Computational Complexity; slides 2, HT 2023 nondeterminism, Cook-Levin theorem, reductions, NP-complete problems

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Nondeterministic Turing Machines

Definition.

A non-deterministic (1-tape) Turing machine is a 6-tuple $(Q, \Sigma, \Gamma, \Delta, q_0, F)$ where

- Q is a finite set of states
- Σ is a finite alphabet of symbols
- $\Gamma \supseteq \Sigma \cup \{\square\}$ is a finite alphabet of symbols
- $\Delta \subseteq (Q \setminus F) \times \Gamma \times Q \times \Gamma \times \{-1, 0, 1\}$ transition **relation**
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is a set of final states

As before, we assume $\Sigma := \{0, 1\}$ and $\Gamma := \Sigma \cup \{\square\}$.

The computation of a non-deterministic Turing machine $M = (Q, \Sigma, \Gamma, \Delta, q_0, F)$ on input w is a "computation tree"
analogy with NFA, (N)PDA

Non-Deterministic Turing Acceptor

Computation path:

Any path from the start configuration to a stop configuration in the configuration tree.

accepting path: the stop configuration is in an accepting state.
(also called an accepting run)

rejecting path otherwise

Language accepted by an NTM M:

$$\mathcal{L}(M) := \{w \in \Sigma^* : \text{ there } \textbf{exists} \text{ an accepting path of } M \text{ on } w\}$$

Simulation: Variants of definition of NTM can be simulated with polynomial (runtime) overhead.
Can also simulate with deterministic TM, but not in poly-time

From **P** to **NP**

NP: languages accepted by NTM in polynomially-many steps; equivalently, decision problems whose yes-instances are accepted by (poly-time) NTM

- e.g. 3-SAT, 3-COLOURABILITY, TSP, SAT, etc
- No polynomial time algorithms for these problems are known
- but are in **NP**

"Guess and test": generic **NP** algorithm. As for **P**, pseudocode algorithms are convenient, but don't forget underlying TM model

Non-Deterministic Complexity Classes

- Time classes:
  - NP (a.k.a. NPTIME) $:= \bigcup_{d \in \mathbb{N}} \text{NTIME}(n^d)$
  - NEXPTIME $:= \bigcup_{d \in \mathbb{N}} \text{NTIME}(2^{n^d})$
- Space classes:
  - NLOGSPACE $:= \bigcup_{d \in \mathbb{N}} \text{NSPACE}(d \log n)$
  - NPSPACE $:= \bigcup_{d \in \mathbb{N}} \text{NSPACE}(n^d)$
  - NEXPSPACE $:= \bigcup_{d \in \mathbb{N}} \text{NSPACE}(2^{n^d})$

where NTIME(T) (etc.) means what you think it means. Note that all accepting/non-accepting computations of a NTIME(T) TM should have length at most T

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We have:

$$\text{P} \subseteq \text{NP} \subseteq \text{PSPACE} \subseteq \text{NPSPACE} \subseteq \text{EXP}$$

(hierarchy: sort-of good news)

More Examples of Problems in **NP**

COMPOSITE (non-prime) NUMBER

Input: A positive integer $n > 1$

Question: Are there integers $u, v > 1$ such that $u \cdot v = n$?

SUBSET SUM

Input: A collection of positive integers $S := \{a_1, \ldots, a_k\}$ and a target integer t.

Question: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Deterministic vs. Non-Deterministic Time

Clearly, $\mathsf{P} \subseteq \mathsf{NP}$.

Question: The question $\mathsf{P} \stackrel{?}{=} \mathsf{NP}$ is among the most important open problems in computer science and mathematics.

- It is equivalent to determining whether or not the existence of a short solution guarantees an efficient way of finding it.
- Most people are convinced that $\mathsf{P} \neq \mathsf{NP}$

  But after $\sim$50 years of effort there is still no proof.
- Resolving the question (either way) would win a prize of \$1 million – see

  `http://www.claymath.org/millennium-problems/`

poly-time reductions amongst NP problems

Recall polynomial-time reduction.

- $\mathcal{A} \leq_p \mathcal{B}$: "$\mathcal{A}$ is poly-time reducible to $\mathcal{B}$": $\mathcal{B}$ is (in a sense) at least as hard as $\mathcal{A}$
- If we have $\mathcal{A} \leq_p \mathcal{B}$ and $\mathcal{B} \leq_p \mathcal{A}$, we can say $\mathcal{A}$ and $\mathcal{B}$ are "inter-reducible", or "polynomial-time equivalent
- Equivalence classes are partially ordered by the reduction relation.
- Problems in the maximal class are called complete for NP (we will see that there is indeed a maximal class!)

NP-Hardness and NP-Completeness

Definition.

1. A language $\mathcal{H}$ is NP-hard, if $\mathcal{L} \leq_p \mathcal{H}$ for every language $\mathcal{L} \in \mathrm{NP}$.
2. A language $\mathcal{C}$ is NP-complete, if $\mathcal{C}$ is NP-hard and $\mathcal{C} \in \mathrm{NP}$.

