Computational Complexity; slides 9, HT 2019
NP search problems, and total search problems

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Examples of total search problems in NP

- FACTORING
- NASH: the problem of computing a Nash equilibrium of a game (comes in many versions depending on the structure of the game)
- PIGEONHOLE CIRCUIT:
  Input: a boolean circuit with n input gates and n output gates
  Output: either input vector x mapping to $\mathbf{0}$ or vectors x, x' mapping to the same output
- NECKLACE SPLITTING
- SECOND HAMILTONIAN CYCLE (in 3-regular graph)
- HAM SANDWICH: search for ham sandwich cut
- Search for *local optima* in settings with neighbourhood structure

The above *seem* to be hard. (of course, many search probs are in P, e.g. input a list L of numbers, output L in increasing order)

Search problems as poly-time checkable relations

NP search problem is modelled as a relation $R(\cdot,\cdot)$ where

- $R(x,y)$ is checkable in time polynomial in $|x|$, $|y|$
- input x, find y with $R(x,y)$ (y as certificate)
- *total* search problem: $\forall x \exists y \quad (|y| = poly(|x|), R(x,y))$

SAT: x is boolean formula, y is satisfying bit vector.
Decision version of SAT is polynomial-time equivalent to search for y.

FACTORING: input (the "x" in $R(x,y)$) is number N, output (the "y") is prime factorisation of N. No decision problem!

NECKLACE SPLITTING (k thieves): input is string of n beads in c colours; output is a decomposition into $c(k-1)+1$ substrings and allocation of substrings to thieves such that they all get the same number of beads of each colour.
(assume the number of beads of each colour is a multiple of k)

contrast with "promise problems"

Reducibility among search problems

FP, FNP: search (or, function computation) problems where output of function is computable (resp., checkable) in poly time.

Any NP problem has FNP version "find a certificate".

Definition

Let R and S be search problems in FNP. We say that R (many-one) reduces to S, if there exist polynomial-time computable functions f, g such that

$$(f(x), y) \in S \implies (x, g(x, y)) \in R.$$

Observation: If S is polynomial-time solvable, then so is R. We say that two problems R and S are (polynomial-time) equivalent, if R reduces to S and S reduces to R.

Theorem: FSAT, the problem of finding a s.a. of a boolean formula, is FNP-complete.

Example

(This slide added subsequently to the lecture)
Consider 2 versions of FACTORING: one using base-10 numbers, and the other version using base-2 numbers. Intuitively, these two problems have the same difficulty: there is a fast algorithm to factor in base 2, if and only if there is a fast algorithm to factor in base 10.

In trying to make that intuition mathematically precise, we get the definition of the previous slide.

Some total search problems seem hard. (F)NP-hard?

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Theorem

There is an FNP-complete problem in TFNP if and only if NP=co-NP.

Proof: "if": if NP=co-NP, then any FNP-complete problem is in TFNP (which is F(NP∩co-NP)).
"only if": Suppose $X \in TFNP$ is FNP-complete, and R is the binary relation for X.
Consider problem FSAT (given formula φ, find a satisfying assignment.)
Any unsatisfiable φ has a certificate of unsatisfiability, namely the string y with $(f(\varphi), y) \in R$ and $g(y) =$"no" (or generally, anything other than a satisfying assignment).

N. Megiddo and C.H. Papadimitriou. On total functions, existence theorems and computational complexity. *Theoretical Computer Science*, **81**(2) pp. 317–324 (1991).

So what can we about the hardness of FACTORING, and others?

FACTORING (for example) cannot be NP-hard unless
NP = co-NP.
Unlikely! So FACTORING is in strong sense "NP-intermediate".

[1] Try to describe "generic" problem/language X in NP∩co-NP as pair of NTMs that accept X and $\overline{X}$: what goes wrong?

