}(\mathrm{Tra}\underline{f}\underline{y})\underline{x}).$$

('Sis' stands for 'Structural ISomorphism', and 'Eso' for 'Extensional definition of structural iSOmorphism'). I believe that in normal mathematical usage refers to Sis rather than to Eso; for example, one might talk of the isomorphism of two topological structures, one of which was defined on a set in terms of neighbourhoods, the other being defined on a different set in terms of closure.

I do not know if a definition similar to our 'Sam' has been given before or not. I think the original definition (due to Russell ?) corresponded to our 'Iso', only n-ary relations being considered as arguments. The most modern definition is that given by Bourbaki (in (2)); starting from a number of basic abstract sets, he defines the <u>ladder</u> of sets based on them as all those sets which are obtained from them by successive applications of the operations of forming the set of all subsets of a given set, and of forming the direct product of any two given sets. A structure is determined by any element of any of the sets of the ladder. Two elements determine the same structure, if there is an explicitly given one-to-one map of a part of the set to which one of the elements belongs onto a part of the set to which the other belongs, the said map carrying one element into the other. If, as I think is intended, we interpret 'explicitly given' as 'representable by a closed formula', then this definition is almost the same as our 'Mud', being possibly a little stronger. Of course both 'Sam' and 'Mud' may be generalised to the case where there is more than one basic type, and to any element of a set of the ladder there corresponds an element of some type and vice versa. Bourbaki's definition of isomorphic structures corresponds exactly to our Iso and therefore two structures which are the same in his sense, are not necessarily isomorphic in his sense. I consider that all the definitions we have given have their uses, and that there is no point in trying to decide which are the 'correct' ones.

We have only discussed the structure defined by a single element, because any finite number of different elements can all be rolled into a single one in a higher type; for instance by forming:

$$\lambda \underline{y}_{\beta} \ldots \underline{x}_{\alpha} . \underline{x} = X_{\alpha} \& \ldots \& \ \underline{y} = Y_{\beta} \ .$$

Thus far we have assumed that the element X_{α} which determined a structure was simply 'given'. We now consider <u>how</u> it may have been given. Were it explicitly given, that is representable by a closed formula, the structure it determined would be merely the logical structure. So we suppose that it was required to satisfy an axiom $\underset{\sim}{A}_{o\alpha} X_{\alpha}$, where $\underset{\sim}{A}_{o\alpha}$ is a closed formula. (For at the beginning of any discussion there must be a <u>definite</u> statement of the subject of discussion, so that if $\underset{\sim}{A}_{o\alpha}$ were not a closed formula, then there would have to be a formula $\underset{\sim}{B}_{o(o\alpha)}$, which indicated the range of $\underset{\sim}{A}_{o\alpha}$; if $\underset{\sim}{B}_{o(o\alpha)}$ were not closed, there would be a formula which limited <u>its</u> range, and so on; the final formula in this series <u>would</u> be closed - though possibly by the trivial form $\lambda \underline{d}_{\delta} . \underline{d} = \underline{d}$ - and we should treat this final formula as the axiom, and the other formula of the series as elements which were given by it.)

The consistency of the axiom might have been established by the method of virtual types; but this is not essential. There is no reason why one should not discuss the consequences of an axiom which is not known to be consistent: all one requires is the assurance that it is not known to be inconsistent. The discussion of an axiomatic system can thus be regarded as an application of the deduction theorem, there being two initial hypotheses:

(H.C.) $(E\underline{x}_{\alpha})(\underset{\sim}{A}_{o\alpha}\underline{x})$;

(H.P) $\underset{\sim}{A}_{o\alpha}X_{\alpha}$ (X).

The first of these - the hypothesis of consistency - is usually made tacitly rather than explicitly. The second - which we shall call the principal hypothesis - is often put in the form of a definition; e.g. 'We define a base of neighbourhoods to be a set of subsets of the given abstract set which satisfy the following conditions ...'. The 'nicknames' which are introduced by (H.P), and which we have denoted by X_{α}, are thus often made to sound rather imposing; but the fact remains that they are just names for variables which are restricted by (H.P) - although when the axiomatic system is applied to a concrete instance (the real numbers, say), there may be elements with proper names which satisfy the axioms.

The reason for giving the restricted variables of (H.P) distinctive names is that they are regarded as the significant quantities of the particular axiomatic system under discussion, and (H.P) is not eliminated until the discussion is over (and then the elimination is usually tacit). There may well be other existentially quantified variables of $\underset{\sim}{A}_{o\alpha}X_{\alpha}$, which could also be restricted by (H.P), but which are not because they are not considered sufficiently important.

