Schedule S1, (Computer Science, CS and Philosophy, Maths and CS)
Schedule B, MSc in Computer Science

Hilary Term 2019

Computational Complexity

Exercise class 4: oracles TMs, space-bounded computation

1. Given two Boolean formulae φ and ψ, we say that φ is *equivalent* to ψ if φ and ψ have the same set of variables, and for any truth assignment β to those variables, β makes φ true if and only if it makes ψ true. We say that a Boolean formula φ is *minimal* if there is no shorter formula ψ such that φ is equivalent to ψ.

   The language MF is the language of minimal Boolean formulae, i.e.:

   $$MF = \{\langle\varphi\rangle \mid \varphi \text{ is a minimal Boolean formula}\}.$$

   (a) Show that $MF \in \text{PSPACE}$.

   (b) Explain the fallacy in the following argument: If $\varphi \notin MF$, then there exists a smaller equivalent formula. An NTM can verify that $\varphi \in \overline{MF}$ by guessing a smaller formula ψ and checking if φ is equivalent to ψ. Therefore, $MF \in$ co-NP.

2. The following definitions will be used in this question:

   Definition. An *oracle Turing Machine* (OTM) is a Turing Machine M that has a special read-write tape (the machine's oracle tape) and three special states: q_{query}, q_{yes}, q_{no}. To execute M, we specify in addition a language $O \subseteq \{0,1\}^*$ that is used as the oracle for M. Whenever during the execution M enters the state q_{query}, the machine moves to the state q_{yes} if $q \in O$ and q_{no} if $q \notin O$, where q denotes the contents of the special oracle tape. Regardless of the choice of O, a membership query to O counts only as a *single computation step*. For every $O \in \{0,1\}^*$, P^O is the class of problems that can be decided by a polynomial-time deterministic TM with oracle access to O. The corresponding class of problems based on non-deterministic Turing machines is denoted as NP^O.

   Definition. For any 3CNF formula φ, let $MAX_{true}(\varphi)$ be the maximum number of variables set to true in a satisfying assignment for φ. We now define the following languages:

   $$\text{MAX-TRUE-3SAT} = \{\langle\varphi, n\rangle \mid MAX_{true}(\varphi) \text{ is at least } n\}$$
   $$\text{ODD-MAX-TRUE-3SAT} = \{\varphi \mid MAX_{true}(\varphi) \text{ is odd}\}$$

   Definition. A propositional formula is minimal if there is no smaller formula equivalent to it (a formula ϕ is smaller than formula ψ if the binary representation of ϕ is smaller than the binary representation of ψ). Then, we define the following language:

   $$\text{NoMinFormula} = \{\varphi \mid \varphi \text{ is not minimal}\}$$

   (a) Show that if $O \in \text{P}$, then $\text{P}^O = \text{P}$.

   (b) Show that $\text{NP} \cup \text{co-NP} \subseteq \text{P}^{\text{SAT}}$.

(c) Let us define $\mathrm{P}^{\mathrm{NP}} = \bigcup_{O \in \mathrm{NP}} P^O$. Show that $\mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\mathrm{SAT}}$.

(d) Show that MAX-TRUE-3SAT is in NP.

(e) Show that ODD-MAX-TRUE-3SAT is contained in $\mathrm{P}^{\mathrm{SAT}}$.

(f) Show that NoMinFormula is in $\mathrm{NP}^{\mathrm{SAT}}$.

(g) Show that NoMinFormula is in APTime by displaying an alternating algorithm that solves the problem.

3. Given a non-deterministic polynomially-time bounded Turing machine M and a word w, denote by $Acc(M, w)$ all the accepting computations of M on w and by $Rej(M, w)$ all the rejecting computations.

   Let PP be the complexity class defined as follows: a language L is in PP if there exists a non-deterministic polynomially-time bounded Turing machine M such that $w \in L$ if and only if
   $$|Acc(M, w)| \geq |Rej(M, w)|.$$
   Show that PP $\subseteq$ PSPACE and NP $\subseteq$ PP.

4. Show that the following problem is in Logspace.

Matched Parenthesis	
Input:	A word w over the alphabet $\Sigma := \{), (\}$.
Problem:	Is every parenthesis in w properly matched?
	(e.g. "(()(()()))" is properly matched whereas "(()))(" is not.)