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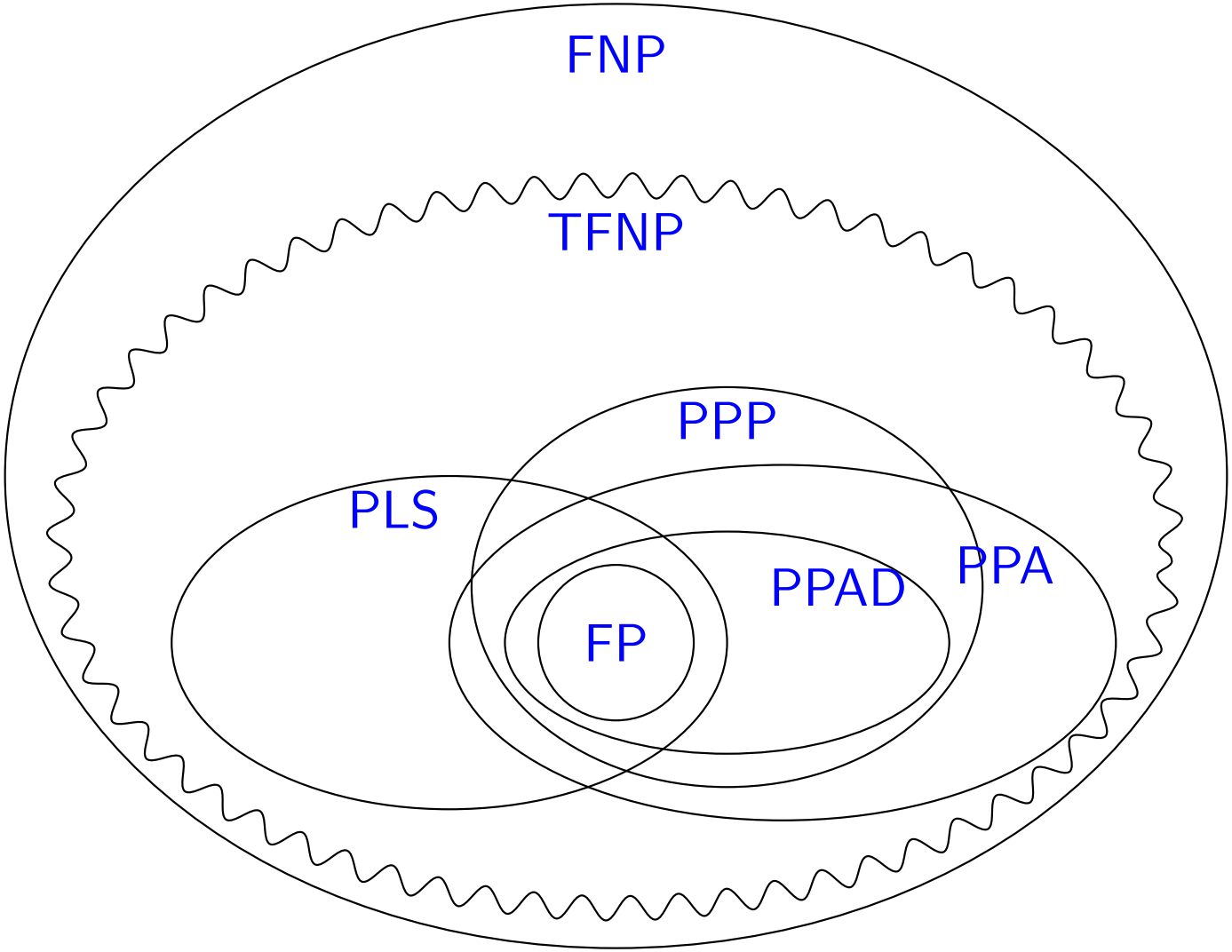
OK can we have, say, FACTORING is TFNP-complete?
Good question! TFNP-completeness is as much as we can hope for, hardness-wise

TFNP doesn't (seem to) have complete problems (which needs syntactic description of "fully general" TFNP problem). (Similarly, RP, BPP, NP$\cap$co-NP don't have complete problems[1])

[1] Try to describe "generic" problem/language X in NP$\cap$co-NP as pair of NTMs that accept X and $\overline{X}$: what goes wrong?

Some syntactic classes



Johnson, Papadimitriou, and Yannakakis. How easy is local search? *JCSS*, 1988.
C.H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *JCSS*, 1994.

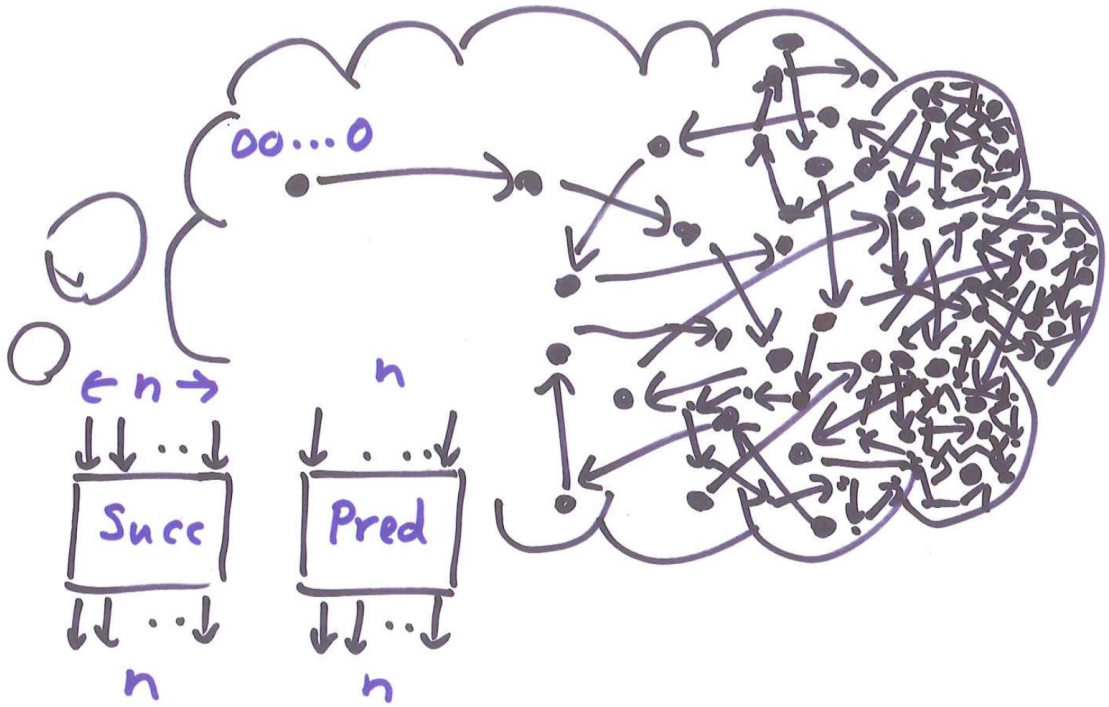
FACTORING, NASH, and the others, as "NP-intermediate"