For example, one axiom governing a base of neighbourhoods $B_{o(o\iota)}$ is:

$$(\underline{u}_{o\iota}, \underline{v}_{o\iota})(E\underline{w}_{o\iota})(B\underline{u} \mathbin{\&} B\underline{v} \;.\supset.\; B\underline{w} \mathbin{\&} \underline{w} \subset (\underline{u} \cap \underline{v}));$$

where we have used the ordinary set theoretical notation. Using the selection axiom the above can be proved equivalent to:

$$(E\underline{f}_{o\iota(o\iota)(o\iota)})(\underline{u}_{o\iota}, \underline{v}_{o\iota})(B\underline{u} \mathbin{\&} B\underline{v} \;.\supset.\; B(\underline{fuv}) \mathbin{\&} \underline{fuv} \quad (\underline{u} \quad \underline{v})),$$

and so $\underline{f}$ might also appear as a restricted variable of the principal hypothesis, but in fact it would not be thought significant enough for this to happen[1].

From an axiom $\underset{\sim}{A}_{o\alpha}X_{\alpha}$ one can deduce a certain amount about the elements which satisfy it. Since $\underset{\sim}{A}_{o\alpha}$ is a closed formula, all the conjugates of an element which satisfies it must also satisfy it. If any two elements which satisfy the axiom are conjugates, the axiom is said to be <u>categorical</u>. One might say that an axiom was <u>isomorphogenetic</u> if all elements satisfying it determined isomorphic structures (taking either the extensional or the intensional definition). Further definitions concerning axiomatic systems will be found in Tarski (3); a perusal of that paper will make it clear how much better the notation of system (C) is suited to the discussion of these problems than that used by Tarski.

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(1) For argument's sake I have assumed in this discussion that the intersection of two neighbourhoods of the base does not necessarily itself belong to the base.

Finally we note that if it is possible to base a model on Clo, then it is also possible to base a model on all those elements representable by formulae whose only free variables are X_{α} and variables of type ι. It would follow then that one could actually produce an enumerable model for every axiomatic system; of course the existence of such models is guaranteed by the Lowenheim-Skolem theorem.

Section 3. Theories.

The purpose of a theory is to bring some sort of order into a mass of given facts, and to make predictions concerning future facts. It must be admitted that part of the difficulty of this process is deciding what is and what is not a fact; but we are going to suppose that this matter has been dealt with, and that the facts with which our theory has to deal have been collected and presented in some kind of standard form, and that there is a presumption that future facts will also be able to be put in this standard form. This presumption is essential to the theoretician because one of his chief activities is the consideration of imaginary facts; if for example one were to sprout an entirely new kind of sense organ every day one would hardly be able to theorise about one's experiences. Our assumption is not so restrictive as might at first be thought; for a sufficiently long and complete motion picture could provide all the factual material required for a wide range of theoretical subjects, and a standard description of such a film, frame by frame could easily be arranged. The facts in this case could, for instance, be described by a function of type $o\nu\nu$; the first argument referring to the frame number, the second to the number of a small cell within that frame, and the value being T or F according as the cell was white or black. We shall suppose then that any conceivable set of facts of the sort with which the theory is to deal is describable by a single element in some type δ. We wish to allow a conceivable set of facts to contain an infinite amount of information, while an actually observed set will, of course, contain only a finite amount of information. We introduce symbols $P_{o\delta}$, $Q_{o\delta}$,..., to refer to observed sets of facts; since they represent only finite amounts of information, they will presumably be representable by closed formulae. For example a $P_{o\delta}$ might be:

$$\lambda \underline{d}_{\rho\nu}.\underline{d}34 \;\&\; \underline{d}35 \;\&\; \underline{d}41.$$

By a <u>theory</u> which deals with facts described by a function of type δ, we mean simply a closed formula of type $o\delta$[1]; we suppose that any requisite virtual types (and in particular ν and ρ) have been included in the logical system. Thus we confine our consideration to theories which are capable of definite logical formulation - so that, for example, the Freudian theory of the censor would fall outside the scope of our remarks.

We give an example of a theory. The facts are described by an element $\underline{d}$ of type $\rho\nu$, which may be interpreted as observations on a number of occasions of a real valued quantity, the value on the nth occasion being $\underline{d}n$. The theory is given by:

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(1) I owe this definition to A.M. Turing.

$$\mathrm{The}_{o(\rho\nu)} \rightarrow \lambda \underline{d}_{\rho\nu}.(\mathrm{E}\underline{m}_{\nu}, \underline{q}_{\rho\nu\rho\nu}, \underline{w}_{\rho\nu})(\mathrm{Mot}\underline{mwq}\ \& \ \underline{d} = \lambda \underline{n}_{\nu}.\mathrm{Dis}\underline{m}(q1_{\nu}(\mathrm{Rea}\underline{n}))(\underline{q}2_{\nu}(\mathrm{Rea}\underline{n})))$$

Here $\mathrm{Mot}_{o(\rho\nu\rho\nu)(\rho\nu)\nu}$, $\mathrm{Dis}_{\rho(\rho\nu)(\rho\nu)\nu}$, $\mathrm{Rea}_{\rho\nu}$, are abbreviations for three closed formulae which we shall not give explicitly; we shall make their meanings clear directly. The first thing that a theoretician does when he has made a theory is to assume that the facts satisfy it; i.e. he makes the hypothesis:

(H.A) $\qquad \mathrm{The}\underline{d}_{\delta} \qquad (\underline{d})$.