NP-Completeness:

- NP-complete problems are the hardest problems in NP.
- They are all equally difficult – an efficient solution to one would solve them all.

Lemma. If $\mathcal{L}$ is NP-hard and $\mathcal{L} \leq_p \mathcal{L}'$, then $\mathcal{L}'$ is NP-hard as well.

Proving NP-Completeness

To show that $\mathcal{L}$ is NP-complete, we must show that every language in NP can be reduced to $\mathcal{L}$ in polynomial time.

But if we know one NP-complete language $\mathcal{C}$, we can show that another language $\mathcal{L}'$ is NP-complete just by showing that

- $\mathcal{C} \leq_p \mathcal{L}'$
- $\mathcal{L}' \in \mathrm{NP}$

Hence: The problem is to find the first one (c.f. undecidable problems)

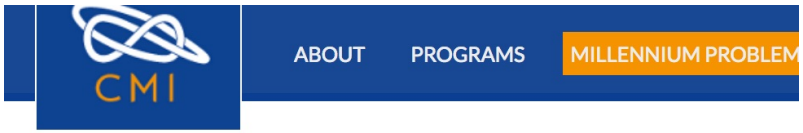
$\rightsquigarrow$ Next: the Cook-Levin Theorem

2 problems involving propositional logic

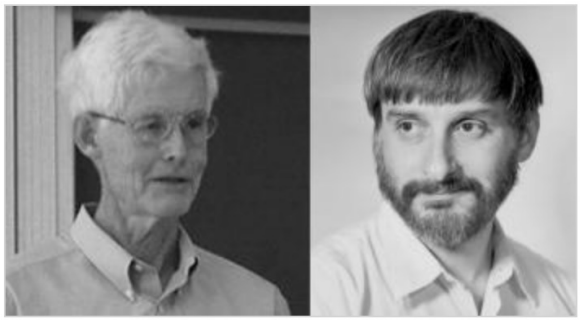
1. Given a formula φ on variables $x_1, \ldots x_n$, and values for those variables, derive the value of φ — easy!
2. Search for values for $x_1, \ldots, x_n$ that make φ evaluate to TRUE — naive algorithm is exponential: 2^n vectors of truth assignments.

Cook's Theorem (1971)
or, Cook-Levin Theorem

The second of these, called SAT, is **NP**-complete.



P vs NP Problem



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Stephen Cook, Leonid Levin

The challenge of solving boolean formulae

(side note:)
There's a HUGE theory literature on the computational challenge of solving various classes of syntactically restricted classes of boolean formulae, also circuits.
Likewise much has been written about their relative *expressive power*
SAT-solver: software that solves input instances of SAT — OK, so it's worst-case exponential, but aim to solve instances that arise in practice.

- "truth table" approach: clearly exponential
- DPLL algorithm; resolution: worst-case exponential, often fast in practice

Next: proof of Cook-Levin, then NP in terms of certificates, verifiers; co-NP

Reducing an **NP** problem to SAT

Goal: fixing non-deterministic TM M, integer k, given w create in poly-time a propositional formula $\textbf{CodesAcceptRun}_M(w)$ that is satisfied by assignments that code an n^k length accepting run of M on w (where $n = |w|$)

Idea: introduce propositional variables

- $HasSymbol_{i,j}(a)$: "at time i, tape has letter a at location j"
- $HasHead_{i,j}(q)$: "at time i, TM is in location j, state q"

We'll assume M has "stay put" transitions for which it can change tape contents; R and L moves don't change tape. Assume also that to accept, M goes to LHS of tape and prints special symbol.

M has a "configuration table"

Tape space j

Time i

	1	2	$\cdots$	n^k
1	(q_0, w_1)	w_2	$\cdots$	
2	w_1'	(q_1, w_2)		
$\vdots$				
$\vdots$				
n^k				

This corresponds to a run where
$HasSymbol_{1,1}(w_1)$
$HasHead_{1,1}(q_0)$
$HasSymbol_{1,2}(w_2)$
$HasSymbol_{2,1}(w_1')$
$HasSymbol_{2,2}(w_2)$
$HasHead_{2,2}(q_1)$
...are true
(Others, e.g. $HasHead_{1,2}(q_0)$ are false)

Idea: the search for "correct" non-deterministic choices for M shall correspond to search for satisfying assignment for $\mathbf{CodesAcceptRun}_M(w)$.
$\mathbf{CodesAcceptRun}_M(w)$ shall be a conjunction of *clauses*.

Getting started

To write the formula **CodesAcceptRun**$_M(w)$, let's start by writing:

$$HasSymbol_{1,j}(w_j)$$

for each $j = 1, ..., |w|$, where w_j is the j-th letter of input w, also

$$\neg HasSymbol_{1,j}(a)$$

for any a where a is *not* the j-th letter of w.

Similarly

$$HasHead_{1,1}(q_0)$$

says M is in state q_0 at time 1, location 1. Add a bunch of negated "HasHead" variables.

TM head "sanity clauses"

Include the following:

$$HasHead_{i,j}(q) \Rightarrow \neg HasHead_{i,j'}(q')$$

...for all states q, q', for all i, j, j' with $j \neq j'$.

