Revenue Maximization in a Bayesian Double Auction Market

Xiaotie Deng, Paul Goldberg*, Bo Tang, and Jinshan Zhang

Dept. of Computer Science, University of Liverpool, United Kingdom
{Xiaotie.Deng,P.W.Goldberg,Bo.Tang,Jinshan.Zhang}@liv.ac.uk

Abstract. We study double auction market design where the market maker wants to maximize its total revenue by buying low from the sellers and selling high to the buyers. We consider a Bayesian setting where buyers and sellers have independent probability distributions on the values of products on the market.

For the simplest setting where each seller has one kind of item that can be sold in whole to a buyer, and each buyer's value can be represented by a single parameter, i.e., single-parameter setting, we develop a maximum mechanism for the market maker to maximize its own revenue.

For the more general case where the product may be different, we consider various models in terms of supplies and demands constraints. For each of them, we develop a polynomial time computable truthful mechanism for the market maker to achieve a revenue at least a constant α times the revenue of any other truthful mechanism.

1 Introduction

We consider a double auction market maker who collects valuations from buyers and sellers about a certain product to decide on the prices each seller gets and each buyer pays. The buyers may want to buy many units and the sellers may have many units to part with. The buyers and sellers may have different valuations of the product, and there is public knowledge of the probability distributions of the valuations (but each valuation, sampled from its distribution, is known only to its own buyer or seller). For simplicity, we assume that the probability distributions are independent. For the sellers and buyers, they know their own private values exactly. The market maker purchases the products from the sellers and sell them to the buyers. Our goal is to design a market mechanism that maximizes the revenue of the market maker. In other words, the market maker is to buy the same amount of products from the sellers as the amount sold to the buyers with the objective of maximizing the difference of its collected payment from the buyers and the total amount paid to the sellers. When in addition we assume public knowledge of distributions of buyers' private values from the previous sales, we call it a revenue maximization Bayesian double auction market maker.

* Supported by EPSRC grant EP/G069239/1 "Efficient Decentralised Approaches in Algorithmic Game Theory".

K.-M. Chao, T.-s. Hsu, and D.-T. Lee (Eds.): ISAAC 2012, LNCS 7676, pp. 690–699, 2012.

There have been many double auction institutions, each of which may be suitable for one type of market environment [9]. Ours is motivated by the growing use of discriminative pricing models over the Internet such as one that is studied in [7] for the prior-free market environment. A possible realistic setting for applications of our model could be Google's ad exchange where Google could play a market maker for advertisers and webpage owners [12]. One may also use it for a market model of Groupon. Our use of the Bayesian model is justified by the repeated uses of a commercial system by registered users. It allows the market maker to gain Bayesian information of the users' valuations of the products being sold. Therefore, the Bayesian model adequately describes the knowledge of the market maker, buyers and sellers for the optimal mechanism design.

Our Results. We provide optimal or constant approximate mechanisms for various settings for double auction design. There are important parameters in the market design issues. The problem can be one or multi dimensional (meaning, one product or multiple different types of products). The buyers can have demand constraints or not, and sellers are supply constrained or not. Players' values are drawn from a continuous or discrete distribution. Our results are summarized in the following table.

Table 1. Results

	Dimension	Demand	Supply	Distribution	Results
Sec. 3	Single	Arbitrary	Arbitrary	Continuous	Optimal
Sec. 4	Multi	Arbitrary	Arbitrary	Continuous	1/4-Approx
Sec. 4	Multi	Arbitrary	Arbitrary	Discrete	1/4-Approx
Sec. 5	Multi	Unlimited	Arbitrary	Discrete	Optimal
Sec. 5	Multi	Arbitrary	Unlimited	Discrete	Optimal

For the demand column, "Arbitrary" refers to the case where buyers can buy at most d_i items where d_i can be an arbitrary number and "Unlimited" means $d_i = +\infty$. The supply column is similar.