This step may be compared with the hypothesis of consistency (H.C), made when studying an axiomatic system; and, as there, the next step is to turn into restricted variables those existentially quantified variables which are considered of importance. We again call this step the <u>principal</u> <u>hypothesis</u>:

(H.P) $\quad \mathrm{Mot}\underline{m}_{\nu}\underline{w}_{\rho\nu}\,\underline{q}_{\rho\nu\rho\nu}$

$$\& \ \underline{d} = \lambda \underline{n}_{\nu}.\mathrm{Dis}\underline{m}_{\nu}(\underline{q}_{\rho\nu\rho\nu}1_{\nu}(\mathrm{Rea}\underline{n}))(\underline{q}_{\rho\nu\rho\nu}2_{\nu}(\mathrm{Rea}\underline{n})) \qquad (\underline{m}, \underline{w}, \underline{q}).$$

As in the case of axiomatic systems this is the statement that would stand at the beginning of a paper or text book; we give a translation of it, thus providing the interpretation of the symbols which appear in it.

'Let there be two particles; let the mass of the ith particle be $\underline{w}_{\rho\nu}\,\underline{i}_{\nu}$; let the kth coordinate of the ith particle at the time t be given by $\underline{q}_{\rho\nu\rho\nu}\,\underline{i}_{\nu}\,\underline{t}_{\rho}\,\underline{k}_{\nu}$. Let the motion of the particles take place in a space of m dimensions, according to the law of motion described by

$$\mathrm{Mot}\underline{m}_{\nu}\underline{w}_{\rho\nu}\underline{q}_{\rho\nu\rho\nu}.$$

Let the distance between two points of the space, whose kth coordinates are respectively $\underline{r}_{\rho\nu}\underline{k}_{\nu}$ and $\underline{s}_{\rho\nu}\underline{k}_{\nu}$ be given by

$$\mathrm{Dis}\underline{m}_{\nu}\underline{r}_{\rho\nu}\underline{s}_{\rho\nu}.$$

Then we suppose that all the above mentioned quantities are such that the real number $\underline{d}_{\rho\nu}\underline{n}_{\nu}$ which is observed on the nth occasion is equal to the distance between the two particles at the time t = n.' ($\mathrm{Rea}_{\rho\nu}\underline{n}_{\nu}$ means just the integer n considered as a real number.)

As for axiomatic systems, the question of just which variables are to be restricted by the principal hypothesis, and thus brought into prominence, is a question which cannot be answered dogmatically. A rough answer is that all dependent variables and constants which are of physical significance should be so restricted.

In order to be able to discuss the general case, we represent the principal hypothesis of an arbitrary theory by:

(H.P) $\qquad \underset{\sim}{H}_{o\alpha\ldots\beta\delta}\,\underline{d}\;\underline{b}_{\beta}\ldots\underline{a}_{\alpha} \qquad (\underline{a},\ldots,\underline{b}).$

We call the variables that are restricted by (H.P) hypotheticals; neither (H.P) nor the hypotheticals are uniquely determined by the theory. '$\underset{\sim}{H}_{o\alpha\ldots\beta\delta}$' stands for a closed formula, and we shall use the same symbol to denote the appropriate formula for the example and for the general case.

We now investigate the nature of the hypotheticals by a study of transformations which leave H invariant. This analysis is more complex than any we have hitherto attempted, because we have to distinguish not only between different mathematical types, but also between the different occurrences of the same mathematical type in the types of the hypotheticals. To this end we define the subtypes of a given type as follows:

a) the value part and the argument parts of α are all distinct subtypes of α ;

b) a type β is a subtype of α, if the value part of β is the same as the value part of α, and all the argument parts of β are argument parts of α; two such subtypes are distinct unless their corresponding argument parts are in each case the same argument part of α ;

c) a subtype of an argument part of α is a subtype of α; two such subtypes are distinct unless they are the same subtype of the same argument part of α .

By 'the subtypes of the hypotheticals' or just 'the subtypes' - we mean all the subtypes of all the hypotheticals. I think it obvious how the transformation induced in type α by a given transformation[1] in a subtype of α, is to be defined.

