Schedule S1, (Computer Science, CS and Philosophy, Maths and CS)
Schedule B, MSc in Computer Science

Hilary Term 2019

Computational Complexity

Exercise class 3: NP reductions

1. A *vertex cover* of a graph G is a set $X \subseteq V(G)$ such that for all edges $\{u, v\} \in E(G)$ at least one of u, v is in X. The problem Vertex Cover is the problem to decide for a given graph G and number $k \geq 1$ if G contains a vertex cover of order $\leq k$.

   A *dominating set* of a graph G is a set $X \subseteq V(G)$ such that for all $u \in V(G)$ either $u \in X$ or there is a $v \in X$ and $\{u, v\} \in E(G)$. The problem Dominating Set is the problem to determine for a given graph G and $k \in \mathbb{N}$ if G has a dominating set of order at most k.

   Assuming that Vertex Cover is NP-complete, show that Dominating Set is also NP-complete.

2. (a) Show that if any co-NP-complete problem is in NP, then co-NP = NP.
   (b) Show that if a language L is NP-complete, then its complement $\overline{L}$ is co-NP-complete.

3. The class DP is defined as follows. We say that a language L belongs to DP if there exist languages $L_1 \in$ NP and $L_2 \in$ co-NP such that $L = L_1 \cap L_2$.

   Let G be an undirected graph and let k be an integer. G contains a clique of order k if there exists some subset $S \subseteq V(G)$ with $|S| = k$ such that there exists an edge $\{x, y\}$ for every pair of distinct vertices $x, y \in S$. The language Clique is then defined as follows:

   $$\textsc{Clique} = \{\langle G, k\rangle \mid \text{ G an undirected graph containing a clique of order } \geq k\}$$

   We define the following languages:

   $$\begin{aligned}
   \textsc{UniqueSat} &= \{\varphi \mid \varphi \text{ is a propositional formula with exactly one satisfying assignment}\}\\
   \textsc{Sat} &= \{\varphi \mid \varphi \text{ is a satisfiable propositional formula}\}\\
   \textsc{Sat-Unsat} &= \{\langle \varphi, \psi\rangle \mid \varphi \text{ and } \psi \text{ propositional formulas}, \varphi \text{ satisfiable}, \psi \text{ unsatisfiable}\}\\
   \textsc{ExactClique} &= \{\langle G, k\rangle \mid \text{ The largest clique in } G \text{ has order exactly } k\}
   \end{aligned}$$

   Answer the following questions:

   (a) Show that NP $\cup$ co-NP $\subseteq$ DP.
   (b) Show that UniqueSAT and ExactClique are in DP.
   (c) Show that Sat-Unsat is DP-complete.
   (d) Show that DP is contained in ExpTime.
   (e) Assume that we could find a polynomial time computable reduction from Sat-Unsat to $\overline{\textsc{Sat}}$. Discuss the complexity-theoretic implications that such finding would have.

4. Given an undirected graph $G = (V, E)$, a *spanning tree* for G is a subgraph of G that is a tree and includes all vertices of G. We define the minimum-leaf spanning tree problem as follows.

$\textsc{MinLeafST} = \{\langle G, k\rangle \mid G$ an undirected graph containing a spanning tree having at most k leaves$\}$

A *simple path* in G is a path with no repeated nodes. Consider also the following problems:

$$\textsc{HamPath} = \{G \mid G \text{ has a simple path involving all nodes in } G\}$$
$$\textsc{LongestSimplePath} = \{\langle G, k\rangle \mid G \text{ has a simple path of length} \geq k\}$$

Do the following:

(a) Assuming that $\textsc{HamPath}$ is NP-complete, show that both $\textsc{MinLeafST}$ and $\textsc{LongestSimplePath}$ are also NP-complete.

(b) Identify the error in the following: "Let

$$\textsc{ShortestSimplePath} = \{\langle G, k\rangle \mid G \text{ has a simple path of length} \leq k\}$$

$\textsc{LongestSimplePath}$ is NP-complete and the complement of $\textsc{ShortestSimplePath}$; hence, $\textsc{ShortestSimplePath}$ is co-NP-complete."