In the Bayesian Mechanism Design problems, there are two computational processes involved. The first one is, given the distribution, to design an optimal or approximate mechanism which can be viewed as a function mapping bidders' profiles to allocation and payment outcomes. Since the function maps potentially exponentially many profiles to outcomes, a succinct representation of the function is also an important part in the Bayesian mechanism design. The second process is the implementation of the mechanism, i.e., given a bid profile, we run the mechanism to compute the outcome. Our results imply that all mechanisms described in the table can be represented in polynomial size and be found and implemented in polynomial time.

Related Works. Auction design play an important role in economics in general and especially in electronic commerce [11]. Of particular interest, a number of research works focus on maximizing the auctioneer's revenue, referred as the optimal auction design problem. Myerson, in his seminal paper [13], characterized the optimal auction for the single-item setting in the Bayesian model. Recently, efforts have been made on extending Myerson's results to border settings [8,15,17].

Unlike Myerson's optimal auction result, finding the optimal solution is not easy for multi-dimensional settings. Recent research interest has turned toward approximate mechanisms [1,5]. Cai et al. [4] presented a characterization of a rather general multi-dimensional setting and proposed an efficient mechanism for the special case where no bidders are demand constrained. Using similar ideas, Alaei et al. [2] present a general framework for reducing multi-agent service problems to single-agent ones.

The double auction design problem becomes more complicated since the market maker acts as the middle man to bring buyers and sellers together. A guide to the literature in micro-economics on this topic can be found in [9]. The profit maximization problem for the single buyer/single seller setting has been studied by Myerson and Satterthwaite [14]. Our optimal double auction is a direct extension of their work and, to our best knowledge, fills a clear gap in the economic theory of double auctions. Deshmukh et al. [7], studied the revenue maximization problem for double auctions when the auctioneer has no prior knowledge about bids. Their prior-free model is essentially different from ours. More auction mechanism design problems were studied by many researchers in recent years, but as far as we know, not in the context of optimal double auction design in the Bayesian setting. While our setting assumes the existence of a monopoly platform, Rochet and Tirole [16] and Armstrong [3] introduced several different models for two-sided markets and studied platform competition.

2 Preliminaries

Throughout the paper we consider Bayesian incentive compatible mechanisms only. Informally, a mechanism is Bayesian incentive compatible if it is optimal for each buyer and each seller to bid its true value of the items. We will formally define this concept later. As a consequence, we should consider their bids to be their true valuations and restrict our discussion to mechanisms that result in less or equal utility if one deviates to report a false value.

Therefore, we will use the notation v_{ij} to represent the ith buyer's (true) bid for one of the jth seller's items and w_j for the jth seller's (true) bid. We will drop the "(true)" subsequently as deviations of bids from the true valuations will be marked. The ith buyer's bid can be denoted by a vector v_i and bids of all buyers can be denoted by v or sometimes $(v_i; v_{-i})$ where v_{-i} is the joint profile of all other bidders. Similarly, we use w and $(w_j; w_{-j})$ for the sellers' bid. [1]

In our model, all players' bids are assumed to be distributed independently according to publicly known distributions, V for buyers, W for sellers. Note that we also assume that V and W should be bounded, i.e. $v_{ij} \in [\underline{v}_{ij}, \overline{v}_{ij}]$ and $w_j \in [\underline{w}_j, \overline{w}_j]$.

[1] We use semi-colon to separate the profile of a special player with others and use comma to separate the buyers' profiles with sellers'.

The outcome of a mechanism M consists of four random variables (x, p, y, q) where x and p are the allocation function and payment functions for buyers, y and q for sellers. That is, buyer i receives item j with probability $x_{ij}(v, w)$ and pays $p_i(v, w)$; seller j sells her item with probability $y_j(v, w)$ and gets a payment $q_j(v, w)$. Thus, the expected revenue of the mechanism is $R(M) = \mathrm{E}_{v,w}[\sum_i p_i(v, w) - \sum_j q_j(v, w)]$ where $\mathrm{E}_{v,w}$ is short for $\mathrm{E}_{v\sim V,w\sim W}$.