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(1) The word transformation is used rather than permutation, because if the particular subtype is not an argument part, nor a subtype of an argument part, the transformation of α can be defined for <u>any</u> map of the subtype into itself.

Suppose now we make some transformations of the various subtypes which induce the transformations

$$\underline{a} \to \overline{\underline{a}}, \ldots, \underline{b} \to \overline{\underline{b}},$$

and suppose that

$$(\underline{d}_{\delta})(\underset{\sim}{H}_{0\alpha\ldots\beta\delta}\, \underline{d\overline{b}} \ldots \overline{\underline{a}} \equiv \underset{\sim}{H}_{0\alpha\ldots\beta\delta}\, \underline{db} \ldots \underline{a}),$$

then we shall say that the given set of transformations forms a <u>permissible</u> set. Now I claim that a complete knowledge of the physical significance of the hypotheticals may be obtained from a consideration of all the sets of permissible transformations.

Let me illustrate this thesis by reference to the example. When I say that such and such a transformation is permissible, I mean that its permissibility could be proved using the full formula for The. The following transformations are permissible.

1) Any transformation

$$\underline{w}_{\rho\nu} \to \overline{\underline{w}}_{\rho\nu}$$

where $\overline{\underline{w}}$ takes the same values as $\underline{w}$ for the arguments 1_ν and 2_ν. This shows that there are just two objects having significance in the argument subtype of $\underline{w}$.

2) Similar transformation in the (14) subtype of $\underline{q}$. (We number parts of a type from left to right in the type symbol, so that 1 always refers to the value part.)

3) The permutation

$$1_\nu \leftrightarrow 2_\nu$$

applied simultaneously to the argument part of $\underline{w}$ and to the

(4) subtype (i.e. the last argument part) of q. This shows that these two subtypes refer to the same <u>physical</u> type; and as there are no further permissible transformations which yield information about this type we can say that it contains just two interchangeable objects. These two objects can be conveniently pictured as particles, which have properties specified by w and q.

4) Transformations corresponding to translations and rotations in an m-dimensional Euclidean space, acting in the (12) subtype of q (assuming that Dis and Mot are suitably defined). This shows that the second argument part of q is not like a particle type, and that $\underline{q}_{\rho\nu\rho\nu}\ \underline{i}_{\nu}\underline{t}_{\rho}$ may be interpreted as a coordinate in an m-dimensional space.

5) Transformations corresponding to the Gallilean transformations of space-time acting in the (123) subtype of q. (Again a suitable definition of Mot is assumed.) In conjunction with 4) this shows that the subtype (3) of q may be interpreted as a time coordinate.

Other transformations may well be possible, according to the exact definition of Mot; but I hope the above brief analysis will serve to show the way in which my thesis could be substantiated. We may say that the principal hypothesis confers a structure on the subtypes of the hypotheticals in rather the same sort of way that the axioms confer a structure on the abstract set of an axiomatic system. It may be noted that we have required invariance for all possible facts; and so the symmetries revealed in the permissible transformations are theoretical symmetries. If instead we considered only one given set of facts then the corresponding transformations would also reveal the factual symmetries.

To show that our way of looking at physical theories is a really useful way we shall discuss briefly its application to one or two problems.

1) Measurement.

The observations on which a measurement is based we represent by a $P_{\sigma\delta}$; what is being measured is usually a hypothetical of a theory, or more likely, a function of the hypotheticals (for example, the ratio of two masses). The theoretical proposition which expresses the fact that the particular observations made mean that the function has a particular value z say, is thus:

$$(\underline{a}_{\alpha}, \ldots, \underline{b}_{\beta}, \underline{d}_{\delta})(P_{\sigma\delta}\,\underline{d}\ \&\ \underset{\sim}{H}\underline{db}\ldots\underline{a}\ .\supset .\ \underset{\sim}{G}\underline{b}\ldots\underline{a} = z_{\gamma}),$$

where $\underset{\sim}{G}_{\gamma\alpha\ldots\beta}$ is the aforesaid function. It is obvious that measurements of this sort - and most measurements <u>are</u> of this sort - depend essentially on the assumption that the facts do satisfy a particular theory.

2) Counter-to-fact conditionals.

There has been a good deal of discussion as to the logical status of such statements as 'If I <u>were</u> to put this lump of sugar into my tea, it <u>would</u> dissolve'. I consider that when such a statement is made, there is always a theory tacitly implied, and the statement is just a deduction from the theory, of the form:

$$(\underline{d}_{\delta})(P_{o\delta}\ \underline{d}\ \&\ \mathrm{The}\underline{d}\ .\supset .\ Q_{o\delta}\underline{d}).$$

If the listener accepts the theory, he will agree with the statement; if he recognises the theory, but does not accept it, he will regard the speaker as superstitious; and if he can recognise no theory behind the statement ('If I open my mouth wide enough the kettle will boil') he will think the speaker dotty.

