# Computational Complexity; slides 7, HT 2022
# Space complexity, time/space hierarchy theorems

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# Road map

I mentioned classes like LOGSPACE (usually called L), SPACE($f(n)$) etc. How do they relate to each other, and time complexity classes?

Next: Various inclusions can be proved, some more easy than others; let's begin with "low-hanging fruit"...

e.g., I have noted: TIME($f(n)$) is a subset of SPACE($f(n)$) (easy!)

We will see e.g. L is a proper subset of PSPACE, although it's unknown how they relate to various intermediate classes, e.g. P, NP

Various interesting problems are complete for PSPACE, EXPTIME, and some of the others.

# Space Complexity

So far, we have measured the complexity of problems in terms of the time required to solve them.

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***Convention:*** Turing machines have a designated read only input tape. So, "logarithmic space" becomes meaningful.

# Space Complexity

***Definition.*** Let $M$ be a Turing accepter and $S : \mathbb{N} \to \mathbb{N}$ a monotone growing function. $M$ is <u>$S$-space bounded</u> if for all input words $w$, $M$ halts and uses at most $S(|w|)$ non-input tape cells.

1. DSPACE($S$): languages $\mathcal{L}$ for which there is an $S$-space bounded $k$-tape deterministic Turing accepter deciding $\mathcal{L}$ for some $k \geq 1$.

2. NSPACE($S$): languages $\mathcal{L}$ for which there is an $S$-space bounded non-deterministic $k$-tape Turing accepter deciding $\mathcal{L}$ for some $k \geq 1$.

# Space Complexity Classes

- Deterministic Classes:
  - LOGSPACE $:= \bigcup_{d\in\mathbb{N}} \text{DSPACE}(d \log n)$
  - PSPACE $:= \bigcup_{d\in\mathbb{N}} \text{DSPACE}(n^d)$
  - EXPSPACE $:= \bigcup_{d\in\mathbb{N}} \text{DSPACE}(2^{n^d})$
- Non-Deterministic versions: NLOGSPACE etc

In the above defs, a single separate work-tape is sufficient.

***Straightforward observation:***

$$
\begin{array}{ccccc}
\text{LOGSPACE} & \subseteq & \text{PSPACE} & \subseteq & \text{EXPSPACE} \\
\cap\!\!| & & \cap\!\!| & & \cap\!\!| \\
\text{NLOGSPACE} & \subseteq & \text{NPSPACE} & \subseteq & \text{NEXPSPACE}
\end{array}
$$

# Elementary relationships between time and space

***Easy observation:***

For all functions $f : \mathbb{N} \to \mathbb{N}$:

$$\begin{aligned} \mathsf{DTIME}(f) &\subseteq \mathsf{DSPACE}(f) \\ \mathsf{NTIME}(f) &\subseteq \mathsf{NSPACE}(f) \end{aligned}$$

***A bit harder:***

For all monotone growing functions $f : \mathbb{N} \to \mathbb{N}$:

$$\begin{aligned} \mathsf{DSPACE}(f) &\subseteq \mathsf{DTIME}(2^{\mathcal{O}(f)}) \\ \mathsf{NSPACE}(f) &\subseteq \mathsf{DSPACE}(2^{\mathcal{O}(f)}) \end{aligned}$$

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***Proof.*** Based on *configuration graphs* (next 2 slides) and a bound on the number of possible configurations.

- Build the configuration graph $\rightsquigarrow$ time $2^{\mathcal{O}(f(n))}$
- Find a path from the start to an accepting stop configuration. $\rightsquigarrow$ time $2^{\mathcal{O}(f(n))}$

# Number of Possible Configurations

Let $M := (Q, \Sigma, \Gamma, q_0, \Delta, F_a, F_r)$ be a 1-tape Turing accepter.
(plus input tape)

*Recall:* Configuration of $M$ is a triple $(q, p, x)$ where

- $q \in Q$ is the current state,
- $p \in \mathbb{N}$ is the head position, and
- $x \in \Gamma^*$ is the tape content.

Let $w \in \Sigma^*$ be an input to $M$, $n := |w|$

If $M$ is $f(n)$-space bounded we can assume that $p \leq f(n)$ and $|x| \leq f(n)$

Hence, there are at most

$$|\Gamma|^{f(n)} \cdot f(n) \cdot |Q| \quad = \quad 2^{\mathcal{O}(f(n))}$$

different configurations on inputs of length $n$.

# Configuration Graphs

Let $M := (Q, \Sigma, \Gamma, q_0, \Delta, F_a, F_r)$ be a 1-tape Turing accepter.
$f(n)$ space bounded

Configuration graph $\mathcal{G}(M, w)$ of $M$ on input $w$:
Directed graph with

| | |
|---|---|
| Vertices: | All possible configurations of $M$ up to length $f(\|w\|)$ |
| Edges: | Edge $(C_1, C_2) \in E(\mathcal{G}(M, w))$, if $C_1 \vdash_M C_2$ |

A computation of $M$ on input $w$ corresponds to a path in $\mathcal{G}(M, w)$ from the start configuration to a stop configuration.

