**Schedule S1, (Computer Science, CS and Philosophy, Maths and CS)**
**Schedule B, MSc in Computer Science**

**Hilary Term 2019**

# Computational Complexity

Exercise class 6: circuit complexity, BPP, RP

1. Show that if $\mathrm{NC}^i = \mathrm{NC}^{i+1}$ for some $i \geq 1$, then $\mathrm{NC}^i = \mathrm{NC}$.

2. A 3-Horn-formula is a propositional logic formula of the form $\phi := \bigwedge_{i=1}^{n} C_i$, where $C_i := \bigvee_{j=1}^{n_i}$ is a clause with at most 3 literals (i.e. $n_i \leq 3$) of which at most one is positive.

   The problem 3-Horn-Sat is defined as the problem, given a 3-Horn formula $\phi$, to decide if $\phi$ is satisfiable.

   Show that 3-Horn-Sat is Ptime-complete (under LogSpace reductions). To prove hardness, use the fact that Monotone-CVP is Ptime-complete.

3. Show that BPP and RP are closed under polynomial-time reductions in the sense that if $P \leq_p Q$ and $Q \in$ BPP (or RP) then $P \in$ BPP (or RP).

4. The aim of this question is to show that if NP $\subseteq$ BPP then NP=RP. We pursue this question in several steps.

   (a) Show that if there is a (deterministic) polynomial-time algorithm for deciding whether a SAT-instance is satisfiable then there is also a (deterministic) polynomial-time algorithm for computing a satisfying assignment.

   (b) Show that the same is true for BPP algorithms. That is, if there is a bounded error polynomial time algorithm for deciding whether an instance to SAT is satisfiable then there is also such an algorithm which constructs a satisfying assignment with bounded error probability.

   *Hint.* Use probability amplification.

   (c) Show that if NP $\subseteq$ BPP then NP $\subseteq$ RP.

   (d) Show that RP $\subseteq$ NP.

5. An undirected graph $G$ is *bipartite* if the vertices of $G$ can be split into two disjoint sets $R \subseteq V(G)$ and $S \subseteq V(G)$ such that $R \cup S = V(G)$, $R \cap S = \emptyset$, and for every edge $\{x, y\} \in E(G)$, either $x \in R$ and $y \in S$, or $x \in S$ and $y \in R$. Let:

$$\textsc{Bipartite} = \{G \mid G \text{ is a bipartite graph}\}.$$

   Show that Bipartite $\in$ NLogspace.