3) Operationalism

It appears to me that what the operationalists (see especially Bridgeman (1)) think they are saying is either:

a) In a good theory the hypotheticals are uniquely determined by the facts

or b) A good theory should be able to be put in a form in which the hypotheticals are uniquely determined by the facts.

But the first of these is contradicted by the fact that all the great theories of physics employ quantities - like coordinates - which are not uniquely determined by the facts; and the second of these can be shown trivially to be true of any theory. For let us modify the general theory we have been considering so that its principal hypothesis becomes:

$$(\mathrm{H.P'})\quad \underline{f}_{o\alpha\ldots\beta} = \underset{\sim}{H}_{o\alpha\ldots\beta\delta}\,\underline{d}\ \&\ (E\underline{a}_{\alpha},\ldots\underline{b}_{\beta})(\underline{f}_{o\alpha\ldots\beta}\,\underline{b}\ldots\underline{a})\quad (\underline{f}).$$

Now the hypothetical $\underline{f}$ here is uniquely determined by the facts; but on the other hand, by considering the transformations of its arguments which leave f invariant one can recover all the structural detail, so that (H.P') as it stands seems an adequate principal hypothesis.

The most famous example of operational criticism is Einstein's dethroning of absolute simultaneity; but the point here is that the theory of an immobile ether had already been exploded by the Michelson-Morley experiment. Had that experiment given the expected positive result, absolute simultaneity would have been operationally definable. Thus one of the things the operationalists are actually saying is: 'Don't use the concepts (i.e. the structure of the hypotheticals) of a theory after that theory has proved unacceptable'; and of course they are right. Another thing they are actually saying is: 'Use a theory with as few hypotheticals as possible'. (See in particular Dingle (1)). And here they are certainly wrong, for if this were taken seriously it would lead to the accumulation of a mass of empirical laws, instead of to those powerful and beautiful theories which are the chief glory of theoretical physics.

4) Constructionalism

Ever since Mach people have tried to construct the fundamental concepts of space and time out of the manifold of possible sensations. (See Mach (1), Russell (1), Nicod (1), Carnap (1)). In terms of a theory, for which the facts are sensations, and of which the hypotheticals are the positions of bodies in space-time, the main principle of these constructions is the formation of the function:

$$\lambda \underline{d}_{\delta}.(E\underline{b}_{\beta}\ldots)(\underset{\sim}{H}\ \underline{db}\ldots\underline{a}'_{\alpha});$$

this set of possible facts (or possible sets of facts) represents the particular value $\underline{a}'_{\alpha}$ of the hypothetical $\underline{a}_{\alpha}$. But it is now clear that this representation only makes sense if the theory is believed to be true; and if one accepts the theory one might as well define the hypotheticals according to their place in the theory, rather than as a set of facts. To give the numbers of the pages on which a particular character in a novel appears does not make him more or less real.

I cannot pretend that the arguments I have given in these brief notes are in any way final, nor that I have been able to do more than skim the surface of some of the problems discussed; but I hope I have said enough to show that our analysis of theories is not only suitable for the discussion of the form and working of actual physical theories, but also helps one to see clearly into the more philosophical problems of physics.

I believe that the first person to give publicity to the fact that the concepts of physics were really hypotheticals introduced by a theory was Poincaré; he emphasised his point by calling theories conventions, and showed by examples that equivalent theories might introduce quite different concepts.

The first statement of a theory in a logical form similar to ours is given by Ramsey in (2), which paper was the starting point of our work. A recent account of a philosophy of physics which is similar to ours, though not logically formulated, is by Margenau in (1); his <u>constructs</u> correspond to objects in the subtypes of our hypotheticals.

There are many subjects and questions we have not discussed, such as: the relations that may exist between different theories; the requirements that are universally demanded by a physical theory; which of these requirements can be satisfied by an appropriate reformulation of any theory; whether the ideas of simplicity and elegance can be given a logical formulation; the formulation of the idea of a fundamental theory. Questions similar to some of these have been discussed in the past in connection with the ultimate physical reality, rather than in connection with theories. They are, in effect metaphysical questions. And I think a benefit of the analysis proposed in this section is that questions which have been dismissed by the positivists as meaningless, can be reformulated in logical terms, and discussed in a logical setting.

Appendix I. Equivalence of ($\underline{C}$) and ($\underline{C'}$).