Hence, to test if $M$ accepts input $w$,

- construct the configuration graph and
- find a path from the start to an accepting stop configuration.

# Basic relationships

Recall: L denotes LOGSPACE; NL=NLOGSPACE

$$
\begin{array}{ccccccc}
\mathsf{L} & & & & & & \\
\rotatebox{90}{$\supseteq$} & & & & & & \\
\mathsf{NL} & \subseteq & \mathsf{P} & \subseteq & \mathsf{PSPACE} & & \\
 & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \\
 & & \mathsf{NP} & \subseteq & \mathsf{NPSPACE} & \subseteq & \mathsf{EXPTIME} \subseteq \mathsf{EXPSPACE} \\
 & & & & & & \rotatebox{90}{$\supseteq$} \quad\quad \rotatebox{90}{$\supseteq$} \\
 & & & & & & \mathsf{NEXPTIME} \subseteq \mathsf{NEXPSPACE}
\end{array}
$$

# Simulating non-deterministic computations with limited space

***Easy observation:*** SAT can be solved in linear space

Just try every possible assignment, one after another, reusing space.

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***Consequence:*** NP $\subseteq$ PSPACE
similarly, NEXPTIME is a subset of EXPSPACE

Generally, non-deterministic time $f(n)$ allows $O(f(n))$ non-deterministic "guesses"; try them all one-by-one, in lexicographic order, over-writing previous attempts.

$$
\begin{array}{ccccccccc}
\mathsf{L} & & & & & & & & \\
\rotatebox{-90}{$\subseteq$} & & & & & & & & \\
\mathsf{NL} & \subseteq & \mathsf{P} & & \mathsf{PSPACE} & & & & \\
 & & \rotatebox{-90}{$\subseteq$} & \rotatebox{-45}{$\subsetneq$} & \rotatebox{-90}{$\subseteq$} & & & & \\
 & & \mathsf{NP} & & \mathsf{NPSPACE} & \subseteq & \mathsf{EXPTIME} & & \mathsf{EXPSPACE} \\
 & & & & & & \rotatebox{-90}{$\subseteq$} & \rotatebox{-45}{$\subsetneq$} & \rotatebox{-90}{$\subseteq$} \\
 & & & & & & \mathsf{NEXPTIME} & & \mathsf{NEXPSPACE}
\end{array}
$$

By the *time hierarchy theorem* (coming up next), P $\subsetneq$ EXPTIME, NP $\subsetneq$ NEXPTIME

By the *space hierarchy theorem*, NL $\subsetneq$ PSPACE, PSPACE $\subsetneq$ EXPSPACE.

# Time Hierarchy theorem

*proper* complexity function $f$: roughly, an increasing function that can be computed by a TM in time $f(n) + n$

For $f(n) \geq n$ a proper complexity function, we have

$$\text{TIME}(f(n)) \text{ is a proper subset of } \text{TIME}((f(2n+1))^3).$$

It follows that P is a proper subset of EXPTIME.

***Proof sketch:*** consider "time-bounded halting language"

$$H_f := \{\langle M, w\rangle \;:\; M \text{ accepts } w \text{ after at most } f(|w|) \text{ steps}\}$$

$H_f$ belongs to $\text{TIME}((f(n))^3)$: construct a universal TM that uses "quadratic overhead" to simulate a step of $\mathcal{M}$. (The theorem can be strengthened by using a more economical UTM, but as stated it's good enough for P$\subsetneq$EXPTIME.)

Next point: $H_f \notin \text{TIME}(f(\lfloor \frac{n}{2} \rfloor))$.

# Time Hierarchy theorem

Reminder: $H_f := \{\langle M, w\rangle \ : \ M$ accepts $w$ after $\leq f(|w|)$ steps$\}$

To prove $H_f \notin \mathrm{TIME}(f(\lfloor \frac{n}{2} \rfloor))$:

- Suppose $M_{H_f}$ decides $H_f$ in time $f(\lfloor \frac{n}{2} \rfloor)$.
- Define "diagonalising" machine:
  $D_f(M) \ : \ \text{if } M_{H_f}(\langle M, M\rangle) =$ "yes" then "no" else "yes"
- Does $D_f$ accept its own description? Contradiction!

**Corollary**

P *is a proper subset of* EXPTIME.

# Space Hierarchy Theorem

***Theorem.*** (Space Hierarchy Theorem)

Let $S, s : \mathbb{N} \rightarrow \mathbb{N}$ be functions such that

1. $S$ is space constructible, and
2. $S(n) \geq n$,
3. $s = o(S)$.

Then $\mathrm{DSPACE}(s) \subsetneq \mathrm{DSPACE}(S)$.

Reminder: item 3 means that $\lim_{n \rightarrow \infty}(s(n)/S(n)) = 0$.

Proof later, but note consequences: LOGSPACE is a proper subset of PSPACE, is proper subset of EXPSPACE