In general, a buyer may buy more than one item from the mechanism. We assume buyers' valuation functions are additive, i.e. $v_i(S) = \sum_{j\in S} v_{ij}$. For each buyer i, let d_i denote the demand constraint for buyer i, i.e. buyer i cannot buy more than d_i items. Similarly, let k_j be the supply constraint for seller j, i.e. seller j cannot sell more than k_j items. By the Birkhoff-von Neumann theorem [10][8][6], it suffices to satisfy $\sum_j x_{ij} \leq d_i$ and $y_j = \sum_i x_{ij} \leq k_j$.

Let $U_i(v, w) = \sum_j x_{ij}(v, w)v_{ij} - p_i(v, w)$ be the expected utility of buyer i when the profile of all players is (v, w) and $T_j(v, w) = q_j(v, w) - y_j(v, w)w_j$ be the expected utility of seller j. We proceed to formally define the concepts of Bayesian Incentive Compatibility of mechanisms and ex-interim Individual Rationality of the buyers and sellers:

Definition 1. *A double auction mechanism M is said to be* Bayesian Incentive Compatible (BIC) *iff the following inequalities hold for all i, j, v, w.*

$$\begin{aligned} \mathrm{E}_{v_{-i},w}[U_i(v, w)] &\geq \mathrm{E}_{v_{-i},w}[U_i((v_i'; v_{-i}), w)] \\ \mathrm{E}_{v,w_{-j}}[T_j(v, w)] &\geq \mathrm{E}_{v,w_{-j}}[T_j(v, (w_j'; w_{-j}))] \end{aligned} \tag{1}$$

We note that, if $U_i(v, w) \geq U_i((v_i'; v_{-i}), w)$ and $T_j(v, w) \geq T_j(v, (w_j'; w_{-j}))$ for all v, w, v_i', w_j', we say M is Incentive Compatible.

Definition 2. *A double auction mechanism M is said to be* ex-interim Individual Rational (IR) *iff the following inequalities hold for all i, j, v, w.*

$$\begin{aligned} \mathrm{E}_{v_{-i},w}[U_i(v, w)] &\geq 0 \\ \mathrm{E}_{v,w_{-j}}[T_j(v, w)] &\geq 0 \end{aligned} \tag{2}$$

Similarly, we note that, if $U_i(v, w) \geq 0$ and $T_j(v, w) \geq 0$ for all v, w, we say M is ex-post Individual Rational.

Finally, we present the formal definition of approximate mechanism.

Definition 3 (α-approximate Mechanism[17]). *Given a set $\mathbb{M}$ of feasible mechanisms, we say mechanism $M \in \mathbb{M}$ is an α-approximate mechanism in $\mathbb{M}$ iff for each mechanism $M' \in \mathbb{M}$, for any set of buyer and sellers $\alpha \cdot R(M') \leq R(M)$. A mechanism is* optimal *in $\mathbb{M}$ if it is an 1-approximate mechanism in $\mathbb{M}$.*

3 Optimal Single-Dimensional Double Auction

In this section, we consider the single-dimensional double auction design problem where all sellers sell identical items, that is for all $j, j' \in [m]$, $v_{ij} = v_{ij'}$. Moreover, as shown in Table 1, in this section we assume the bidders' bids are drawn from continuous distributions. Let f_i, F_i be the probability density function (PDF) and cumulative distribution function (CDF) for buyer i's value, g_j, G_j be the PDF and CDF for seller j's value.

Our mechanism can be viewed as a generalization of the classical Myerson's Optimal Auction [13]. It is well known that Myerson's approach is powerful and extensive in the single-dimensional setting. We strengthen this by showing that a similar optimal double auction can be found in this single-dimensional setting. In addition, in Section 4 this optimal mechanism will be used to construct a constant approximate mechanism for a multi-dimensional setting.