Moving head clauses: leftward-moving State

Leftward moving state. If M has transition rule $(q, a) \to \{(q_1, a, L), (q_2, a, L)\}$ then we write:

$$HasHead_{i,j}(q) \Rightarrow [HasHead_{i+1,j-1}(q_1) \vee HasHead_{i+1,j-1}(q_2)]$$

Write the above for all $i, j \in \{1, 2, 3, \ldots, n^k\}$.

Tape space

Time	1	$\cdots$	$j-1$	j	$\cdots$	n^k
1						
i			w_2	(q, a)		
$i+1$			$(q_1/q_2, w_2)$	a		
$\vdots$						
n^k						

Moving head clauses: Rightward-moving State or Leftward-moving State

For every rightward or leftward state q, for every a we add the clause:

$$HasSymbol_{i,j}(a) \wedge HasHead_{i,j}(q) \Rightarrow HasSymbol_{i+1,j}(a)$$

meaning: if the head is at place j at step i and we are in a rightward- or leftward moving state, symbol in place j at step $i+1$ is the same.

Tape space

	1	$\cdots$	j		$\cdots$	n^k
1						
Time i			(q, a)	w_2	$\cdots$	
$i+1$			a	(q_1, w_2)	$\cdots$	
$\vdots$						
n^k						

Moving head clauses: stay-same-place state

For every stay-and-write state q, if we have (say) transition $(q, w_0) \to \{(q_1, w_1, Stay), (q_2, w_1, Stay)\}$ then we add:

$$HasSymbol_{i,j}(w_0) \wedge HasHead_{i,j}(q) \Rightarrow HasSymbol_{i+1,j}(w_1)$$

(new symbol is written) and also:

$$HasHead_{i,j}(q) \Rightarrow [HasHead_{i+1,j}(q_1) \vee HasHead_{i+1,j}(q_2)]$$

(head does not move, although state may change)

	1	$\cdots$	j	$\cdots$	$\cdots$	n^k
1						
i			(q, w_0)	$\cdots$		
$i+1$			(q_1, w_1)	$\cdots$		
$\vdots$						
n^k						

More sub-formulae for Transitions: away from head clauses

Clauses stating that if the head is not close to place j at time i, then symbol in place j is unchanged in the next time.
For any state q and symbol w_3, any $i \leq n_k$ and number h in a certain range we have

$$HasHead_{i,j}(q) \wedge HasSymbol_{i,j+h}(w_3) \Rightarrow HasSymbol_{i+1,j+h}(w_3)$$

If q is a rightward-moving state, do this for $n^k - j \geq h \geq 2$ **and** $-(j-1) \leq h < 0$
If q is a leftward-moving state do this for $n^k - j \geq h \geq 1$ **and** $-(j-1) \leq h < -1$

If q is a stay put state, do this for $h \neq 0$

	1	$\cdots$	j	$\cdots$	$j+h$	$\cdots$	n^k
1							
i			(q, w_0)	$\cdots$	w_3		
$i+1$			(q_1, w_1)	$\cdots$	w_3		
$\vdots$							
n^k							

Reducing an **NP** problem to SAT (conclusion)

Final configuration clause: let's assume that whenever M accepts, it accepts at LHS of tape and prints special symbol $\square$ there

$$HasSymbol_{n^k,1}(\square) \wedge HasHead_{n^k,1}(q_{accept})$$

At time n^k, head is at the beginning and state is accepting with special termination symbol

	1	$\cdots$		$\cdots$	n^k
1	q_0	w_1	w_2	$\cdots$	
$\vdots$					
n^k	$(q_{accept}, \square)$				

Proof of the construction (overview, not details)

We started with M, w, constructed formula **CodesAcceptRun**$_M(w)$. Two items to establish:

- **CodesAcceptRun**$_M(w)$ is constructed in polynomial time
- **CodesAcceptRun**$_M(w)$ is satisfiable iff M accepts w

For the first item, as I pointed out, many clauses were added, but polynomially-many. (large polynomial blow-up may be counter-intuitive)

For the second, the main point is that an accepting run gives rise to a satisfying assignment of the formula (and vice versa) in a direct way, according to our understanding of what the **HasHead** and **HasSymbol** variables mean, for runs of M.

Certificates

Every yes-instance of such problems has a short and easily checkable certificate that proves it is a yes-instance.

- SAT – a satisfying assignment
- k-COLOURABILITY – a k-colouring
- HAMILTONIAN CIRCUIT – a Hamiltonian circuit
- TSP (decision-problem version) – a round trip (i.e. permutation)

Verifiers

Definition.

1. A Turing acceptor M which halts on all inputs is called a verifier for language $\mathcal{L}$ if

$$\mathcal{L} = \{w : M \text{ accepts } \langle w, c \rangle \text{ for some string } c\}$$

   The string c is called a certificate (or witness) for w.

2. A polynomial time verifier for $\mathcal{L}$ is a polynomially time bounded Turing acceptor M such that

$$\mathcal{L} = \{w : M \text{ accepts } \langle w, c \rangle \text{ for some string } c \text{ with } |c| \leq p(|w|)\}$$

   for some fixed polynomial $p(n)$.

