Computational Complexity; slides 14, HT 2022 Space Hierarchy Theorem, Gap Theorem, NP-intermediate problems

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Overview of next 2 lectures

Recall: Relation between complexity classes covered so far:

$$\begin{array}{l} \mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \\ \quad \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXP} \subseteq \mathsf{NEXP} \subseteq \\ \quad\quad \mathsf{EXPSPACE} = \mathsf{NEXPSPACE} \subseteq \ldots \end{array}$$

Next: a closer look at the space hierarchy theorem, and strict containments it gives us.
Then: "NP-intermediate" problems – Ladner's theorem; search problems where solutions are guaranteed to exist

recall: Time Hierarchy theorem

proper complexity function f: roughly, an increasing function that can be computed by a TM in time $f(n) + n$

For $f(n) \geq n$ a proper complexity function, we have

$$\text{TIME}(f(n)) \text{ is a proper subset of } \text{TIME}((f(2n+1))^3).$$

It follows that P is a proper subset of EXP.

Proof used "time-bounded halting language" H_f and a "diagonalising machine"

$$H_f := \{\langle M, w\rangle \ : \ M \text{ accepts } w \text{ after } \leq f(|w|) \text{ steps}\}$$

Space Hierarchy Theorem

Theorem. (Space Hierarchy Theorem)

Let $S, s : \mathbb{N} \to \mathbb{N}$ be functions such that

1. S is "space constructible", and
2. $S(n) \geq n$,
3. $s = o(S)$.

Then $\mathrm{DSPACE}(s) \subsetneq \mathrm{DSPACE}(S)$.

Reminder: item 3 means that $\lim_{n\to\infty}(s(n)/S(n)) = 0$.

Space-constructible functions

Definition.

$f : \mathbb{N} \to \mathbb{N}$ is space constructible if $f(n) \geq \log n$ and $f(n)$ can be computed from input $1^n := \underbrace{1 \ldots 1}_{n \text{ times}}$ in space $\mathcal{O}(f(n))$.

Most standard functions are space-constructible:

- All polynomial functions (e.g. $3n^3 - 5n^2 + 1$)
- All exponential functions (e.g. 2^n)

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For any space-constructible function f we can build a counter that goes off after $f(n)$ cells have been used on inputs of length n.

Consequence: As polynomials are space constructible:

We can enforce that in an n^k-space bounded NTM M all computations halt after using $\mathcal{O}(n^k)$ space.

(Let M and a "counter" run in parallel. Stop if the counter goes off.)

slight digression: contrast with time-constructible functions

Definition.

$f : \mathbb{N} \to \mathbb{N}$ is time constructible if $f(n) \geq n \log n$ and $f(n)$ can be computed from input $1^n := \underbrace{1 \dots 1}_{n \text{ times}}$ in time $\mathcal{O}(f(n))$.

Similar points apply for time constructible functions (as for space constructible ones, previous slide).

Proof of Space Hierarchy Theorem — Part I

Construct S-space bounded TM $\mathcal{D}$ as follows.

1. On input $\langle M, w \rangle$, let $n = |\langle M, w \rangle|$.
2. If the input is not of the form $\langle M, w \rangle$, then reject.
3. Compute $S(n)$ and mark off this much tape. If later stages ever exceed this allowance, then reject.
4. Simulate M on input $\langle M, w \rangle$ while counting number of steps used in simulation; if count ever exceeds $2^{S(n)}$, then reject.

   The simulation introduces only a constant factor c space overhead.
5. If M *accepts*, then reject; otherwise accept.

$\mathcal{L}(\mathcal{D}) = \{\langle M, w \rangle : \mathcal{D} \text{ accepts } \langle M, w \rangle\}$.

By construction, $\mathcal{L}(\mathcal{D}) \in \mathrm{DSPACE}(S)$

Proof of Space Hierarchy Theorem — Part II

Claim. $\mathcal{L}(\mathcal{D}) \notin \mathsf{DSPACE}(s)$

Towards a contradiction,

let $\mathcal{B}$ be a s space bounded TM with $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{D})$.

- As $s = o(S)$ there is $n_0 \in \mathbb{N}$ such that $S(n) \geq c \cdot s(n)$ for all $n \geq n_0$.
- Hence, for almost all inputs $\langle \mathcal{B}, w \rangle$ (length of $\langle \mathcal{B}, w \rangle \geq n_0$)

  $\mathcal{D}$ completely simulates the run of $\mathcal{B}$ on $\langle \mathcal{B}, w \rangle$
- Hence, for almost all $w \in \{0,1\}^*$

  $\langle \mathcal{B}, w \rangle \in \mathcal{L}(\mathcal{D}) \iff \mathcal{B}$ does not accept $\langle \mathcal{B}, w \rangle$ (Def of $\mathcal{D}$)

  $\langle \mathcal{B}, w \rangle \in \mathcal{L}(\mathcal{B}) \iff \mathcal{B}$ accepts $\langle \mathcal{B}, w \rangle$. (Def of "$\mathcal{L}(\mathcal{B})$")

A Hierarchy of Complexity Classes

Consequence of hierarchy theorems:

- LOGSPACE $\subsetneq$ PSPACE $\subsetneq$ EXPSPACE
- P $\subsetneq$ EXP

Relation between complexity classes covered so far:

$$
\begin{array}{ccccccccc}
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\neq & & \neq & & \neq & & \neq & \\
\mathsf{PSPACE} & = & \mathsf{NPSPACE} & \subseteq & \mathsf{EXP} & \subseteq & \mathsf{NEXP} & \subseteq \\
\neq & & \neq & & & & & \\
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\end{array}
$$

The Gap Theorem

Question. Given more resources, can we always solve more problems?