Advantage of (problems arising in) Ladner's theorem: you just have to believe P≠NP, to have NP-intermediate. For us, we have to believe that FACTORING (say) is not in FP, also that NP≠co-NP.

Disadvantage of Ladner's theorem: the NP-intermediate problems are unnatural (did not arise independently of Ladner's thm; problem definitions involve TMs/circuits)

PPAD

PPAD "given a source in a digraph having in/outdegree at most 1, there's another degree-1 vertex"



The End-of-Line problem

given Boolean circuits S, P with n input bits and n output bits and such that $P(0) = 0 \neq S(0)$, find x such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0$.

PPA, PPP

PPA: Like PPAD, but the implicit graph is undirected.

The END-OF-UNDIRECTED-LINE problem

given boolean circuit C with n inputs, $2n$ outputs. regard input as one of 2^n vertices, output as 2 neighbouring vertices.

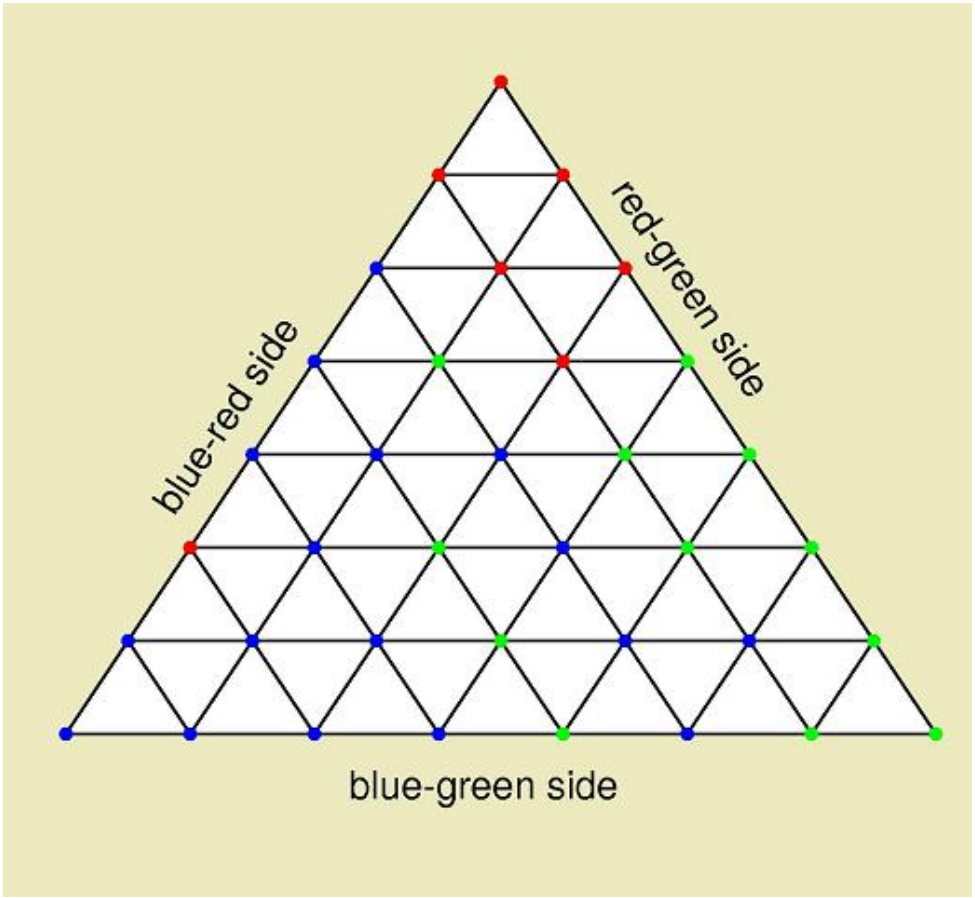
PPP ("polynomial pigeonhole principle"): defined in terms of the PIGEONHOLE CIRCUIT problem.