All problems for the previous slide have verifiers that run in polynomial time.

Equivalent definition of **NP**

The class of languages that have polynomial-time verifiers

Examples.

- SAT is in **NP**

  For any formula that can be satisfied, the satisfying assignment can be used as a certificate.

  It can be verified in polynomial time that the assignment satisfies the formula.

- k-COLOURABILITY is in **NP**

  For any graph that can be coloured, the colouring can be used as a certificate.

  It can be verified in polynomial time that the colouring is a proper colouring.

A Problem (probably) not in **NP**

NO HAMILTONIAN CYCLE

Input: A graph G

Question: Is it true that G has no Hamiltonian cycle?

Note. Whereas it is easy to certify that a graph has a Hamiltonian cycle, there does not seem to be a (general purpose) certificate that it has not.

co-NP

co-NP problem: complement of an NP problem
In a co-NP problem, no-instances have (concise) certificates
Believed that NP is not equal to co-NP

The following result justifies guess and test approach to establishing membership of NP:

NP as languages having concise certificates

Theorem. NP as just defined, is languages having concise certificates

Proof. Suppose $\mathcal{L} \in$ NP.

Hence, there is an NTM M such that

$$w \in \mathcal{L} \iff \text{there is an accepting run of } M \text{ of length} \leq n^k$$

for some k. This path can be used as a certificate for w

(A DTM can check in polynomial time that a candidate for a certificate is a valid accepting computation path.)

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for some k. This path can be used as a certificate for w

(A DTM can check in polynomial time that a candidate for a certificate is a valid accepting computation path.)

Conversely: If $\mathcal{L}$ has a polynomial-time verifier M, say of length at most n^k,

then we can construct an NTM M^* deciding $\mathcal{L}$ as follows:

1. M^* guesses a string of length $\leq n^k$
2. M^* checks in deterministic polynomial-time if this is a certificate.

NP-Completeness Proofs

To prove that a problem $\mathcal{X}$ is NP-complete, we now just have to perform two steps:

1. Show that $\mathcal{X} \in$ NP usually easy
2. Find a known NP-complete problem $\mathcal{X}'$ and reduce $\mathcal{X}' \leq_p \mathcal{X}$.
   the FUN part

Thousands of problem have now been shown to be NP-complete (See Garey and Johnson for an early survey); Karp 1972, "reducibility among combinatorial problems" kicked-off this work

A relevant quote

Proving NP-completeness results is an important ingredient of our methodology for studying computational problems. It is also something of an art form.

start of chapter 9 of Papadimitriou's textbook

NP-Completeness and some follow-up topic

Coming up next:

- Some NP reductions
- dealing with NP-hardness
- NP vs. co-NP
- Search and optimisation problems, FNP, reducibility

NP-Completeness Proofs

Coming up next: some examples.

CNF-SAT$\leq_p$3-SAT (BTW, goes back to Cook's paper)

$\{X_1, X_2, \ldots, X_n\} \mapsto \{X_1, X_2, X_{new}\}, \{\neg X_{new}, X_3, \ldots, X_n\}$
repeat until clause lengths ≤ 3

3-SAT is a more convenient starting-point of reductions than unrestricted SAT.

- 3-SAT$\leq_p$INTEGER PROGRAMMING (simple but important)
- 3-SAT$\leq_p$IND SET$\leq_p$CLIQUE
- 3-SAT$\leq_p$DIRECTED HAMILTONIAN PATH
- 3-SAT$\leq_p$SUBSET SUM$\leq_p$KNAPSACK

NP-Completeness of INTEGER PROGRAMMING

IP: Input: a set of linear constraints, Question: can we satisfy them with integer values?
3-SAT$\leq_p$IP

X_i in 3-SAT instance $\mapsto$ x_i in IP instance.

$\forall i$, include constraints $0 \leq x_i \leq 1$
(idea: 0 means F, 1 means T)

$\{X_i, X_j, X_k\} \mapsto x_i + x_j + x_k \geq 1$
$\{X_i, \neg X_j, X_k\} \mapsto x_i + (1 - x_j) + x_k \geq 1$
and similarly for more than one negated literal

Example

$\{\{X_1, X_2, X_3\}, \{\neg X_1, \neg X_2, X_4\}\}$
is reduced to the following IP:
$0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$
$x_1 + x_2 + x_3 \geq 1$, $(1 - x_1) + (1 - x_2) + x_4 \geq 1$

NP-Completeness of CLIQUE

CLIQUE: Given G, k, does G contain a clique of order $\geq k$?

Theorem

CLIQUE *is* NP-*complete.*

It's convenient to reduce from 3-SAT to IND SET and from there to CLIQUE.

3-SAT$\leq_p$IND SET: each clause of a 3-SAT instance becomes a triangle in the graph. Label vertices with the literals.
If the formula had m clauses, the graph now has $3m$ vertices. Is there an independent set of size m?

(idea: choice of vertex in each triangle corresponds to choice of literal that gets satisfied)

Add new edges between any pair of vertices labelled by a variable X_i and its negation $\neg X_i$.
Any n-independent set corresponds to a satisfying assignment.

