Computational Complexity; slides 15, HT 2022
Search problems, and total search problems

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Overview

We noted that NP problems have "search" counterparts that are of equal difficulty.

(recall FSAT: find a satisfying assigment)

For search problems having guaranteed solutions, we'll see that a novel classification is needed...

Search versus decision

For NP-complete problems, e.g. SAT, suppose we want to compute a satisfying assignment, not just test for satisfiability. This is at least as challenging as SAT...

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If we had a SAT-oracle, proceed as follows.
For φ over variables $x_1, \ldots, x_n$, check if φ is satisfiable, if so, try φ with $x_1 \mapsto 0$ alternatively $x_1 \mapsto 1$, then proceed to x_2 etc.

Conclude that in a sense, computing a s.a. is no harder than SAT.

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Complexity class **FNP**: functions checkable in poly-time.

- FSAT is FNP-complete (via Cook-Levin)
- So are function versions of other NP-complete problems

Some apparently-hard *total* search problems in **FNP**

- many problems of local optimisation, e.g. LOCAL-MAX-CUT of a weighted graph.
- FACTORING
- NASH: the problem of computing a Nash equilibrium of a game (comes in many versions depending on the structure of the game)
- PIGEONHOLE CIRCUIT:
  Input: a boolean circuit with n input gates and n output gates
  Output: either input vector x mapping to $\mathbf{0}$ or vectors x, x' mapping to the same output
- many other problems associated with "non-constructive" existence results

Search problems as poly-time checkable relations

NP search problem is modelled as a relation $R(\cdot, \cdot)$ where

- $R(x, y)$ is checkable in time polynomial in $|x|$, $|y|$
- input x, find y with $R(x, y)$ (y as certificate)
- *total* search problem: $\forall x \exists y \quad (|y| = poly(|x|), R(x, y))$

SAT: x is boolean formula, y is satisfying bit vector.
Decision version of SAT is polynomial-time equivalent to search for y.

FACTORING: input (the "x" in $R(x, y)$) is number N, output (the "y") is prime factorisation of N. No decision problem!

contrast with "promise problems"

Reducibility among search problems

FP, FNP: search (or, function computation) problems where output of function is computable (resp., checkable) in poly time.

Any NP problem has FNP version "find a certificate".

Definition

Let R and S be search problems in FNP. We say that R (many-one) reduces to S, if there exist polynomial-time computable functions f, g such that

$$(f(x), y) \in S \implies (x, g(x, y)) \in R.$$

Observation: If S is polynomial-time solvable, then so is R. We say that two problems R and S are (polynomial-time) equivalent, if R reduces to S and S reduces to R.

Theorem: FSAT, the problem of finding a s.a. of a boolean formula, is FNP-complete.

Example

(To help motivate/understand that definition of reducibility)

Consider 2 versions of FACTORING: one using base-10 numbers, and the other version using base-2 numbers. Intuitively, these two problems have the same difficulty: there is a fast algorithm to factor in base 2, if and only if there is a fast algorithm to factor in base 10.

In trying to make that intuition mathematically precise, we get the definition of the previous slide.

TFNP

TFNP: "Total" function computation problems in NP

As we shall see, it looks like we really do need to introduce a new complexity class, in fact a collection of complexity classes...

Contrast with "promise problems", e.g. $\textsc{Promise SAT}$: SAT-instances where you've been promised there is a satisfying assignment. But such a promise isn't directly checkable.

Some total search problems seem hard. (F)NP-hard?

Theorem

There is an FNP-complete problem in TFNP if and only if NP=co-NP.

Proof: "if": if NP=co-NP, then any FNP-complete problem is in TFNP (which is F(NP$\cap$co-NP)).

"only if": Suppose X$\in$TFNP is FNP-complete, and R is the binary relation for X.

Consider problem FSAT (given formula φ, find a satisfying assignment.) We have $\mathrm{FSAT}\leq_p$X.

Any unsatisfiable φ would get a certificate of unsatisfiability, namely the string y with $(f(\varphi), y) \in R$ and $g(y) =$"no" (or generally, anything other than a satisfying assignment).

N. Megiddo and C.H. Papadimitriou. On total functions, existence theorems and computational complexity. *Theoretical Computer Science*, **81**(2) pp. 317–324 (1991).

So what can we say about the hardness of FACTORING, and others?

FACTORING (for example) cannot be NP-hard unless NP = co-NP. Unlikely! So FACTORING is in strong sense "NP-intermediate".

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OK can we have, say, FACTORING is TFNP-complete? Good question! TFNP-completeness is as much as we can hope for, hardness-wise

TFNP doesn't (seem to) have complete problems (which needs syntactic description of "fully general" TFNP problem). (Similarly, RP, BPP, NP$\cap$co-NP don't have complete problems)

Try to describe "generic" problem/language X in NP$\cap$co-NP as pair of NTMs that accept X and $\overline{X}$: what goes wrong?