We denote by (C') the system described in Church (1) as defined by rules I - VI, and axioms 1) - 10), 12). System (C) omits axiom 6), restricts axiom 9) to types o and ι, and adds the constants C_o and C_ι and axioms (D2). Church regards axiom 12) (our axiom (T)) as a 'strong' addition to the system, but Turing has shown that if the system is consistent without it, it is consistent with it (see footnote in Newman and Turing (1)).

First we show that elements C can be defined in (C') which satisfy (D2). We denote by $\iota'_{\alpha(o\alpha)}$ the descriptions operators of system (C'). Then we introduce:

$$C'_o \rightarrow (\imath' \underline{p}_o)(\underline{p} \neq \underline{p})$$

$$C'_\iota \rightarrow (\imath' \underline{x}_\iota)(\underline{x} \neq \underline{x})$$

$$\iota''_{\kappa(o\kappa)} \rightarrow \lambda \underline{f}_{o\kappa} \cdot (\imath' \underline{x}_\kappa)(J\underline{f} \supset \underline{x} = \iota' \underline{f} \ .\&.\ {\sim} J\underline{f} \supset \underline{x} = C'_\kappa)$$

where κ is o or ι, and $(\imath' \underline{x}_\kappa)$ is associated with $\iota'_{\kappa(o\kappa)}$. Then it is easy to prove the following (in (C'))

$$J\underline{f}_{o\kappa} \supset \underline{f}_{o\kappa}(\iota''_{\kappa(o\kappa)} \underline{f}_{o\kappa}),$$

and $\quad {\sim} J\underline{f}_{o\kappa} \supset \iota''_{\kappa(o\kappa)} \underline{f}_{o\kappa} = C'_\kappa \quad$ (where κ is o or ι);

i.e. we have shown that ι'' and C' satisfy (D). Further, it is easy to show (in (C)) that the constants $\iota_{\alpha(o\alpha)}$ and C_α, defined on page 8, satisfy the propositions (D) for each complex type α, and thus that the axioms 9) of (C') can be satisfied in (C).

We now show that axiom 6) of (C'), namely

$$(\underline{x}_\alpha)(\underline{p}_o \text{ v } \underline{f}_{o\alpha}\underline{x}) \supset \underline{p}_o \text{ v } (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x}),$$

can be proved from the other axioms of (C').

We introduce the abbreviations:

$T'_o \rightarrow (\underline{p}_o)(\underline{p} \equiv \underline{p})$

$F'_o \rightarrow \sim T'$.

Now we prove a number of lemmas:

(A)	T'	P.C., VI.
(B)	$T' = \underline{p}_o$ .v. $F' = \underline{p}_o$	
For	$\underline{q}_o \equiv \underline{p}_o$ .v. $\sim\underline{q}_o \equiv \underline{p}_o$	P.C.
	(B)	12) and IV.
(C)	$\underline{f}_{oo}T'$ & $\underline{f}_{oo}F'$ $. \supset \underline{f}_{oo}\underline{p}_o$	
For	$T' = \underline{p}_o \supset . \underline{f}_{oo}T' \supset \underline{f}_{oo}\underline{p}_o$	Definition of
	$F' = \underline{p}_o \supset . \underline{f}_{oo}F' \supset \underline{f}_{oo}\underline{p}_o$	'=' and 5^{oo}).
	(B) $\supset$ (C)	P.C.
	(C)	
(D)	$(\underline{x}_\alpha)(T' \text{ v } \underline{f}_{o\alpha}\underline{x}) \supset T' \text{ v } (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x})$	For both sides are provable, using (A) and VI.
(E)	$(\underline{x}_\alpha)(F' \text{ v } \underline{f}_{o\alpha}\underline{x}) \supset F' \text{ v } (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x})$	
For	$F' \text{ v } \underline{f}_{o\alpha}\underline{x} = \underline{f}_{o\alpha}\underline{x}$	P.C. and 12).
	$(\underline{x}_\alpha)((\lambda\underline{y}_\alpha . F' \text{ v } \underline{f}_{o\alpha}\underline{y})\underline{x} = \underline{f}_{o\alpha}\underline{x})$	III and VI.
	$\lambda\underline{y}_\alpha(F' \text{ v } \underline{f}_{o\alpha}\underline{y}) = \underline{f}_{o\alpha}$	$10^{o\alpha}$).
	$\prod_{o(o\alpha)}(\lambda\underline{y}_\alpha . F' \text{ v } \underline{f}_{o\alpha}\underline{y}) \supset \prod_{o(o\alpha)}\underline{f}_{o\alpha}$	Definition of '=', $5^{o(o\alpha)}$) and IV.
	$(\underline{x}_\alpha)(F' \text{ v } \underline{f}_{o\alpha}\underline{x}) \supset . F' \text{ v } (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x})$	P.C.