NP-Completeness of DIRECTED HAMILTONIAN PATH

DIRECTED HAMILTONIAN PATH

Input: G: directed graph.

Problem: Is there a directed path in G containing every vertex exactly once?

Theorem

DIRECTED HAMILTONIAN PATH *is* NP-*complete*

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Input: G: directed graph.

Problem: Is there a directed path in G containing every vertex exactly once?

Theorem

DIRECTED HAMILTONIAN PATH *is* NP*-complete*

Proof.

1. DIRECTED HAMILTONIAN PATH $\in$ NP.

   Take the path to be the certificate.

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Theorem

DIRECTED HAMILTONIAN PATH *is* NP-*complete*

Proof.

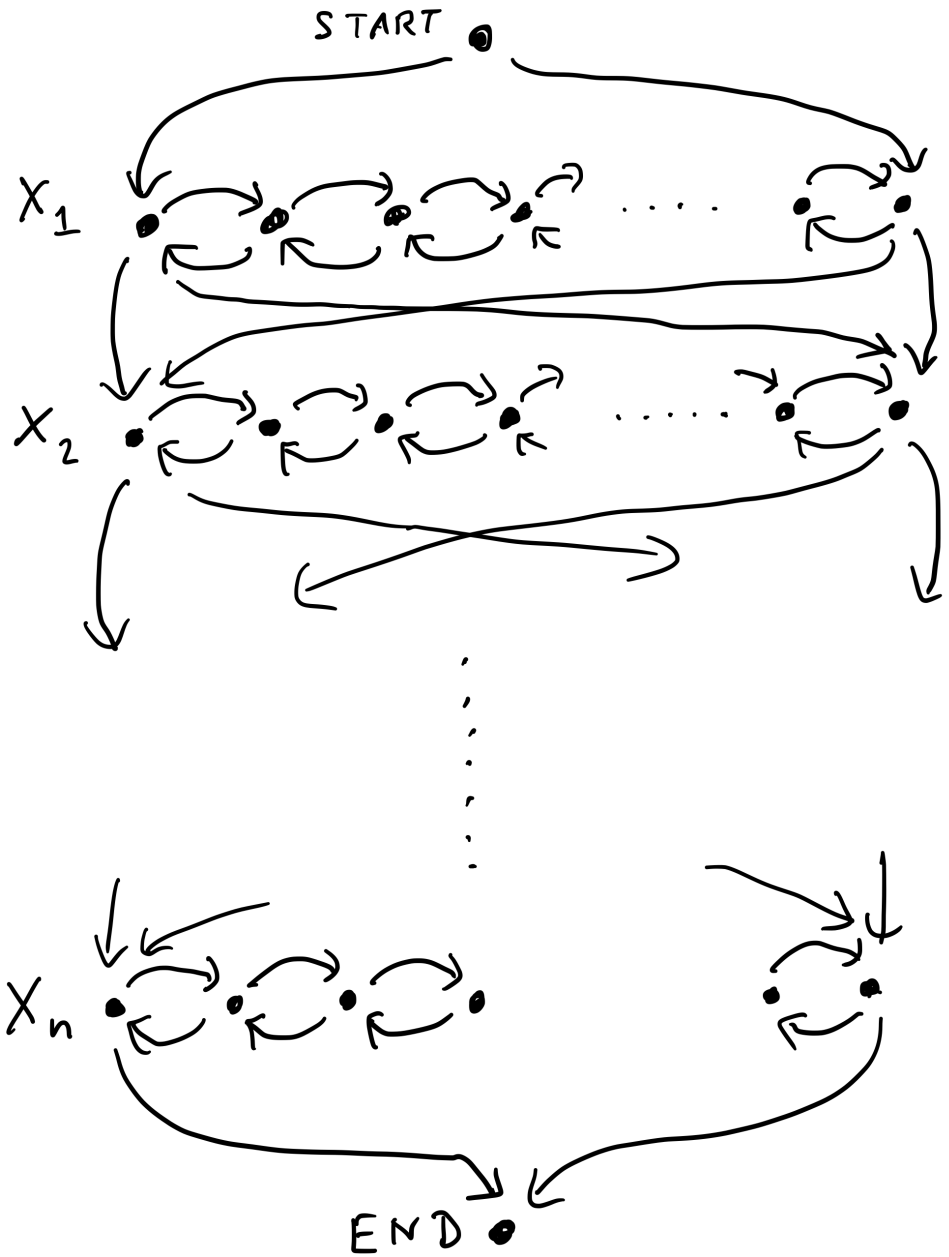
1. DIRECTED HAMILTONIAN PATH $\in$ NP.

   Take the path to be the certificate.