FACTORING, NASH, and the others, as "NP-intermediate"

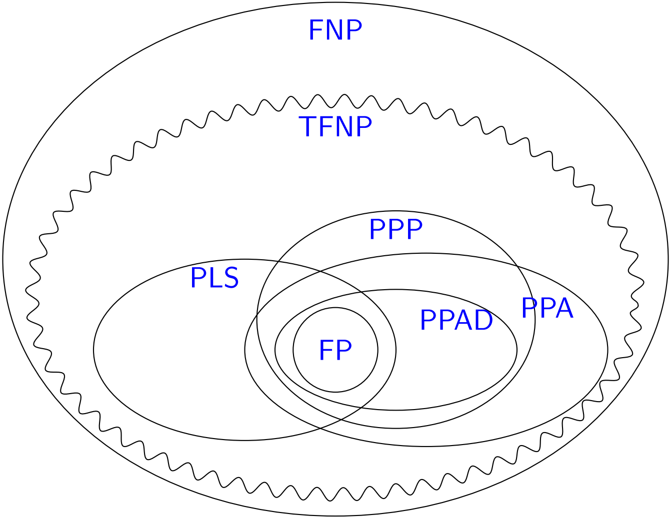
Advantage of (problems arising in) Ladner's theorem: you just have to believe P$\neq$NP, to have NP-intermediate. For us, we have to believe that FACTORING (say) is not in FP, also that NP$\neq$co-NP.

Disadvantage of Ladner's theorem: the NP-intermediate problems are unnatural (did not arise independently of Ladner's thm; problem definitions involve TMs/circuits)

Next: subclasses of TFNP that have complete problems

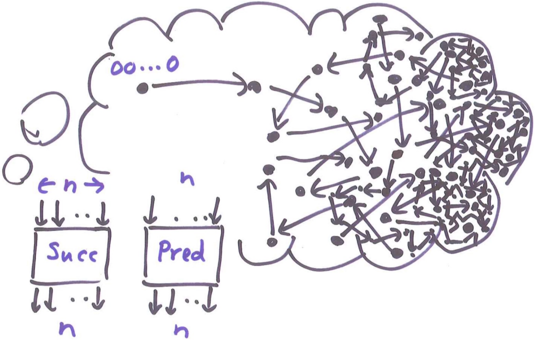
General idea: define classes in terms of "non-constructive" existence principles

Some syntactic classes



Johnson, Papadimitriou, and Yannakakis. How easy is local search? *JCSS*, 1988.
C.H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *JCSS*, 1994.

PPAD "given a source in a digraph having in/outdegree at most 1, there's another degree-1 vertex"



The END-OF-LINE problem

given Boolean circuits S, P with n input bits and n output bits and such that $P(0) = 0 \neq S(0)$, find x such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0$.

PPAD

A problem X belongs to PPAD if $X \leq_P$ End-of-Line.

X is PPAD-complete if in addition, End-of-Line$\leq_P X$

Work by myself and others: Nash equilibrium computation is PPAD-complete

...which is taken to indicate it's a hard problem! Let's see why we believe "PPAD is hard"

PPA, PPP

PPA: Like PPAD, but the "implicit" graph is undirected.

The LEAF problem

given boolean circuit C with n inputs, $2n$ outputs. regard input as one of 2^n vertices, output as 2 neighbouring vertices.
If $\mathbf{0}$ has degree 1, find some other degree-1 vertex.

PPP ("polynomial pigeonhole principle"): defined in terms of:

The PIGEONHOLE CIRCUIT problem

given boolean circuit C with n inputs, n outputs.
Find *either* a bit-string that is mapped to $\mathbf{0}$, *or* two bitstrings that are mapped to the same bit-string

We have

- END-OF-LINE $\leq_p$ LEAF (hence PPAD$\subseteq$PPA)
- END-OF-LINE $\leq_p$ PIGEONHOLE CIRCUIT

PPAD is a subset of PPP

END-OF-LINE reduces to PIGEONHOLE CIRCUIT:

Given S, P, circuits representing an END OF LINE instance, build a circuit C_{PPP} that does the following:

C_{PPP} uses S, P to identify any neighbours of a vertex v in the END-OF-LINE graph, then

- If v has no outgoing edge in the END-OF-LINE graph, C_{PPP} maps v to itself.
  (so all isolated vertices are mapped to themselves)
- Otherwise, let (v, w) be a directed edge in the END-OF-LINE graph.
  C_{PPP} maps v to w

Evidence of hardness

- Failure to find poly-time algorithms for most of these problems, indeed even sub-exponential algorithms.
- cryptographic hardness

- **Separation oracles**

Circuits viewed as proxies for unrestricted boolean functions: the search problems stay total even if the circuits in the defs are allowed to be any functions (not necessarily having small circuits)

Warm-up: in the context of END OF LINE/PPAD, if the circuits S and P were replaced with unrestricted boolean functions allowing "black-box access", the problem becomes impossible.
Call this "oracle PPAD"

Now define a "PPAD machine" to be a notional machine that, given black-box access to S and P, identifies a solution...
Such a machine can't used to solve oracle PPA!

Paul Beame, Stephen A. Cook, Jeff Edmonds, Russell Impagliazzo, Toniann Pitassi: The Relative Complexity of NP Search Problems. *JCSS* (1998)

Conclusion

This is ongoing work! The hardness of some of these complexity classes has been derived from various cryptographic assumptions (that are stronger than P$\neq$NP).

Further question include: do we "need" any other as-yet undefined classes of TFNP problems?
Can we base the hardness of (say) PPAD on weaker assumptions, ideally P$\neq$NP?

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This is ongoing work! The hardness of some of these complexity classes has been derived from various cryptographic assumptions (that are stronger than P$\neq$NP).

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Thanks!