We have

- END-OF-LINE $\leq_p$ END-OF-UNDIRECTED-LINE (hence PPAD$\subseteq$PPA)
- END-OF-LINE $\leq_p$ PIGEONHOLE CIRCUIT (hence PPAD$\subseteq$PPP)

Next: SPERNER, a PPAD-complete problem

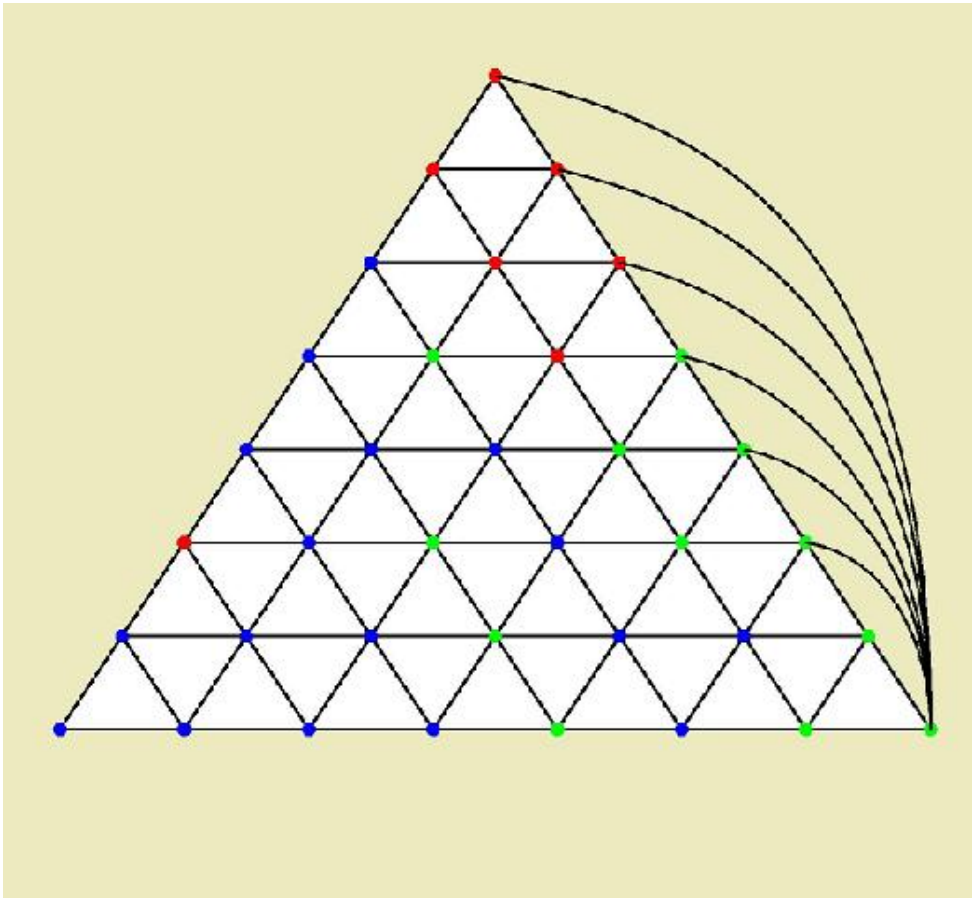
Sperner's Lemma



SPERNER: given a triangular grid coloured subject to certain boundary conditions. Find a "trichromatic triangle". (in "exponentially-fine" resolution, assume some circuit colours each point.)