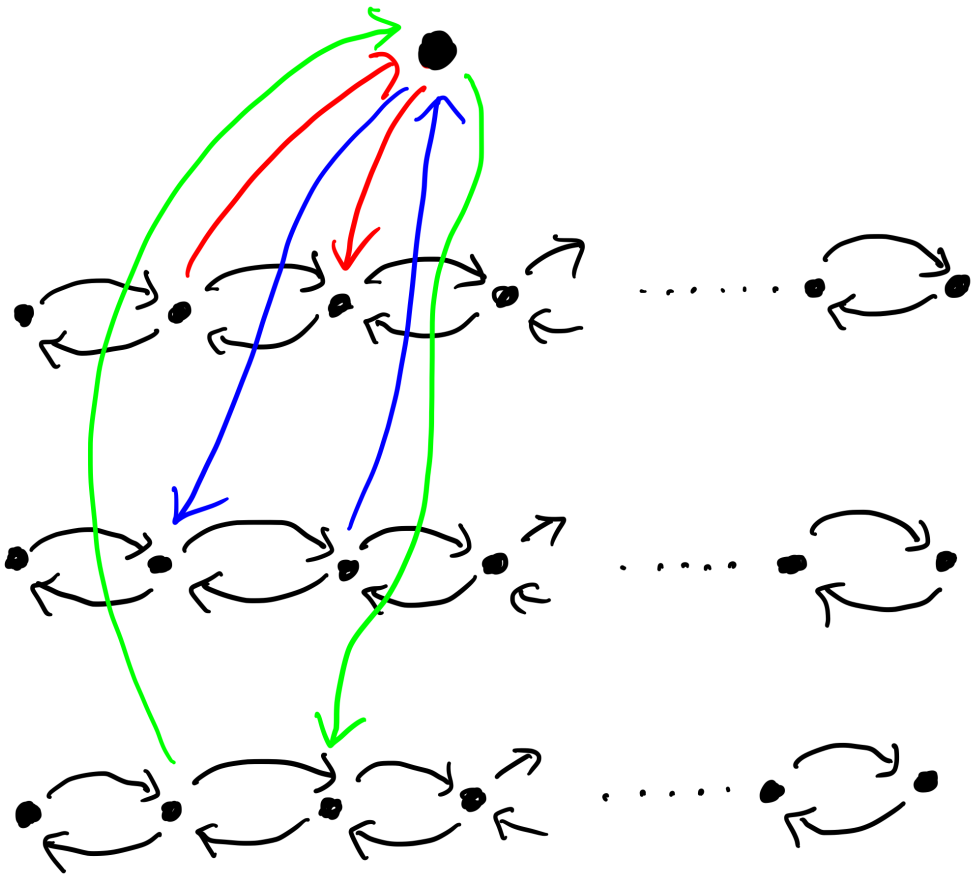
2. DIRECTED HAMILTONIAN PATH is NP-hard.

   3-SAT $\leq_p$ DIRECTED HAMILTONIAN PATH

from 3-SAT to DIRECTED HAMILTONIAN PATH



from 3-SAT to DIRECTED HAMILTONIAN PATH



Digression: how to design reductions

Show that problem $\mathcal{X}$ (DIR. HAMILTONIAN PATH) is NP-hard.

Which problem to reduce to $\mathcal{X}$:

- Arguably, the most important part is to decide where to start from; e.g. which problem to reduce to DIRECTED HAMILTONIAN PATH — something graph-theoretic?
- Considerations:
  - Is there an NP-complete problem similar to $\mathcal{X}$?
    (E.g. CLIQUE and IND SET)
  - It is not always beneficial to choose a problem of the same type
    (E.g. reducing a graph problem to a graph problem)
    - For instance, CLIQUE, IND SET are "local" problems (is there a set of vertices inducing some structure)
    - Hamiltonian Path is a "global" problem
      (find a structure containing all vertices)

How to design the reduction:

- Does your problem come from an optimisation problem?
  If so: a maximisation problem? a minimisation problem?

NP-Completeness of SUBSET SUM

SUBSET SUM

Input: A collection of positive integers $S := \{a_1, \ldots, a_k\}$ and a target integer t.

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Theorem. SUBSET SUM is NP-complete

Proof.

1. SUBSET SUM $\in$ NP.

   Take T to be the certificate.

2. SUBSET SUM is NP-hard.

   CNF-SAT $\leq_p$ SUBSET SUM (example next slide)

Example

$$(X_1 \vee X_2 \vee X_3) \wedge (\neg X_1 \vee \neg X_4) \wedge (X_4 \vee X_5 \vee \neg X_2 \vee \neg X_3)$$

		X_1	X_2	X_3	X_4	X_5	C_1	C_2	C_3
t_1	=	1	0	0	0	0	1	0	0
f_1	=	1	0	0	0	0	0	1	0
t_2	=		1	0	0	0	1	0	0
f_2	=		1	0	0	0	0	0	1
t_3	=			1	0	0	1	0	0
f_3	=			1	0	0	0	0	1
t_4	=				1	0	0	0	1
f_4	=				1	0	0	1	0
t_5	=					1	0	0	1
f_5	=					1	0	0	0
$m_{1,1}$	=						1	0	0
$m_{1,2}$	=						1	0	0
$m_{2,1}$	=						0	1	0
$m_{3,1}$	=						0	0	1
$m_{3,2}$	=						0	0	1
$m_{3,3}$	=						0	0	1
t	=	1	1	1	1	1	3	2	4

SAT $\leq_p$ SUBSET SUM (the general construction)

Given: $\varphi := C_1 \wedge \cdots \wedge C_k$ in conjunctive normal form.