If we now substitute

$$\lambda \underline{p}_o((\underline{x}_\alpha)(\underline{p} \vee \underline{f}_{o\alpha}\underline{x}) \supset .\underline{p} \vee (\underline{x}_\alpha)(\underline{f}_{o\alpha}\underline{x}))$$

for $\underline{f}_{oo}$ in (C), use II, and detach (D) and (E) by V, we conclude with 6^α). This completes the proof of the equivalence of (C) and (C').

Appendix III. Elementary formal theorems.

1. $\underline{p}_o = T \equiv \underline{p}_o$.

2. $\underline{p} = F \equiv \sim \underline{p}_o$.

3. $V^{\alpha} Q_{o\alpha\alpha}$.

4. $\underline{f}_{\alpha\beta} = \underline{g}_{\alpha\beta} \;\&\; \underline{x}_{\beta} = \underline{y}_{\beta} \;.\supset.\; \underline{f}_{\alpha\beta}\underline{x}_{\beta} = \underline{g}_{\alpha\beta}\underline{y}_{\beta}$.

5. $\underline{f}_{\alpha\beta} = \underline{g}_{\alpha\beta} \equiv (\underline{x}_{\beta})(\underline{f}_{\alpha\beta}\underline{x} = \underline{g}_{\alpha\beta}\underline{x})$.

6. $\underline{p}_o \supset (\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x}) \;.\equiv.\; (\underline{x}_{\alpha})(\underline{p}_o \supset \underline{f}_{o\alpha}\underline{x})$.

7. $\underline{p}_o \supset (E\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x}) \;.\equiv.\; (E\underline{x}_{\alpha})(\underline{p}_o \supset \underline{f}_{o\alpha}\underline{x})$.

8. $(\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x}) \supset \underline{p}_o \;.\equiv.\; (E\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x} \supset \underline{p}_o)$.

9. $(E\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x}) \supset \underline{p}_o \;.\equiv.\; (\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x} \supset \underline{p}_o)$.

10. $(\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x} \supset \underline{g}_{o\alpha}\underline{x}) \;.\supset.\; (\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x}) \supset (\underline{x}_{\alpha})(\underline{g}_{o\alpha}\underline{x})$.

11. $(E\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x}) \supset (E\underline{x}_{\alpha})(\underline{g}_{o\alpha}\underline{x}) \;.\supset.\; (E\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x} \supset \underline{g}_{o\alpha}\underline{x})$.

12. $J^{\alpha}\underline{f}_{o\alpha} \supset \underline{f}_{o\alpha}(\iota^{\alpha}\underline{f}_{o\alpha})$.

13. $\sim J^{\alpha}\underline{f}_{o\alpha} \supset \iota^{\alpha}\underline{f}_{o\alpha} = C_{\alpha}$.

14. $\underline{f}_{o\alpha}C_{\alpha} \supset \iota^{\alpha}\underline{f}_{o\alpha} = C_{\alpha}$.

15. $\underline{p}_o \supset (\text{℩}\underline{x}_{\alpha})(\underline{p}_o \;\&\; \underline{x} = \underline{u}_{\alpha}) = \underline{u}_{\alpha}$.

16. $\sim\underline{p}_o \supset (\text{℩}\underline{x}_{\alpha})(\underline{p}_o \;\&\; \underline{x} = \underline{u}_{\alpha}) = C_{\alpha}$.

17. $\underline{p}_o \;\&\; \sim\underline{q}_o \;.\supset.\; (\text{℩}\underline{x}_{\alpha})(\underline{p}_o \supset \underline{x} = \underline{u}_{\alpha} \;.\&.\; \underline{q}_o \supset \underline{x} = \underline{v}_{\alpha}) = \underline{u}_{\alpha}$.

18. $J^{\alpha}\underline{f}_{o\alpha} \supset (\text{℩}\underline{x}_{\beta})(E\underline{y}_{\alpha})(\underline{f}_{o\alpha}\underline{y} \;\&\; \underline{x} = \underline{g}_{\beta\alpha}\underline{y}) = \underline{g}_{\beta\alpha}(\iota^{\alpha}\underline{f}_{o\alpha})$.

19. $\underline{g}_{o\alpha}(\iota^{\alpha}\underline{f}_{o\alpha}) \;.\equiv.\; (E\underline{x}_{\alpha})(\underline{f}_{o\alpha}\underline{x} \;\&\; (\underline{y}_{\alpha})(\underline{f}_{o\alpha}\underline{y} \supset \underline{y} = x)$

$\;\&\; \underline{g}_{o\alpha}\underline{x}) \vee \underline{g}_{o\alpha}C_{\alpha}$.

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Note: J.S.L. stands for Jour. Symbolic Logic. Those references marked * are known to me only by reviews.

Index of signs and symbols.