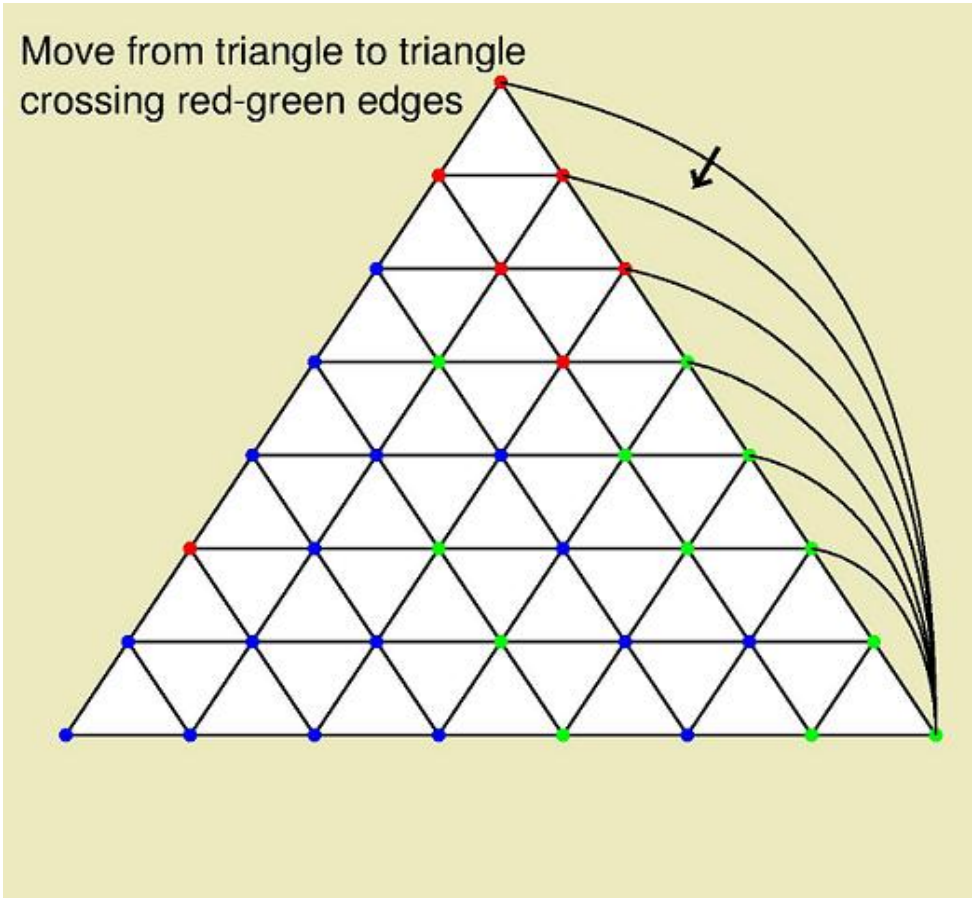
Suppose we color the grid points under the constraint shown in the diagram. Why can we be *sure* that there is a trichromatic triangle?

Reduction to End-of-Line



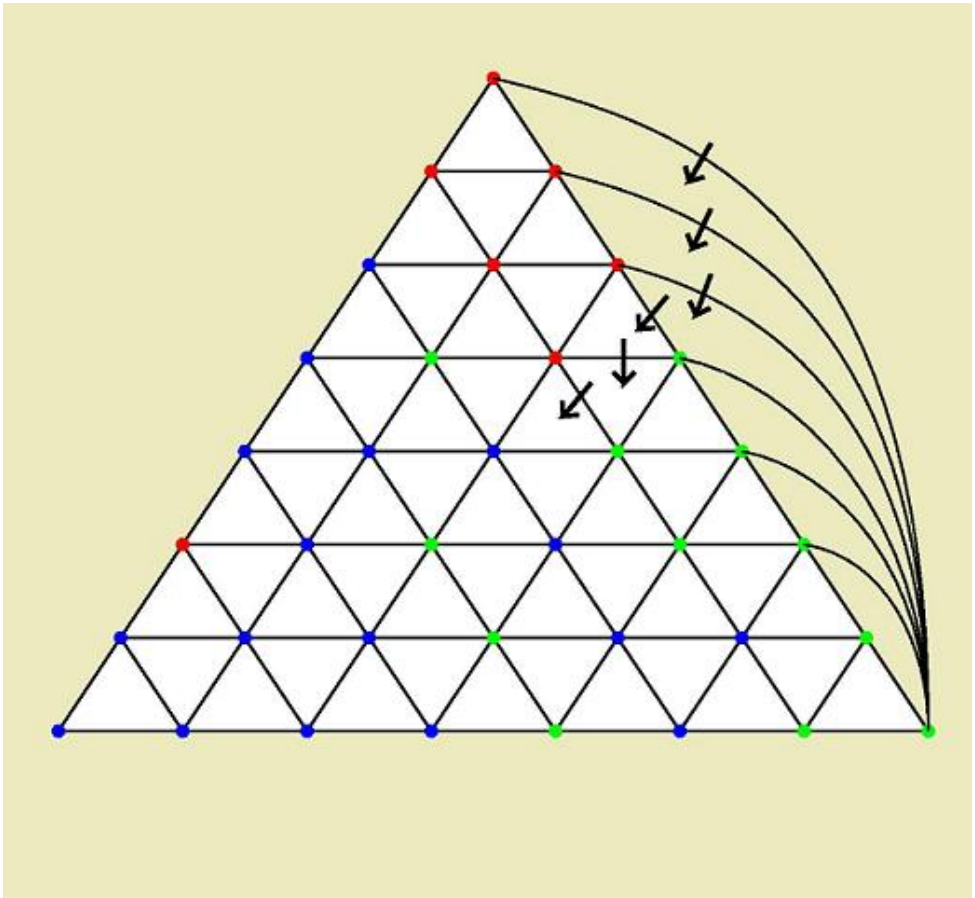
Add some edges such that only one red/green edge is open to the outside

Reduction to End-of-Line



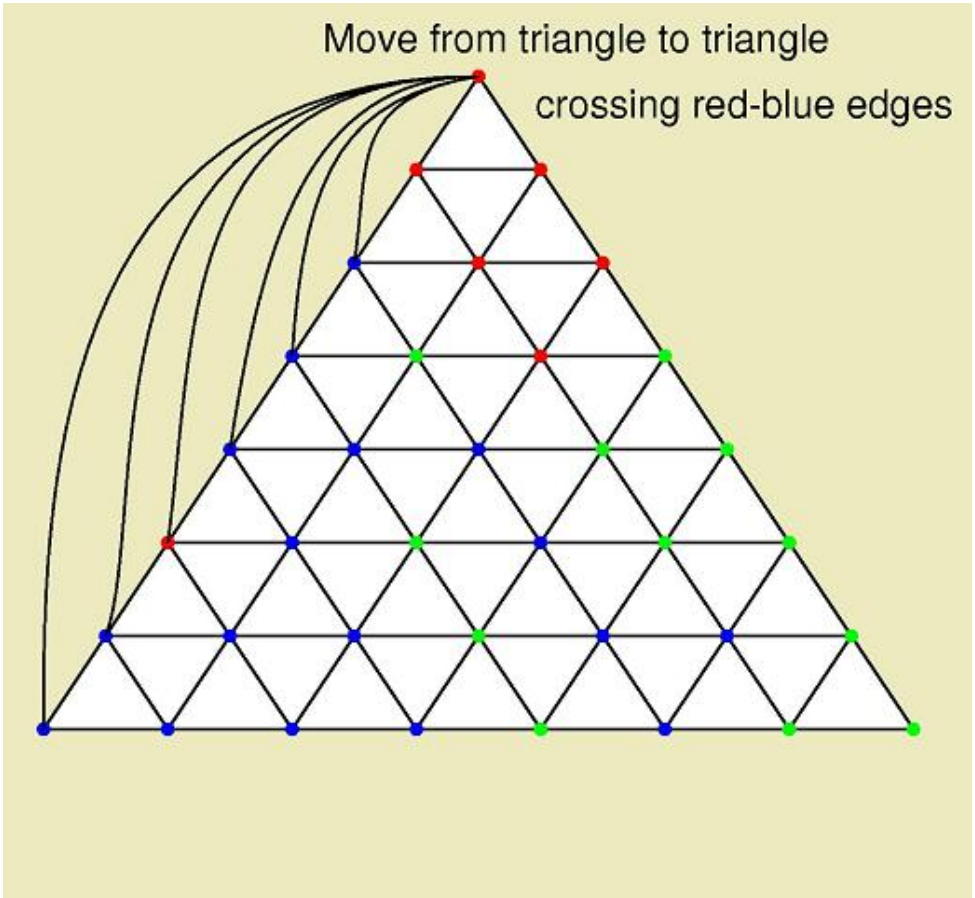
red/green edges are "doorways" that connect the triangles

Reduction to End-of-Line



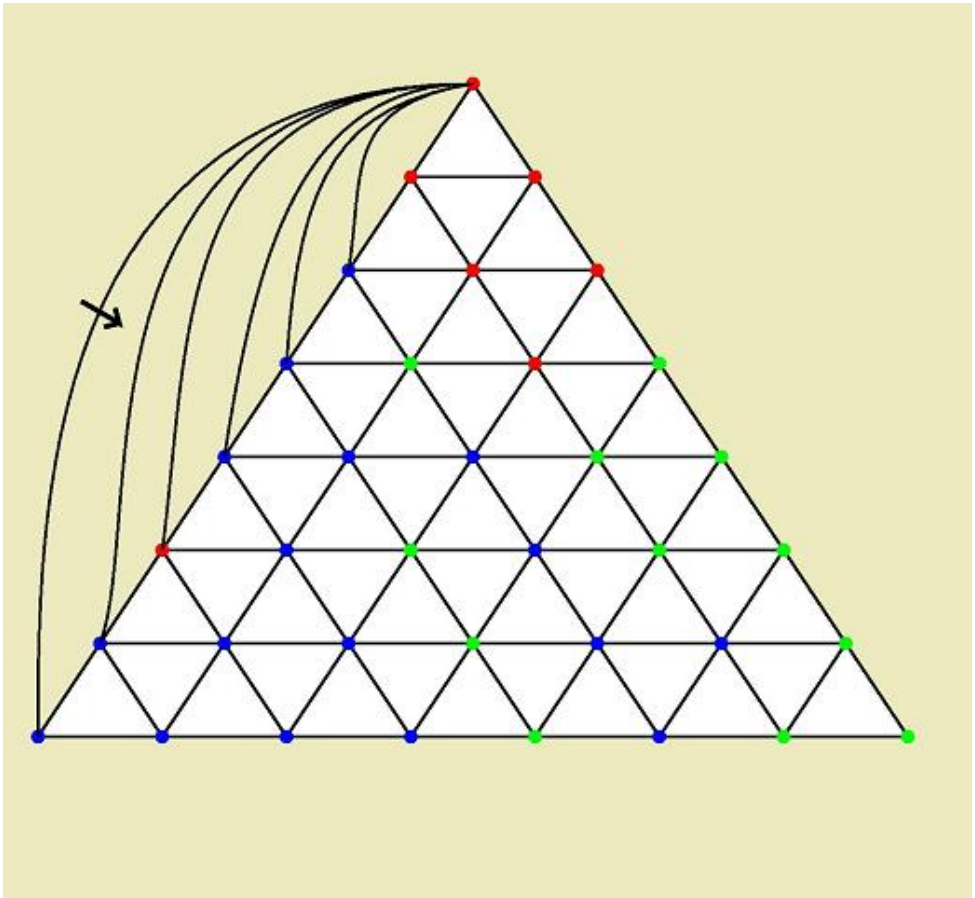
Keep going — we end up at a trichromatic triangle!

Reduction to End-of-Line



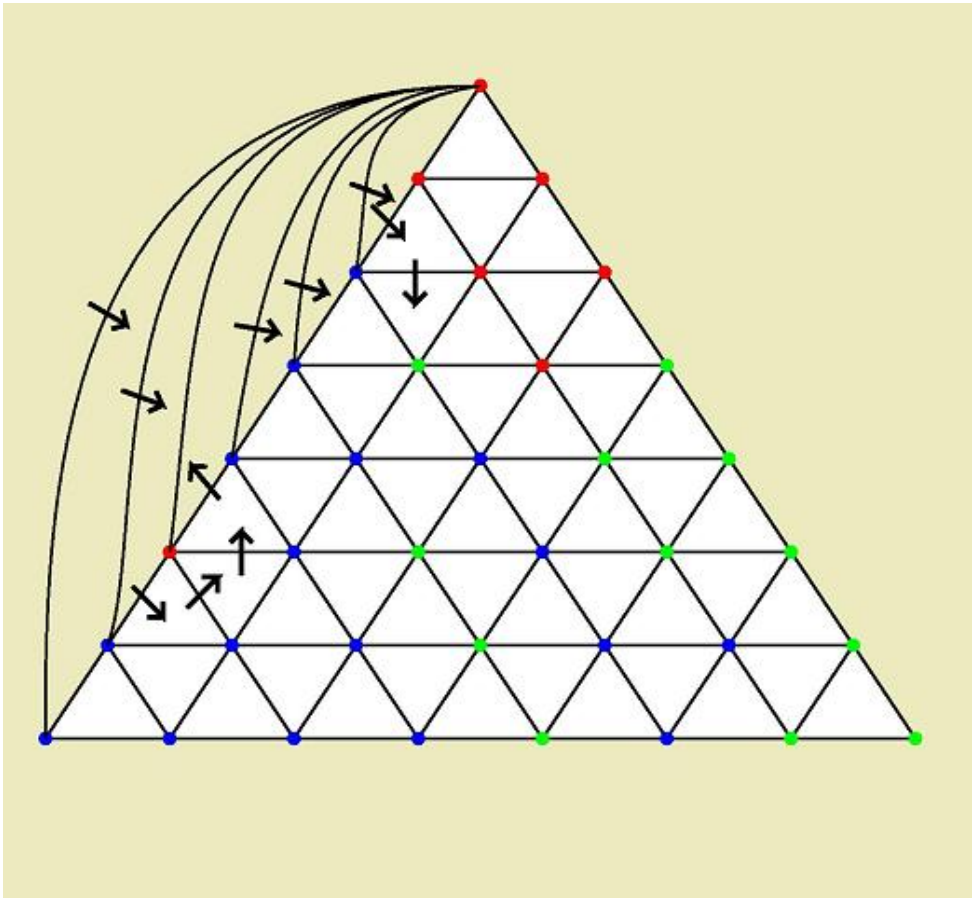
We can do the same trick w.r.t. the red/blue edges