(for numbers in base 10: at most 9 literals per clause)

Let $X_1, \ldots, X_n$ be the variables in φ. For each X_i let

$t_i := a_1 \ldots a_n c_1 \ldots c_k$ *where*

$$a_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$c_j := \begin{cases} 1 & X_i \text{ occurs in } C_j \\ 0 & \text{otherwise} \end{cases}$$

$f_i := a_1 \ldots a_n c_1 \ldots c_k$ *where*

$$a_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$c_j := \begin{cases} 1 & \neg X_i \text{ occurs in } C_j \\ 0 & \text{otherwise} \end{cases}$$

SAT $\leq_p$ SUBSET SUM (the general construction)

Further, for each clause C_i take $r := |C_i| - 1$ integers $m_{i,1}, \ldots, m_{i,r}$

where $m_{i,j} := c_i \ldots c_k$ with $c_j := \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

Definition of S: Let

$$S := \{t_i, f_i : 1 \leq i \leq n\} \cup \{m_{i,j} : 1 \leq i \leq k, \quad 1 \leq j \leq |C_i| - 1\}$$

Target: Finally, choose as target

$$t := a_1 \ldots a_n c_1 \ldots c_k \text{ where } a_i := 1 \text{ and } c_i := |C_i|$$

Claim: There is $T \subseteq S$ with $\sum_{a_i \in T} a_i = t$ iff φ is satisfiable.

NP-Completeness of SUBSET SUM

Let $\varphi := \bigwedge C_i$ C_i: clauses

Show. If φ is satisfiable, then there is $T \subseteq S$ with $\sum_{s \in T} s = t$.

Let β be a satisfying assigment for φ

Set $T_1 := \{t_i : \beta(X_i) = 1 \quad 1 \leq i \leq m\} \cup$
$\{f_i : \beta(X_i) = 0 \quad 1 \leq i \leq m\}$

Further, for each clause C_i let r_i be the number of satisfied literals in C_i

(with resp. to β).

Set $T_2 := \{m_{i,j} : 1 \leq i \leq k, \quad 1 \leq j \leq |C_i| - r_i\}$

and define $T := T_1 \cup T_2$.

It follows: $\sum_{s \in T} s = t$

NP-Completeness of SUBSET SUM

Show. If there is $T \subseteq S$ with $\sum_{s \in T} s = t$, then φ is satisfiable.

Let $T \subseteq S$ s.th. $\sum_{s \in T} s = t$

Define $\beta(X_i) = \begin{cases} 1 & \text{if } t_i \in T \\ 0 & \text{if } f_i \in T \end{cases}$

This is well defined as for all i: $t_i \in T$ or $f_i \in T$ but not both.

Further, for each clause, there must be one literal set to 1 as for all i, the $m_{i,j} : m_{i,j} \in S$ do not sum up to the number of literals in the clause.

NP-completeness of KNAPSACK

KNAPSACK

Input: A set $I := \{1, \ldots, n\}$ of items
each of value v_i and weight w_i for $1 \leq i \leq n$
target value t weight limit ℓ

Problem: Is there $T \subseteq I$ such that

- $\sum_{i \in T} v_i \geq t$
- $\sum_{i \in T} w_i \leq \ell$

Theorem. KNAPSACK is NP-complete

1. KNAPSACK $\in$ NP
   Take T as certificate.
2. KNAPSACK is NP-hard
   By reduction SUBSET SUM $\leq_p$ KNAPSACK

Key point: KNAPSACK is "more general/expressive" than SUBSET SUM

SUBSET SUM $\leq_p$ KNAPSACK (the details)

reminder: SUBSET SUM

Given: $S := \{a_1, \ldots, a_n\}$ collection of positive integers

t target integer

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

SUBSET SUM $\leq_p$ KNAPSACK (the details)

reminder: SUBSET SUM

Given: $S := \{a_1, \ldots, a_n\}$ collection of positive integers
t target integer

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Reduction: From this input to SUBSET SUM construct

- $I := \{1, \ldots, n\}$: set of items
- $v_i = w_i = a_i$ for all $1 \leq i \leq n$
- target value $t' := t$ weight limit $\ell := t$

SUBSET SUM $\leq_p$ KNAPSACK (the details)

reminder: SUBSET SUM

Given: $S := \{a_1, \ldots, a_n\}$ collection of positive integers
t target integer

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Reduction: From this input to SUBSET SUM construct

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- $v_i = w_i = a_i$ for all $1 \leq i \leq n$
- target value $t' := t$ weight limit $\ell := t$

Clearly: For every $T \subseteq S$

$$\sum_{a_i \in T} a_i = t \quad \Longleftrightarrow \quad \begin{array}{ll} \sum_{a_i \in T} v_i \geq t' & = t \\ \sum_{a_i \in T} w_i \leq \ell & = t \end{array}$$

Hence: The reduction is correct and in polynomial time.