A. Constant elements of the system.

Page No.	Symbol	Interpretation
120	$\mathrm{Eso}_{o\alpha\beta}$	Extensionally isomorphic (of structures)
117	$\mathrm{Exu}_{o(\iota)\alpha}$	Extensional Structure
8	F_o	Falsehood
84	F_ι	
72	$\mathrm{Fin}_{o(o\alpha)}$	Of finite cardinal
91	$\mathrm{Fir}_{\sigma\sigma\nabla}$	
92	α-$\mathrm{Fus}_{\iota_2\sigma\eta}$	
92	$\mathrm{Fus}_{\iota_2\sigma\eta}$	
8	$I_{\alpha\alpha}$	Identity operator
5,8	$\iota_{\alpha(o\alpha)}$	Descriptions operator
84	ι', ι''	
88	$\bar{\iota},$	
34	$\mathrm{Inv}_{o\alpha}$	Invariant
119	$\mathrm{Iso}_{o\alpha\bar{\alpha}}$	Isomorphic (of elements)
8	$J_{o(o\alpha)}$	Uniqueness (of a set)
8	$K_{\alpha\beta\alpha}$	Constant operator
92	$\mathrm{Las}_{\eta\sigma}$	
85	$\mathrm{Map}^n_{\eta\alpha}$	Map of elements of type $\leqslant$ n into η
92	$\mathrm{Mix}_{\iota_2\sigma\eta(\iota_2\sigma)}$	
117	$\mathrm{Mon}_{o\alpha}$	Monomorphic
63	$\mathrm{Mod}_{o\alpha}$	Belonging to the model
118	$\mathrm{Mud}_{o\alpha\beta}$	Mutually interdefinable
5	N_{oo}	Negation

Page No	Symbol	Interpretation
89	α-Nar_{ν}	
9	$\mathrm{Num}_{o\alpha'}$	Being an integer
28	$\mathrm{Ont}_{o(\alpha\beta)}$	One-to-one map onto
41	$\mathrm{P}^{\beta}_{o\bar{\beta}}$	
5	$\prod_{o(o\alpha)}$	Universality
84	$\mathrm{Pai}^{n}_{\iota_n \iota_{n-1} \iota_{n-1}}$	
28	$\mathrm{Per}_{o(\alpha\alpha)}$	Permutation
93	$\mathrm{Pic}_{o(\iota_2\sigma)\bar{\nu}}$	
92	$\mathrm{Pro}_{o(\iota_2\sigma)}$	
5	$\mathrm{Q}_{o\alpha\alpha}$	Relation of identity
53	$\mathrm{Quo}_{\beta'\beta(o\alpha\alpha)}$	
41	$\mathrm{R}^{\beta}_{o\bar{\beta}\bar{\beta}}$	
29	$\mathrm{Rec}^{\alpha\beta}_{\beta\alpha(\alpha\beta)}$	
9	$\mathrm{S}_{\alpha'\alpha'}$	Successor operator
8	$\sum_{o(o\alpha)}$	Existence (of a set)
116	$\mathrm{Sam}_{o\alpha\beta}$	Extensional sameness of structure
92	$\mathrm{Sap}_{\iota_2\sigma(\iota_2\sigma)(\iota_2\sigma)}$	
94	$\mathrm{Sei}_{\iota_2\sigma\alpha}$	
91	$\mathrm{Seq}_{o\sigma}$	
120	$\mathrm{Sis}_{o\alpha\beta}$	Intensionally isomorphic (of structure)
88	α-$\mathrm{Sub}_{o\eta}$	
8	T_{o}	Truth
84	T_{ι}	

Page No.	Symbol	Interpretation
88	$\mathrm{Tot}_{o\eta}$	
31	$\mathrm{Tra}_{\bar{\gamma}\gamma(\alpha\beta)}$	Transportation of a map to a higher type
88	$\mathrm{Typ}_{o(o\eta)}$	
28	$\mathrm{Uni}_{o(\alpha\beta)}$	One-to-one map
8	$\mathrm{V}_{o(o\alpha\alpha)}$	Property of being an equivalence relation
8	$\mathrm{W}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$	
117	$\mathrm{Wea}_{o\alpha\beta}$	Weaker (of structures)
84	X_{ι}	Nonsense element
90	Y_{η}	Nonsense element

B. Special type symbols

Page No.	Symbol
5	o
5	ι
50a	ν
60	ρ
84	ι_η
90	η
90	σ
60	μ
90	$\bar{\nu}$
31,41	$\bar{\gamma}$
52,8	β^1
52	β_1

C. Connectives and miscellaneous

Page No.	Symbol
8	$\sim$
8	&
8	v
8	$\supset$
8	$\equiv$
8	=
41	$\cong$
54	$\dot{\cong}$
28	: