

## MULTI-UNIT BAYESIAN AUCTION WITH DEMAND OR BUDGET CONSTRAINTS

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We consider the problem of revenue maximization on multi-unit auctions where items are distinguished by their relative values; any pair of items has the same ratio of values to all buyers. As is common in the study of revenue maximizing problems, we assume that buyers' valuations are drawn from public known distributions and they have additive valuations for multiple items. Our problem is well motivated by sponsored search auctions, which made money for Google and Yahoo! in practice. In this auction, each advertiser bids an amount  $b_i$  to compete for ad slots on a web page. The value of each ad slot corresponds to its click-through-rate, and each buyer has her own per-click valuations, which is her private information. Obviously, a strategic bidder may bid an amount that is different with her true valuation to improve her utility. Our goal is to design truthful mechanisms avoiding this misreporting.

We develop the optimal (with maximum revenue) truthful auction for a *relaxed demand* model (where each buyer  $i$  wants at most  $d_i$  items) and a *sharp demand* model (where buyer  $i$  wants exactly  $d_i$  items). We also find an auction that always guarantees at least half of the revenue of the optimal auction when the buyers are budget constrained. Moreover, all of the auctions we design can be computed efficiently, that is, in polynomial time.

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### 1. INTRODUCTION

Internet markets have opened up many opportunities for applications of different markets and their pricing models. Search engine advertising, as an example, makes the matching market model an everyday's practice: advertisement slots of Google and Yahoo! are created as products for advertisers (the buyers) who want to display their ads to users searching keywords related to their business. Each such ad slot may be of different importance, which is measured by the click-through-rates (CTR), the average number of clicks on the ad placed at the slot for a unit time. Slots with higher CTR are more likely to be clicked by customers. On the other hand, users' interest expressed by a click on an ad may have different values to different advertisers. Combining the two major factors, we have a standard model of a sponsored search market (Edelman, Ostrovsky, and Schwarz 2007; Varian 2007). Besides the standard sponsored search auctions, our model is also motivated by TV advertising where inventories of a commercial break are usually divided into slots of five seconds each, and slots have various qualities measuring their expected number of viewers and corresponding attraction. Note that the private information about the value for each advertiser creates an asymmetry among the participants and the market maker. Truthful market design relies on the general revelation principle (Myerson 1979) to simplify the search for mechanisms with desirable properties, such as one that brings in the maximum revenue. Therefore, our focus will be on considering market mechanisms that bring in the optimum revenue (or failing that, an approximation to the optimal revenue), while ensuring the participants' incentives to speak the truth about their private values.

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We model the problem as a multi-unit auction where the auctioneer holds  $m$  indivisible items and each item  $j$  has a parameter  $q_j$ , measuring the quality of the item. On the other side,  $n$  potential buyers participate in the auction and each buyer  $i$  has a value  $v_i$  for an item of unit quality. Thus, the valuation that  $i$  obtains from item  $j$  is given by  $v_{ij} = v_i q_j$ . The  $v_i q_j$  valuation model has been considered by Edelman et al. (2007) and Varian (2007) in their seminal work for keywords advertising. We will explore this model by considering buyers that have specific demand or are budget constrained. Demand is a practical consideration and has occurred in a number of applications. For instance, in TV (or radio) advertising (Nisan et al. 2009), advertisers may request different lengths of advertising slots for their ads programs. As another example, in banner (or newspaper) advertising, advertisers may request different sizes or areas for their displayed ads, which may be decomposed into a number of base units. Simultaneously, in several scenarios, such as the Google TV ad auction (Nisan et al. 2009) and the Federal Communications Commission spectrum auctions (Brusco and Lopomo 2009), where auctions have been applied in the past few years, bidders are constrained by the amount of money they can spend.

Instead of the standard deterministic model, we consider a Bayesian model for the private values  $v_i$  of the buyers, where  $i \in [n]$ . To obtain the performance guarantees we study here, we need to use this kind of prior knowledge of what the valuations are likely to be, and in the context of ad auctions, it is realistic to assume that distributions on valuations and click-through rates have been previously learned from data (Graepel et al. 2010; Richardson, Dominowska, and Ragno 2007). Thus, the private value  $v_i$  of advertiser  $i$  follows a publicly known distribution. Therefore, an advertiser knows its exact per-click value  $v_i$ , but other advertisers as well as the seller of the slots only know that  $v_i$  is generated by the given public probability distribution. Therefore, we adapt Myerson's (1981) classic setting, where each buyer's private value is independently drawn from a publicly known distribution, and the buyers are risk neutral. We focus on Bayesian incentive compatible (BIC) mechanisms, that is, bidding his true value is a Bayesian dominant strategy for every buyer (that is, it maximizes the expected gain for a buyer, where expectation is taken over the random selection of values for the other buyers). We are interested in obtaining mechanisms to optimize or approximate the expected revenue, taking into consideration the demand constraints and the budget constraints for all the buyers.

## 2. RELATED WORK

The theoretical study of sponsored search under the generalized second price auction was initiated in Edelman et al. (2007) and Varian (2007). The items for sale are called "positions," referring to positions of ads on a web page. Some positions have more value than others; usually, the ones toward the top of the web page are considered to be more valuable. Under this model, the utility of an agent is its private per unit value times the quality of the position it acquires through the auction, minus the price that it pays. Each buyer is assigned to at most one position, and in this article, we study a more general model that does not have that limitation. In our model, each buyer may demand multiple items, so that we have a special kind of combinatorial auction. This represents a natural extension of the sponsored search auction model, and is motivated by applications such as TV ad slot bidding, rich medium web page ads, and so on. The total value of a buyer is determined by its private per unit value times the total quality of the positions it is assigned. In the relaxed demand version, a buyer's demand is an upper bound on the number of positions it may be assigned. In the sharp demand version, the number of positions assigned to any successful buyer must be equal to that buyer's demand. In either case, the demand may vary from buyer to buyer.