How much more resources do we need to be able to solve more problems? (Can we solve strictly more problems in time $2^{2^{g(n)}}$ than in $g(n)$?)

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Theorem. (Gap theorem for time complexity)

For every total computable function $g : \mathbb{N} \to \mathbb{N}$ with $g(n) \geq n$ there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{DTIME}\big(f(n)\big) = \mathrm{DTIME}\big(g(f(n))\big)$$

Analogously for space complexity.
contrast with Time hierarchy theorem

For $f(n) \geq n$ a proper complexity function, we have TIME($f(n)$) is a proper subset of TIME($(f(2n+1))^3$).

The Gap Theorem

Special case (Papadimitriou's book, theorem 7.3): There is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that TIME($f(n)$)=TIME($2^{f(n)}$). Proof works by constructing f such that no TM, on input of length n, halts between $f(n)$ and $2^{f(n)}$ steps.

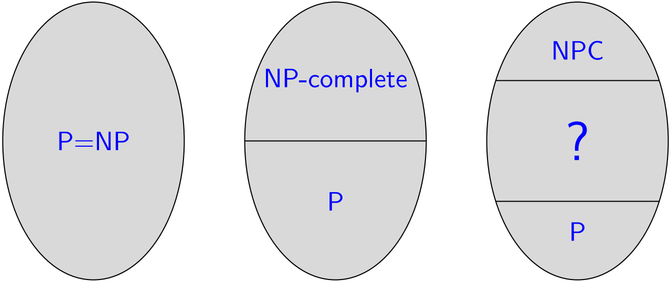
Corollaries of Gap theorem. There are computable functions f such that

- $\mathrm{DTIME}(f) = \mathrm{DTIME}(2^f)$
- $\mathrm{DTIME}(f) = \mathrm{DTIME}(2^{2^f})$
- $\mathrm{DTIME}(f) = \mathrm{DTIME}\Big(\left. 2^{2^{\cdot^{\cdot^{2}}}} \right\} f(n) \text{ times}\Big)$

However, the functions f are not time (space) constructible.

Question.

- Can we classify any problem in NP as polynomial or NP-complete?
- Which of the following diagrams corresponds to a true picture of NP?



Ladner's theorem

background

Cook/Levin (1971): SAT is NP-complete
Karp (1972): many other diverse NP problems of interest also NP-complete

Ladner's Theorem (1975)
If P $\neq$ NP then there is a language in NP that is neither in P not NP-complete.

Proof. Non-constructive argument (using diagonalisation). (details in Papadimitriou Chapter 14; Arora/Barak Ch.3).

Proof idea

Diagonalisation; let M_i be i-th Turing machine...

For $f : \mathbb{N} \longrightarrow \mathbb{N}$ let $\mathrm{SAT}_f = \{\varphi 1^{n^{f(n)}} \ : \ \varphi \in \mathrm{SAT} \text{ and } n = |\varphi|\}$

Q: How hard is SAT_f for f constant? $f(n) = n$?

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Q: How hard is SAT_f for f constant? $f(n) = n$?

Let $f(n)$ be smallest $i < \log\log n$ such that for every bit-string x with $|x| < \log n$, M_i on input x outputs $\mathrm{SAT}_f(x)$ within $i|x|^i$ steps; if no such i, set $f(n) = \log\log n$.

$f(n)$ can be computed from n in $O(n^3)$ time

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Claim. $\mathrm{SAT}_f \in \mathrm{P}$ iff $f = O(1)$.

Then if $\mathrm{SAT}_f \in \mathrm{P}$, solved by some TM M_i — for $n > 2^{2^i}$, $f(n) \leq i$ — f never gets larger than a constant.

If SAT_f is NP-complete, consider reduction from SAT to SAT_f. Reduction must map instances of SAT to instances of SAT_f only polynomially larger...

NP-Intermediate Problems

Ladner's theorem gives an *artificial* problem between P and NP. Other candidates exist, however. Keep in mind, *unconditional* NP-intermediateness is too much to hope for...
We can base this property on stronger assumptions than P$\neq$NP.

Garey and Johnson 1979.

In their text book they highlight three problems whose complexity was undecided:

- Linear Programming
- Primes/Composite
- Graph Isomorphism

The first 2 of these now known to belong to P.
Total search problems (FACTORING, Nash equilibrium computation, and others) are NP-intermediate assuming they're not in P, and NP$\neq$co-NP.