Reduction to End-of-Line



Now the red/blue edges are doorways

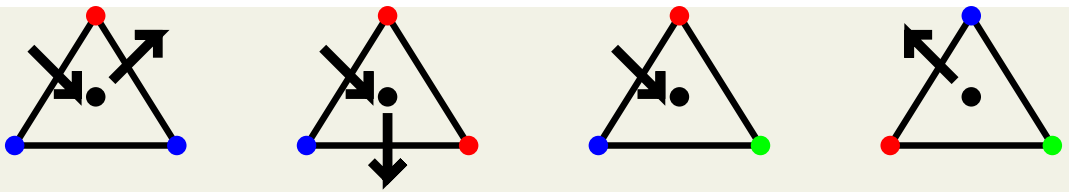
Reduction to End-of-Line

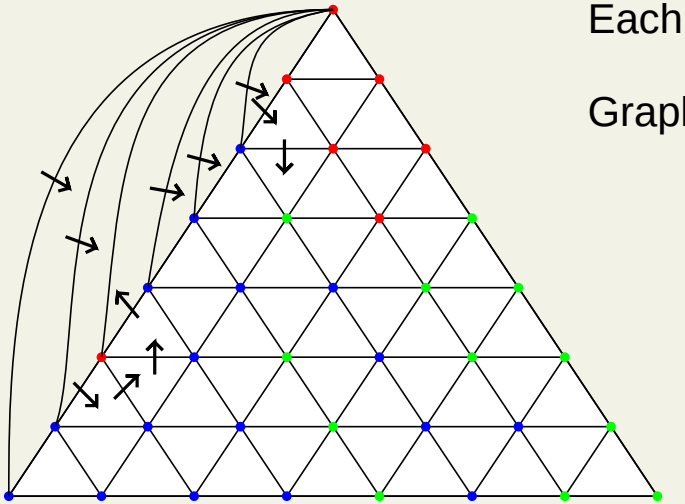


Keep going through them — eventually find a panchromatic triangle!

Reduction to End-of-Line

Degree-2 Directed Graph





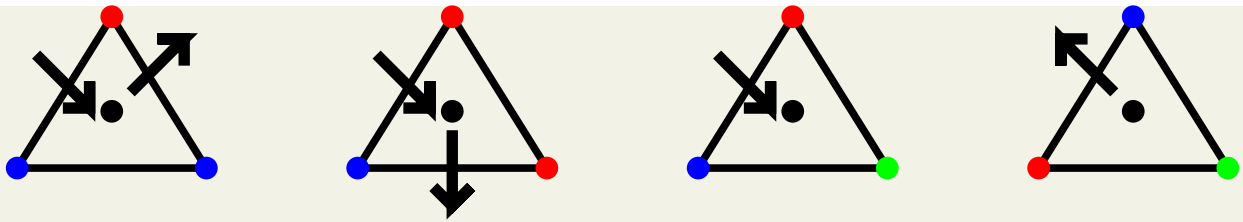
Each little triangle is a vertex

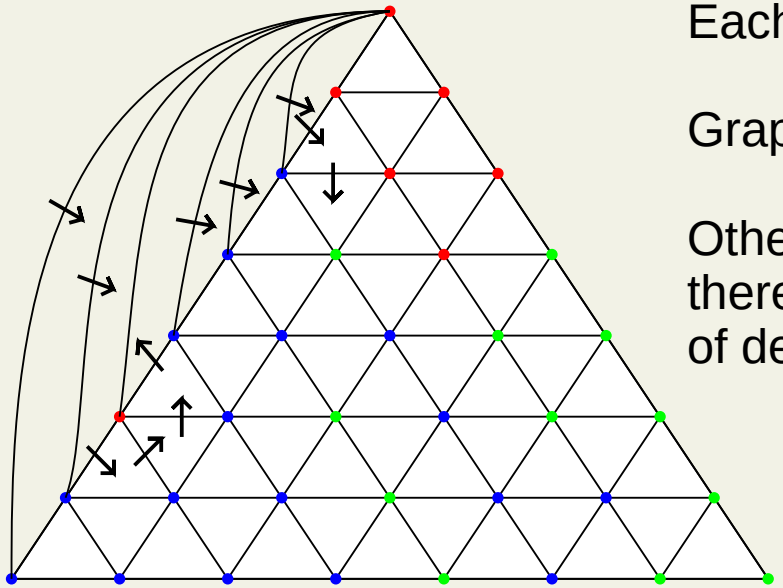
Graph has one known source

Essentially, Sperner's lemma converts the colouring function into an End of line graph!

Reduction to End-of-Line

Degree-2 Directed Graph





Each little triangle is a vertex

Graph has one known source

Other than the known source, there must be an odd number of degree-1 vertices.

Classifying some of these problems

PPAD-completeness:

The Complexity of Computing a Nash Equilibrium. Daskalakis, G, and Papadimitriou (STOC 2006, SICOMP 2009)

Settling the Complexity of Computing Two-Player Nash Equilibria. Chen, Deng, and Teng (FOCS 2006, JACM 2009)

Settling the Complexity of Arrow-Debreu Equilibria in Markets with Additively Separable Utilities Chen, Dai, Du, Teng (FOCS 2009)

PPA-completeness

The Complexity of Splitting Necklaces and Bisecting Ham Sandwiches. Filos-Ratsikas and G (STOC 2019)

PPP-completeness

PPP-Completeness with Connections to Cryptography. Sotiraki, Zampetakis, Zirdelis (FOCS 2018)

Various problems involving circuits are complete for these classes. "Integer Factoring and Modular Square Roots" (Jeřábek, JCSS 2016): randomised reduction from FACTORING to PPA.