There have been a series of studies of position auctions in deterministic settings (Lahaie 2006). Chen et al. (2012) introduce an ad auction setting having the same combinatorial demand structure we study here (each buyer wanting a specific quantity of items). Chen et al. (2012) differs from this article by focusing on the search for prices that constitute market equilibrium solutions, based on known valuations of the buyers. In contrast, here, we address the challenge of eliciting truthful bids from the buyers, based on partial knowledge.

Our consideration of position auctions in the Bayesian setting fits in the general single-parameter auction design framework (Hartline and Roughgarden 2009) with demand constraints and budget constraints. The work in Hartline and Roughgarden (2009) compares the Vickrey-Clarke-Groves auction revenue with reserve prices versus optimal revenues in many one-dimensional settings. For a single item with budget constraints, the problem is reduced to the case of single item without budget (Chawla, Malec, and Malekian 2011) where a 2-approximate mechanism was introduced. Their method does not directly apply to our problem.

Our study considers continuous distributions on buyers' values. For discrete distributions, Cai, Daskalakis, and Weinberg (2012a) present an optimal mechanism for budget-constrained buyers without demand constraints in multiparameter settings, and very recently, they also give a general reduction from revenue to welfare maximization in Cai, Daskalakis, and Weinberg (2012b); for buyers with both budget constraints and demand constraints, 2-approximate mechanisms (Alaei 2011) and 4-approximate mechanisms (Bhattacharya et al. 2010) exist in the literature. However, their techniques based on solving a related linear programming cannot be extended to the buyers with continuous distributions and budget constraints. Their techniques' limits are addressed in Cai et al. (2012a, 2012b), which in some sense imply that the problem with continuous distribution is more difficult than that with discrete distribution when other constraints are the same, for example, multi parameters, demand constraints.

### 3. OUR CONTRIBUTION

We construct new mechanisms that we analyze mathematically and prove the following performance guarantees (detailed definitions are given in the next section). Following Myerson's works, we compare our mechanism with the optimal truthful mechanism (with maximal expected revenue) in Bayesian setting. For the relaxed demand or the sharp demand case without budget constraints, an optimal truthful mechanism (that is, one that maximizes the expected revenue) can be constructed efficiently (this is, the mechanism is computable in polynomial time). For the case with the budget constraint but without demand constraints, a 2-approximate truthful mechanism (one that guarantees revenue within a factor 2 of optimal) can be constructed efficiently.

### 4. PRELIMINARIES

#### 4.1. The Auction Domain

In our auction design problem, we want to sell  $m$  items to  $n$  buyers. Each buyer has a private number  $v_i$  representing her valuation, and each item is characterized by a number  $q_j$ , which can be viewed as the quality or desirability of the item. Thus, the  $i$ th buyer's value for item  $j$  is  $v_i q_j$ . In other words, the valuation matrix for  $n$  buyers and  $m$  items is the outer product of  $\vec{v}$  and  $\vec{q}$ . Buyers are also assumed to abide by additional constraints as follows. We consider three specific constraints of this problem.

- (1) *Relaxed demand constraint*: Buyer  $i$ 's demand is relaxedly constrained by  $d_i$  if  $i$  may buy any number of items up to a maximum  $d_i$  in this auction.
- (2) *Sharp demand constraint*: Buyer  $i$ 's demand is sharply constrained by  $d_i$  if  $i$  must buy exactly  $d_i$  items in this auction or alternatively buys nothing.
- (3) *Budget constraint*: Buyer  $i$ 's budget is constrained by a publicly known number  $B_i$  if  $i$  cannot pay more than  $B_i$ .

The vector of all the buyers' values is denoted by  $\vec{v}$  or sometimes  $(v_i; v_{-i})$  where  $v_{-i}$  is the joint bids of all bidders other than  $i$ . We assume that all buyers' values are distributed independently according to publicly known bounded distributions, that is,  $v_i \in [v_i, \bar{v}_i]$  and  $\vec{v} \in V = \prod_i [v_i, \bar{v}_i]$ . For each buyer  $i$ , let  $F_i$  be the cumulative distribution function of buyer  $i$ 's value distribution and let  $f_i$  be the probability density function of this distribution. In addition, we assume that the concave closure or convex closure or integration of those functions can be computed efficiently.

We represent a feasible assignment by a vector  $x = (x_{ij})_{i,j}$ , where  $x_{ij}$  is simply the probability that item  $j$  is assigned to buyer  $i$ ; thus, we have  $x_{ij} \in [0, 1]$  for every  $i, j$ ,  $\sum_i x_{ij} \leq 1$  for every item  $j$ . We say an assignment is deterministic if every  $x_{ij}$  is 1 or 0. Given a fixed assignment  $x$ , we use  $t_i$  to denote the expected total quality of items that buyer  $i$  is assigned, precisely,  $t_i = \sum_j q_j x_{ij}$ . In general, when  $x$  is a function of buyers' bids  $\vec{v}$ , we define  $t_i$  to be a function of  $\vec{v}$  such that  $t_i(\vec{v}) = \sum_j q_j x_{ij}(\vec{v})$ .

## 4.2. Mechanism Design

We consider the revenue maximization problem in the aforementioned auction domains in the context of mechanism design. Mechanism design studies algorithmic procedures where the input data are not always objective but reported from selfish agents. Following the work of Myerson (1981), we consider this problem in a Bayesian setting where the seller has a prior knowledge about the buyers' distribution of valuations. This probability distribution can be regarded as a model of the seller's beliefs about which values are more likely than others. This has been shown to be a necessary assumption if one wants to optimize the auctioneer's revenue. More specifically, in the setting (called prior-free) without this assumption, any performance guarantees for an auction would have to apply to worst-case choices of buyers' valuations. In such a setting, we cannot approximate the optimal revenue within any factor, that is, the performance of any mechanism will be arbitrary bad for some bidder profile. The auctioneer holds the set of items that can be sold, but does not know the (true) valuations of these items for different buyers. Each buyer is a selfish entity, which privately knows her own valuation for each item (which constitutes the buyer's type). Obviously, a strategic buyer may choose to misreport her valuations, which are private, to increase her utility, that is, the valuation of assigned item minus her payment. Because we only consider the auction setting, we may use auction and mechanism interchangeably when there is no ambiguity.

We consider direct-revelation mechanisms: each buyer reports her valuation, and the mechanism must be designed such that it is in the buyers' best interest to report their true valuation. The mechanism then computes a feasible assignment and charges the players (i.e., buyers) the payment for the items they have been assigned. An auction  $M$  thus consists of a tuple  $(x, p)$ , where  $x$  specifies the allocation of items and  $p = (p_i)_i$  specifies the buyers' payments where both  $x$  and  $p$  are functions of the reported valuations  $\vec{v}$ . Thus, the expected revenue of the mechanism is  $Rev(M) = E_{\vec{v}} [\sum_i p_i(\vec{v})]$  where  $E_{\vec{v}}$  denotes the expectation with respect to components of  $\vec{v}$  sampled from their respective distributions. From the viewpoint of a single buyer  $i$  with private value  $v_i$ , her expected

utility is given by  $E_{v_{-i}} [v_i t_i(\vec{v}) - p_i(\vec{v})]$ . The goal of an auctioneer is to maximize her expected revenue. A buyer  $i$  is however only interested in maximizing her own expected utility. Considering the possibility that a buyer may declare a false value if this could increase her utility, the mechanism therefore needs to incentivize the buyers/players to truthfully reveal their values. This is made precise using the following notion of Bayesian incentive compatibility.

*Definition 1 (Incentive compatibility).* A mechanism  $M = (x, p)$  is called BIC iff the following inequalities hold for all player  $i$  and his valuation  $v_i, v'_i$ . Recall that  $t_i(\vec{v}) = \sum_j q_j x_{ij}(\vec{v})$ .

$$E_{v_{-i}} [v_i t_i(\vec{v}) - p_i(\vec{v})] \geq E_{v_{-i}} [v_i t_i(v'_i; v_{-i}) - p_i(v'_i; v_{-i})]. \quad (1)$$

If  $v_i t_i(\vec{v}) - p_i(\vec{v}) \geq v_i t_i(v'_i; v_{-i}) - p_i(v'_i; v_{-i})$ , for all  $\vec{v}, i, v'_i$ , we say  $M$  is deterministically *incentive compatible (IC)*.

To put it in words, in a BIC mechanism, no player can improve her *expected* utility (expectation taken over other players' bids) by misreporting her value. An IC mechanism satisfies the stronger requirement that no matter what the other players declare, no player has incentives to deviate.

In addition to Bayesian incentive compatibility, the desired mechanism should also satisfy another constraint named individual rationality, which guarantees the participation of all players. It requires that the (expected) utility of each player cannot be negative no matter what valuation she has.

*Definition 2 (Individual Rationality).* A mechanism  $M = (x, p)$  is called *ex-interim Individual Rational (IR)* iff the following inequalities hold for all player  $i$  and his valuation  $v_i$ . Recall that  $t_i(\vec{v}) = \sum_j q_j x_{ij}(\vec{v})$ .

$$E_{v_{-i}} [v_i t_i(\vec{v}) - p_i(\vec{v})] \geq 0. \quad (2)$$

If  $v_i t_i(\vec{v}) - p_i(\vec{v}) \geq 0$  for all  $\vec{v}, i$ , we say  $M$  is *ex-post individual rational*.

Obviously, an ex-post individual rational mechanism must be ex-interim individual rational. The term "ex-interim" here indicates that the nonnegativity of each agent's utility holds for every possible valuation of this agent, averaged over the possible valuations of the other agents. Ex-post IR holds if and only if the utility of each player cannot be negative for any bidding profile  $\vec{v}$ .

### 4.3. Goal and Objectives

Given the buyers' value distributions, our goal is to design BIC and ex-interim IR mechanisms to allocate items to buyers so as to maximize the auctioneer's expected revenue. As is common in Computer Science, the optimal solution may be hard to compute efficiently; thus, we also consider the mechanisms that implement this objective *approximately*. More precisely, our aim is to devise a mechanism that for any distributions of buyers' values, the mechanism guarantees at least  $1/\alpha$  times the optimum, where  $\alpha$  is a constant. We call such mechanisms  $\alpha$ -approximate mechanisms.

*Definition 3 ( $\alpha$ -approximate Mechanism).* We say a BIC and ex-interim IR mechanism  $M$  is an  $\alpha$ -approximate mechanism if and only if for any BIC and ex-interim IR mechanism  $M'$ ,  $Rev(M) \geq 1/\alpha \cdot Rev(M')$ . We say a mechanism is optimal if it is a 1-approximate mechanism.

We are also interested in obtaining *computationally efficient* mechanisms, which is made precise by requiring that they should be computable in polynomial time. That is of course a standard requirement in the context of algorithmic mechanism design.

## 5. OPTIMAL MECHANISM FOR DEMAND CONSTRAINTS

In this section, we consider the case where the buyers have relaxed demand or sharp demand. We show that, in these auction domains, the optimal randomized, BIC and ex-interim IR auction can be represented by a simple deterministic, IC and ex-post IR auction. Furthermore, this optimal auction can be constructed efficiently.

Our constructions and proofs are simple and based on a basic idea of converting the optimization problem with allocation rules and payment rules to a problem only involving allocations. This can be carried out in two steps. First, because of the fact that our mechanism design problems fall within the single-parameter domain where each player can be represented by a single parameter, we can replace the complicated BIC conditions with a much simpler requirement of monotonicity on allocation rules. After that, all of the constraints are related with allocation functions instead of payments. Second, although the objective of our auction is to maximize the revenue, we can show that maximizing the auctioneer's revenue in a BIC auction is equivalent to maximizing a specific function of allocations, more precisely, the virtual surplus, which is developed in Myerson (1981). Thus, we can get rid of the payments in our optimizing goal as well.

After this transformation, the original revenue optimization problems can be viewed as simple combinatorial optimization problems. As we will show later, both of the problems can be solved efficiently and even in a deterministic way. Following the sketch described earlier, we will review the classical characterization of Bayesian incentive compatibility in Section 5.1. And then, in Section 5.2, we will show how the payments can be discarded in the objective by incorporating Myerson's virtual value functions. At the end, we will solve the pure optimization problems for relaxed demand and sharp demand settings in Section 5.3 and Section 5.4, respectively.

### 5.1. Monotonicity

Although the incentive compatibility is defined in the terms of payments, it can be boiled down to a simple condition of monotonicity in single-parameter settings. The proof can be sketched as follows. Fix a player  $i$  and all other players' bids  $v_{-i}$ . Recall that we use  $t_i$ , a function of  $\vec{v}$  to denote the total quality of items assigned to  $i$ . Consider two possible values  $v_i$  and  $v'_i$  player  $i$  may hold. By the definition of IC, we have  $v_i t_i(v_i; v_{-i}) - p_i(v_i; v_{-i}) \geq v_i t_i(v'_i; v_{-i}) - p_i(v'_i; v_{-i})$  and similarly  $v'_i t_i(v'_i; v_{-i}) - p_i(v'_i; v_{-i}) \geq v'_i t_i(v_i; v_{-i}) - p_i(v_i; v_{-i})$ . Summing up these two inequalities, we obtained  $(v_i - v'_i)(t_i(v_i; v_{-i}) - t_i(v'_i; v_{-i})) \geq 0$ . It follows that  $t_i(x; v_{-i})$  must be a monotone nondecreasing function of  $x$  for any given  $v_{-i}$ . Regarding the Bayesian setting, the BIC condition can be similarly characterized in the following Lemma 1 adapted from Myerson (1981). For convenience in the Bayesian model, let  $T_i(v_i)$  be the expectation of  $t_i(\vec{v})$  over all other players' bids, more precisely,  $T_i(v_i) = E_{v_{-i}}[t_i(\vec{v})]$ . Similarly, we define an expected version of payment rules, thus  $P_i(v_i) = E_{v_{-i}}[p_i(\vec{v})]$ . Let  $U_i(v_i, T, P) = v_i T_i(v_i) - P_i(v_i)$ .

*Lemma 1 (From Myerson (1981)).* A mechanism  $M = (x, p)$  is BIC if and only if

- (a)  $T_i(x)$  is monotone nondecreasing for any agent  $i$ .
- (b)  $P_i(v_i) = v_i T_i(v_i) - \int_{\underline{v}_i}^{v_i} T_i(z) dz - \underline{v}_i T_i(\underline{v}_i) + P_i(\underline{v}_i)$ .

The proof of the aforementioned lemma is similar to the IC condition except we need to take integration for continuous distributions (Myerson 1981). For completeness, we provide a proof of Lemma 1 in the succeeding text.

*Proof.* If  $M = (x, p)$  is BIC, then for any  $v_i$  and  $v'_i$ ,

$$\begin{aligned} v_i T_i(v_i) - P_i(v_i) &\geq v_i T_i(v'_i) - P_i(v'_i) \\ v'_i T_i(v'_i) - P_i(v'_i) &\geq v'_i T_i(v_i) - P_i(v_i). \end{aligned}$$

Adding the aforementioned two inequalities, we could obtain  $(v_i - v'_i)(T_i(v_i) - T_i(v'_i)) \geq 0$ ; hence,  $T_i(x)$  is monotone nondecreasing for any buyer  $i$ . In addition,  $v_i T_i(v_i) - P_i(v_i) \geq v_i T_i(v'_i) - P_i(v'_i)$  is equivalent to  $U_i(v_i, T, P) \geq (v_i - v'_i) T_i(v'_i) + U_i(v'_i, T, P)$ , for any  $i, v_i, v'_i$ . Therefore,  $U_i(v_i, T, P) = U_i(\underline{v}_i, T, P) + \int_{\underline{v}_i}^{v_i} T_i(z) dz$ . Thus,

$$P_i(v_i) = v_i T_i(v_i) - \int_{\underline{v}_i}^{v_i} T_i(z) dz - \underline{v}_i T_i(\underline{v}_i) + P_i(\underline{v}_i).$$

Similarly, if (a) and (b) hold, then  $v_i T_i(v_i) - P_i(v_i) \geq v_i T_i(v'_i) - P_i(v'_i)$  is equivalent to (by b))  $\int_{v'_i}^{v_i} T_i(z) dz \geq (v_i - v'_i) T_i(v'_i)$ , which is true by the monotonicity of  $T_i(x)$ . ■

## 5.2. Virtual Surplus

For single-item settings where the auctioneer has only one item to be sold, Myerson (1981) showed that to maximize the seller's revenue is equivalent to maximizing the social welfare when each agent's bid is his virtual value defined as  $\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$ , where recall that  $F_i(x)$  and  $f_i(x)$  are, respectively, the cumulative distribution function and probability density function of the agent  $i$ 's value distribution. That is, the virtual value of an agent is her true value minus the hazard rate of her value and distribution. Then given buyers' distributions, we define the virtual surplus as the expectation of the summation of every buyer's virtual value times her allocation, more precisely,  $E_{\vec{v}} [\sum_i \phi_i(v_i) t_i(\vec{v})]$ . Then we can show that in both of our auction domains in this section, expected revenue is equal to expected virtual surplus. Recall that  $U_i(v_i, T, P)$  is defined as  $v_i T_i(v_i) - P_i(v_i)$ .

*Lemma 2.* For any BIC mechanism  $M = (x, p)$ , the expected revenue  $E_{\vec{v}} [\sum_i P_i(v_i)]$  is equal to the virtual surplus  $E_{\vec{v}} [\sum_i \phi_i(v_i) t_i(\vec{v})]$  minus  $\sum_i U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i))$ .

The proof is based on standard mathematical calculations. We express expectation as an integration over densities, that is,  $p_i, x_i$ , and  $f_i$ , apply the Myerson characterization of payments, and simplify.

*Proof.*

$$\begin{aligned} E_{\vec{v}}[P_i(v_i)] &= E_{\vec{v}}[-U_i(v_i, T, P) + v_i T_i(v_i)] \\ &= E_{\vec{v}}[-U_i(v_i, T, P)] + E_{\vec{v}}[v_i T_i(v_i)] \\ &= -E_{\vec{v}} \left[ \int_{\underline{v}_i}^{v_i} T_i(z) dz \right] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) + E_{\vec{v}}[v_i t_i(\vec{v})] \\ &= - \int_{\underline{v}_i}^{\bar{v}_i} dv_i \int_{\underline{v}_i}^{v_i} f_i(v_i) T_i(z) dz + E_{\vec{v}}[v_i t_i(\vec{v})] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) \end{aligned}$$

$$\begin{aligned}
&= - \int_{\underline{v}_i}^{\bar{v}_i} T_i(z) dz \int_z^{\bar{v}_i} f_i(v_i) dv_i + E_{\vec{v}}[v_i t_i(\vec{v})] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) \\
&= - \int_{\underline{v}_i}^{\bar{v}_i} T_i(z)(1 - F_i(z)) dz + E_{\vec{v}}[v_i t_i(\vec{v})] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) \\
&= - \int_{\underline{v}_i}^{\bar{v}_i} T_i(v_i) \frac{1 - F_i(v_i)}{f_i(v_i)} f_i(v_i) dz + E_{\vec{v}}[v_i t_i(\vec{v})] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) \\
&= -E_{\vec{v}} \left[ T_i(v_i) \frac{1 - F_i(v_i)}{f_i(v_i)} \right] + E_{\vec{v}}[v_i t_i(\vec{v})] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) \\
&= E_{\vec{v}}[\phi_i(v_i) t_i(\vec{v})] - U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)).
\end{aligned}$$

■

*Remark 1.* Because the mechanism we consider is ex-post IR,  $U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) \geq 0$ , for any  $i$ . Thus, to maximize the total revenue, what we need to do is to find allocations  $x$  to maximize the virtual surplus, and set  $U(\underline{v}_i, T(\underline{v}_i), P(\underline{v}_i)) = 0$  for all  $i$  and then use the payment rule  $P_i(v_i) = v_i T_i(v_i) - \int_{\underline{v}_i}^{v_i} T_i(z) dz$ .

### 5.3. Relaxed Demand Case

For the relaxed demand case, the only problem we need to address is the additional demand constraint. Note that our mechanism only considers the allocation probability  $x$ , not the realized allocations. To convert the randomized mechanism to a realized allocation, we need a randomized rounding procedure satisfying the demand constraints. Fortunately, such a procedure is explicit in the Birkhoff–von Neumann theorem (Johnson et al. 1960). Thus, the relaxed demand constraint can be rewritten as  $\sum_j x_{ij} \leq d_i$  for each buyer  $i$ . By using the characterization of BIC and virtual surplus, we can transform the revenue optimization problem to an essentially simpler combinatorial optimization problem. The following lemma follows from Lemma 2 and Remark 1.

*Lemma 3.* Suppose that  $x$  is the allocation function that maximizes  $E_{\vec{v}}[\phi_i(v_i) t_i(\vec{v})]$  subject to the constraints that  $T_i(v_i)$  is monotone nondecreasing and inequalities

$$\sum_j x_{ij}(\vec{v}) \leq d_i, \quad \sum_i x_{ij}(\vec{v}) \leq 1, \quad x_{ij}(\vec{v}) \geq 0. \quad (3)$$

Suppose also that

$$p_i(\vec{v}) = v_i t_i(\vec{v}) - \int_{\underline{v}_i}^{v_i} t_i(v_{-i}, s_i) ds_i. \quad (4)$$

Then  $(x, p)$  represents an optimal mechanism for the relaxed demand case.

We assume  $\phi_i(t)$  is monotone increasing, that is, the distribution is regular. This assumption is without loss of generality, otherwise Myerson's ironing technique can be utilized to make  $\phi_i(t)$  monotone—it is here that we invoke our assumption that we can efficiently compute the convex closure and integration of a continuous function.

A main observation on Lemma 3 is that all inequalities in equation (3) only constrain  $\vec{v}$  independently, not correlatively with different  $\vec{v}$ 's. This allows us to consider the optimization problem for each  $\vec{v}$  separately. After that, we will prove  $T_i$  is still monotone



increasing in the resulting mechanism. In other words, we consider the problem of maximizing  $\sum_i \phi_i(v_i)t_i(\vec{v})$  for each  $\vec{v}$  separately instead of maximizing its expectation overall. This problem can be solved by a simple greedy algorithm in the spirit of assigning items with good quality to buyers with higher virtual value. For completeness, we describe our mechanism for the relaxed demand case in Algorithm 1.

**Input:** Demands  $d_i$ , CDFs  $F_i$ , PDFs  $f_i$ , qualities  $q_j$  and bids  $\vec{v}$   
**Output:** Allocation  $x_{ij}$   
 $\phi_i \leftarrow v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ ;  
Sort buyers in decreasing order of  $\phi_i$ ;  
Sort items in decreasing order of  $q_j$ ;  
 $x_{ij} \leftarrow 0$ ;  
**for each buyer  $i$  do**  
    **for each item  $j$  do**  
        **if  $\phi_i > 0$  and  $\sum_i x_{ij} < 1$  and  $\sum_j x_{ij} < d_i$  then**  
             $x_{ij} \leftarrow 1$ ;  
        **end**  
    **end**  
**end**  
return  $x$ ;

**Algorithm 1:** RELAXED

Ultimately, we prove that the  $T_i$  deduced from our mechanism is monotone nondecreasing in the following theorem—our summary statement.

*Theorem 1.* The mechanism that applies the allocation rule according to Algorithm 1 and payment rule according to equation (4) is an optimal mechanism for the multi-unit auction design problem with relaxed demand constrained buyers.

*Proof.* As mentioned earlier, it suffices to prove that  $T_i(v_i)$  is monotone nondecreasing. More specifically, we prove a stronger fact that  $t_i(v_i, v_{-i})$  is nondecreasing as  $v_i$  increases. In Algorithm 1, given  $v_{-i}$ , the monotonicity of  $t_i$  is equivalent to  $t_i(v_i, v_{-i}) \leq t_i(v'_i, v_{-i})$  given  $v'_i > v_i$ . If  $v'_i > v_i$ , without loss of generality, suppose  $\phi_i(v_i) < \phi_i(v'_i)$  (otherwise, if  $\phi_i(v_i) = \phi_i(v'_i)$  because  $\phi_i$  is regular, then  $t_i(v_i, v_{-i}) = t_i(v'_i, v_{-i})$  because the algorithm is deterministic, we are done). Let  $Q$  and  $Q'$  denote the total quantities obtained by all the other buyers except buyer  $i$  in Algorithm 1 when buyer  $i$  bids  $v_i$  and  $v'_i$ , respectively. Then we have

$$\begin{aligned} \phi_i(v'_i)t_i(v'_i, v_{-i}) + Q' &\geq t_i(v_i, v_{-i})\phi_i(v'_i) + Q \\ &\geq t_i(v_i, v_{-i})\phi_i(v_i) + Q \geq t_i(v'_i, v_{-i})\phi_i(v_i) + Q'. \end{aligned}$$

The first and last inequalities are due to the optimality of allocations found by the greedy Algorithm 1 when  $i$  bids  $v_i$  and  $v'_i$ , respectively, and the second one comes from  $\phi_i(v_i) < \phi_i(v'_i)$ . Thus, we have

$$\phi_i(v'_i)(t_i(v_i, v_{-i}) - t_i(v'_i, v_{-i})) \leq Q' - Q \leq \phi_i(v_i)(t_i(v_i, v_{-i}) - t_i(v'_i, v_{-i})).$$

By the fact that  $\phi_i(v_i) < \phi_i(v'_i)$ , it follows  $t_i(v_i, v_{-i}) - t_i(v'_i, v_{-i}) \leq 0$ . ■

#### 5.4. Sharp Demand Case

We now describe how to design an optimal mechanism for sharp demand cases. The only difference between this problem with the one with relaxed constraints is that buyers' demands are sharply constrained. To address this, we replace the inequalities (3) in Lemma 3 with the following inequalities.<sup>1</sup>

$$\sum_j x_{ij}(\vec{v}) = d_i \text{ or } 0, \quad \sum_i x_{ij}(\vec{v}) \leq 1, \quad x_{ij}(\vec{v}) \geq 0 \quad \forall i, j, \vec{v}. \quad (5)$$

Similar to the relaxed demand case, we can convert the revenue optimization problem in the sharp demand case to a simple combinatorial optimization problem. Recall that  $\phi_i(t)$  is monotone nondecreasing without loss of generality.

*Lemma 4.* Suppose that  $x$  is the allocation function that maximizes  $E_{\vec{v}}[\sum_i \phi_i(v_i)t_i(\vec{v})]$  subject to the constraints that  $T_i(v_i)$  is nondecreasing monotone and inequalities (5). Suppose also that

$$p_i(v) = v_i t_i(\vec{v}) - \int_{\underline{v}_i}^{v_i} t_i(s_i, v_{-i}) ds_i.$$

Then  $(x, p)$  represents an optimal mechanism for the sharp demand case.

Considering each bidding profile  $v_{-i}$  separately, we observe that the optimal mechanism always maximizes  $\sum_i \phi_i(v_i)t_i(\vec{v})$  for all  $\vec{v}$  subject to sharp demand constraints. By incorporating the definition of  $t_i$ , our goal is to maximize  $\sum_i \sum_j \phi_i(v_i)q_j x_{ij}(\vec{v})$  subject to  $\sum_j x_{ij}(\vec{v}) = d_i$  or 0 and  $\sum_i x_{ij}(\vec{v}) \leq 1$ . It is not hard to see that this problem is equivalent to a maximum weighted matching problem on a bipartite graph with  $n$  left nodes and  $m$  right nodes. For any pair of nodes  $(i, j) \in [n] \times [m]$ , there exists an edge with weight  $\phi_i(v_i)q_j$ . Besides, the matching should satisfy an additional constraint that each left node must be matched with exact  $d_i$  right nodes or nothing. We call this problem maximum weighted matching with sharp constraints. An essential observation our algorithm relies on is a property of the optimal solution as we will show in Lemma 5. For convenience, we sort all left nodes in decreasing order of their  $\phi_i(v_i)$  and all right nodes in decreasing order of their  $q_j$ .

*Lemma 5.* There must exist an optimal solution for the maximum weighted matching problem with sharp constraints such that each left node is matched with consecutive  $d_i$  right nodes or nothing.

*Proof.* Assume by contradiction, there exists a left node that the optimal match it with a set of nonconsecutive right nodes. Let  $i$  be the first left node (with regard to the decreasing order of  $\phi_i(v_i)$ ) with this property and  $U_i$  be the set of right nodes assigned to  $i$ . By our assumption,  $U_i$  is not consecutive. Thus, there exists a right node  $j$  not in  $U_i$  such that  $\min_{k \in U_i} \{q_k\} \leq q_j \leq \max_{k \in U_i} \{q_k\}$ . It is easy to see that  $j$  must be assigned to a left node with smaller  $\phi$  than  $i$ ; otherwise,  $i$  is not the first left node with nonconsecutive matching set. Let  $r$  be the last node of  $U_i$ , that is, with the largest index in  $U_i$ . Thus  $q_j \geq q_r$ . After that, we can refine the optimal solution by exchanging the assignment of node  $j$  and node  $r$ .

<sup>1</sup>The formula  $\sum_j x_{ij}(\vec{v}) = d_i$  or 0 here is not precise because in the random mechanism  $\sum_j x_{ij}(\vec{v})$  may be an arbitrary number between 0 and  $d_i$ . A more precise definition may need to be in terms of a distribution over deterministic mechanisms. However, we did not explicitly use the randomized value of  $\sum_j x_{ij}(\vec{v})$  in our algorithm, and our mechanism is deterministic implying  $x_{ij} \in \{0, 1\}$ , and  $\sum_j x_{ij}(\vec{v}) = d_i$  or 0 is correct if  $x_{ij} \in \{0, 1\}$ ; hence, we still use this formula here.

The resulting assignment is still feasible and has larger weight. If we keep doing this, we can obtain the desired optimal solution. ■

By using this property, the problem can be solved by dynamic programming precisely. Let  $w[i, j]$  denote the weight of the maximum weighted matchings with first  $i$  left nodes and all first  $j$  right nodes being assigned. Then we have the transition function

$$w[i, j] = \max \left\{ w[i - 1, j], w[i - 1, j - d_i] + \sum_{k=j-d_i+1}^j \phi_i(v_i)c_k \right\}.$$

Finding the maximum  $w[i, j]$  over  $i \in [n]$  and  $j \in [m]$  gives the maximum weighted matchings and optimal solutions. We describe the mechanism in Algorithm 2.

**Input:** Demands  $d_i$ , CDFs  $F_i$ , PDFs  $f_i$ , qualities  $q_j$  and bids  $\vec{v}$   
**Output:** Allocation  $x_{ij}$

$\phi_i \leftarrow v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ ;  
 Sort buyers  $i$  in decreasing order of  $\phi_i$ ;  
 Sort items  $j$  in decreasing order of  $q_j$ ;  
 $w[i, j] \leftarrow -\infty$ ;  $w[0, 0] \leftarrow 0$ ;  
 $r[i, j] \leftarrow 0$ ;  $x_{ij} \leftarrow 0$ ;  
**for each buyer  $i$  with positive  $\phi_i$  do**  
     **for each item  $j$  do**  
          $tmp \leftarrow w[i - 1, j - d_i] + \sum_{k=j-d_i+1}^j \phi_i q_k$ ;  
          $w[i, j] \leftarrow w[i - 1, j]$ ;  
         **if  $w[i, j] < tmp$  then**  
              $r[i, j] \leftarrow 1$ ;  
              $w[i, j] \leftarrow tmp$ ;  
         **end**  
     **end**  
**end**  
 $w[i^*, j^*] = \max_{i,j} \{w[i, j]\}$ ;  
**while  $i^* > 0$  do**  
     **if  $r[i^*, j^*] = 1$  then**  
         **for each item  $k$  from  $j^* - d_{i^*} + 1$  to  $j^*$  do**  
              $x_{i^*,k} \leftarrow 1$ ;  
         **end**  
          $j^* \leftarrow j^* - d_{i^*}$ ;  
     **end**  
      $i^* \leftarrow i^* - 1$ ;  
**end**  
 return  $x$ ;

**Algorithm 2:** SHARP

*Theorem 2.* The mechanism that applies the allocation rule with regard to the aforementioned dynamic programming and payment rule with regard to equation (4) is an optimal

mechanism for the multi-unit auction design problem with sharp demand constrained buyers.

To complete the proof of Theorem 2, it is sufficient to show  $T_i(v_i)$  is nondecreasing, where the proof is similar to the relaxed demand case.

## 6. APPROXIMATE MECHANISM FOR BUDGET CONSTRAINTS

In this section, we will present a 2-approximate mechanism for the multi-item auction with budget-constrained buyers. It should be noted that there are no demand constraints for all the buyers considered in this section. Recall that a mechanism  $M = (x, p)$  satisfies the buyer  $i$ 's budget constraint iff  $p_i(\vec{v}) \leq B_i$  for all buyer profiles  $v$ . If  $m = 1$ , that is, the auctioneer only has one slot, a 2-approximate mechanism has been suggested in Alaei (2011) and Bhattacharya et al. (2010). Thus, our approach is to reduce the multi-item auction to single-item auction, that is, the case for  $m = 1$ . Recall that  $B_i$  denotes bidder  $i$ 's budget,  $x_{ij}(\vec{v})$  denote the probability of allocating item  $j$  to buyer  $i$  when the buyers' bids revealed type is  $\vec{v}$  and recall we use  $t_i(\vec{v}) = \sum_j q_j x_{ij}(\vec{v})$ , a function of  $\vec{v}$  to denote the total quality of items assigned to  $i$ . Then the multi-item auction problem can be formalized as the following optimization problem.

$$\begin{aligned}
 \text{Max: } & E_v \left[ \sum_i p_i(\vec{v}) \right] \\
 \text{s.t. } & E_{v_{-i}} [v_i t_i(\vec{v}) - p_i(\vec{v})] \geq E_{v_{-i}} [v_i t_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i})], \quad \forall \vec{v}, i, v'_i \\
 & E_{v_{-i}} [v_i t_i(\vec{v}) - p_i(\vec{v})] \geq 0, \quad \forall \vec{v}, i \\
 & p_i(\vec{v}) \leq B_i, \quad \forall \vec{v}, i \\
 & x_{ij}(\vec{v}) \geq 0 \quad \forall \vec{v}, i, j \\
 & \sum_i x_{ij}(\vec{v}) \leq 1 \quad \forall \vec{v}, j
 \end{aligned}$$

(MULTI-ITEM)

Now consider the following single-item problem. Denote  $B'_i = \frac{B_i}{\sum_j q_j}$ , and let  $y_i(\vec{v})$  be the allocation function for bidder  $i$  and  $s_i(\vec{v})$  be the payment function for bidder  $i$ . The single-item auction with budget constraints can be formalized as following optimization problem.

$$\begin{aligned}
 \text{Maximize: } & E_v \left[ \sum_i s_i(\vec{v}) \right] \\
 \text{s.t. } & E_{v_{-i}} [v_i y_i(\vec{v}) - s_i(\vec{v})] \geq E_{v_{-i}} [v_i y_i(v'_i, v_{-i}) - s_i(v'_i, v_{-i})], \quad \forall \vec{v}, i, v'_i \\
 & E_{v_{-i}} [v_i y_i(\vec{v}) - s_i(\vec{v})] \geq 0, \quad \forall \vec{v}, i \\
 & s_i(\vec{v}) \leq B'_i, \quad \forall \vec{v}, i \\
 & y_i(\vec{v}) \geq 0 \quad \forall \vec{v}, i \\
 & \sum_i y_i(\vec{v}) \leq 1 \quad \forall \vec{v}
 \end{aligned}$$

(SINGLE)

Our main observation for the aforementioned optimization problems is the following proposition.

*Proposition 1.* The problems MULTI-ITEM and SINGLE are equivalent as follows:

- For any feasible mechanism  $M(\vec{v}) = (x(\vec{v}), p(\vec{v}))$  of problem MULTI-ITEM, the following mechanism  $\hat{M}(\vec{v}) = (y(\vec{v}), s(\vec{v}))$  is a feasible mechanism for problem SINGLE where  $y_i(\vec{v}) = \frac{t_i(\vec{v})}{\sum_j q_j}$ ,  $s_i(\vec{v}) = \frac{p_i(\vec{v})}{\sum_j q_j}$ ,  $\forall i \in [n]$ .
- For any feasible mechanism  $\hat{M}(\vec{v}) = (y(\vec{v}), s(\vec{v}))$  of problem SINGLE, the following mechanism  $M(\vec{v}) = (x(\vec{v}), p(\vec{v}))$ , where  $x_{ij}(\vec{v}) = y_i(\vec{v}) \forall i, j$  and  $p_i(\vec{v}) = s_i(\vec{v})(\sum_j q_j) \forall i$ , is a feasible mechanism for problem MULTI-ITEM.

Ultimately, we reduce the multi-item auction design problem to the single-item auction design problem. By the results of Alaei (2011) and Bhattacharya et al. (2010), there exists a 2-approximate mechanism for problem SINGLE. Thus, we have a 2-approximate mechanism for problem MULTI-ITEM.

*Remark 2.* For the discrete distribution case, Cai et al. (2012a) presents an optimal mechanism, for multibuyers with multi-items. Their algorithm can be extended to the case where buyers are budget constrained but not demand constrained. Given buyers' discrete distribution and bid profiles, a revised version of their mechanism is an optimal mechanism and runs in time polynomial in  $\sum_i |T_i|$ , where  $|T_i|$  is the number of types of buyer  $i$  for all the items. Hence, restricting their results to MULTI-ITEM auction, that optimal mechanism is indeed an optimal mechanism for each buyer having a budget constraint but no demand constraint, with values independently drawn from discrete distribution, running in time polynomial in the input.

## 7. CONCLUSION

In this work, we study the optimal mechanism design issues for the multi-item auction problem with correlated valuations  $v_{ij} = v_i q_j$ . We focus on two demand models, the relaxed demand and the sharp demand models. We develop optimal (revenue) mechanisms for the seller. In addition, for the budget-constrained model (without demand constraints), we develop a 2-approximate truthful mechanism. We prove that the solution is polynomial time solvable. Our results have potential application to a wide range of areas, such as sponsored search or TV advertising. Moreover, the sharp demand model is related to interesting applications such as sponsored search market for rich media ad pricing. Our work serves as a modest step toward an efficient algorithmic mechanism design and can be further investigated to deal with applications in more complicated settings.

A major open problem is to find a constant approximation scheme when both the demand constraints and the budget constraints apply simultaneously. For discrete distributions, Alaei (2011) and Bhattacharya et al. (2010) suggested a constant approximate mechanism for multi-unit auctions with budget and relaxed demand constrained buyers. However, their approach based on solving an associated linear program cannot be extended to the continuous distribution case. Of course, another direction is to improve the approximation ratio for budget constrained cases.

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