Pseudo-Polynomial Time

KNAPSACK can be solved in time $\mathcal{O}(n\ell)$ using dynamic programming
(recall ℓ is weight limit, n is number of items)
... but, ℓ can be exponential in the input description length!

Pseudo-Polynomial Time: Algorithms polynomial in the maximum of the input length and the value of numbers occurring in the input.

If KNAPSACK is restricted to instances with $\ell \leq p(n)$ for some polynomial p, then we obtain a problem in P.

Equivalently: KNAPSACK is in polynomial time for unary encoding of numbers.

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Strong NP-completeness: Problems (e.g. CLIQUE, SAT) which remain NP-complete even if all numbers are bounded by a polynomial in the input length
(equivalently, for unary encoding of numbers).

digression: dealing with NP-hardness

- Maybe a pseudo-polynomial time algorithm is OK
- Move from exact to approximate optimisation: it may be hard to find optimal solution, but finding one within fact 2 (say) of optimal of optimal, is in P.
- fixed-parameter tractability
- model data as noisy (e.g. in smoothed analysis)

NP and co-NP

Notation. For a language $\mathcal{L} \subseteq \Sigma^*$ let $\overline{\mathcal{L}} := \Sigma^* \setminus \mathcal{L}$ be its complement.

Definition.

If $\mathcal{C}$ is a complexity class, we define

$$\text{co-}\mathcal{C} := \{\mathcal{L} : \overline{\mathcal{L}} \in \mathcal{C}\}.$$

coNP: In particular, $\text{co-NP} := \{\mathcal{L} : \overline{\mathcal{L}} \in \text{NP}\}$

A problem belongs to co-NP, if no-instances have short certificates.

co-NP-completeness

Examples of problems in co-NP:

NO HAMILTONIAN CYCLE

Given: Graph G

Question: Is it true that G contains no Hamiltonian cycle?

TAUTOLOGY

Given: Formula φ

Question: Is φ a tautology, i.e. satisfied by all assignments?

co-NP-completeness

Examples of problems in co-NP:

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TAUTOLOGY
Given: Formula φ
Question: Is φ a tautology, i.e. satisfied by all assignments?

Definition. A language $\mathcal{C} \in$ co-NP is *co-NP-complete*, if $\mathcal{L} \leq_p \mathcal{C}$ for all $\mathcal{L} \in$ co-NP.

TAUTOLOGY is co-NP-complete: any reduction from an NP problem to SAT can convert to a reduction from a co-NP problem to TAUTOLOGY

co-NP-complete decision problems are as hard as NP-complete ones.

P, NP, and co-NP

Proposition.

1. P = co-P
2. Hence, P $\subseteq$ NP $\cap$ co-NP

Question:

- NP = co-NP?

  Most people do not think so (c.f. research in propositional proof theory).

- P = NP $\cap$ co-NP?

  Again, most people do not think so.

Later: Ladner's theorem: assuming P$\neq$NP, there are "NP-intermediate" problems.

Search versus decision

Problem 1: boolean formula φ — is φ satisfiable?

Problem 2: Given a formula φ, find a satisfiable assignment, or answer "no".

Problem 2 is at least as hard.
But — we can say: "problem 2 is no harder than NP"; solve problem 2 with **oracle** for problem 1.

"oracle": an imaginary black-box that supports queries to a computational problem: given an input, will (in one step) tell you correct output.

FNP, reducing search to decision

FNP: problems of computing a function that can be checked in polynomial time: find a certificate, not just answer "yes"

An FNP problem comprises a *polynomially balanced relation* R for which $R(x, y)$ can be checked in time polynomial in $|x|$, $|y|$.
Given x, search for y with $R(x, y)$.
Note: it's asking too much to solve a NP search problem X using a single decision oracle (why?).
But can solve X using multiple oracle calls to corresponding decision problem.

So, "FNP is as hard as NP"

Reducibility amongst FNP problems

Informally, $X \leq_p Y$ means: given an oracle for problem Y, can reconstruct a solution to X.

In detail, reduction needs 2 functions f, g, where f maps instances of X to instances of Y, and g maps solutions of Y to solutions of X.
If X and Y correspond with relations R_1 and R_2 respectively, want

$$(x, g(z)) \in R_1 \quad \text{iff} \quad (f(x), z) \in R_2.$$

I'll come back to this in the final lecture. Meanwhile, think about how to compare FACTORING in base-2 with FACTORING in base-10.

FSAT problem: given a boolean formula, compute a satisfying assignment.
FSAT is FNP-complete. FACTORING seems to be hard, but is unlikely to be FNP-complete!

Optimisation

"Is there a k-clique" is (in a sense) equally hard as "find a k-clique"

"What's the size of the largest clique?" is (in a sense) harder!

If we told the answer is some value k, an NP machine can verify k-clique(s) exist

Need also a co-NP machine to verify: no $k+1$-clique exists!

NP, or co-NP alone, don't seem to be sufficient, more later.

In exercises later, will make a start at classifying problems like this

Definition: For complexity classes A and B let A^B denote problems solved by an A-machine with oracle access to B.

As a start, we can put the problem in P^{NP}