Recall that Myerson's virtual value function is defined as $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ for each buyer. In the double auction, we define the virtual value functions for buyers and sellers as $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ and $r_j(w_j) = w_j + \frac{G_j(w_j)}{g_j(w_j)}$. If $c_i(v_i)$ is not an increasing function of v_i or r_j is not decreasing, by Myerson's ironing technique, we can use the ironed virtual value function $\bar{c}_i$ and $\bar{r}_j$. W.l.o.g, we assume the buyers are sorted in decreasing order with respect to $\bar{c}_i(v_i)$ and all sellers are in increasing order with respect to $\bar{r}_j(w_j)$. Let $D = \max_{i,j}\{\min\{\sum_{s=1}^{i} d_s, \sum_{t=1}^{j} k_j\} | \bar{c}_i(v_i) > \bar{r}_j(w_j)\}$. Thus, we can define the optimal auction in the spirit of maximizing virtual surplus.

$$x_i(v,w) = \begin{cases} d_i & \text{if } \sum_{s\le i} d_s \le D \\ D - \sum_{s<i} d_s & \text{if } \sum_{s<i} d_s < D < \sum_{s\le i} d_s \\ 0 & \text{otherwise} \end{cases}$$

$$y_j(v,w) = \begin{cases} k_j & \text{if } \sum_{t\le j} k_t \le D \\ D - \sum_{s<j} k_s & \text{if } \sum_{t<j} k_t < D < \sum_{t\le j} k_t \\ 0 & \text{otherwise} \end{cases}$$

$$p_i(v,w) = x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i((s;v_{-i}),w)ds$$

$$q_j(v,w) = y_j(v,w)w_j + \int_{w_j}^{\overline{w}_j} y_j(v,(t;w_{-j}))dt$$

Theorem 1. *The above mechanism is an optimal (revenue) mechanism for the single-dimensional double auction setting. Under the assumption that the integration and convex hull of f, g can be computed in polynomial time, the mechanism can be found and implemented. Moreover, the mechanism is deterministic, incentive compatible and ex-post Individual Rational.*

4 Approximate Multi-dimensional Double Auction

In this section, we provide a general framework for approximately reducing the double auction design problem for multiple buyers and sellers to single pair of buyer and seller sub-problems. As an application, we apply the framework to construct a 1/4-approximate mechanism for the multi-dimensional setting. Our approach is inspired by the work of Alaei [1] which provide a general framework for the one sided auction.

Recall that all bids are drawn from public known distributions and our goal is to maximize the expected revenue for the auctioneer. It should be emphasized that, in this section, we assume the buyers' values for different items are independent, i.e. v_{ij} and $v_{ij'}$ are independent.

First of all, we introduce the concept of Primary Mechanism which can be viewed as a mechanism between one buyer and one seller.

Definition 4 (Primary Mechanism/Primary Benchmark). *A* primary mechanism *denoted by M_{ij} for buyer i and seller j is a single buyer and single seller mechanism which allows specifying an upper bound on the ex-ante expected probability $\bar{k}_{ij}$ of allocating jth item to buyer i. A* primary benchmark *denoted by $\bar{R}_{ij}$ is a concave function such that the optimal revenue of any primary mechanism M_{ij} subject to $\bar{k}_{ij}$ is upper bounded by $\bar{R}_{ij}(\bar{k}_{ij})$.*

Intuitively, for any allocation rule, define the ex-ante probability of assigning jth seller's items to buyer i as $\bar{k}_{ij} = \mathrm{E}_{v_i,w_j}[x_{ij}(v_i, w_j)]$. Then we can relax the supply constraints $\sum_i x_{ij}(v, w) \le k_j$ and demand constraints $\sum_j x_{ij}(v, w) \le d_i$ to the ex-ante probability constraints, $\sum_i \bar{k}_{ij} \le k_j$ and $\sum_j \bar{k}_{ij} \le d_i$. Then we compute the optimal ex-ante probability by convex programming. Obviously, the optimal solution of the relaxed problem must be an upper bound for any original solution. Unfortunately, the solution solved by convex programming may not be a feasible solution of the original problem. To solve this problem, Alaei introduced the following rounding process to round the relaxed solution to a feasible one.

Lemma 1 (γ-Conservative Magician (Theorem 2 in [1])). *In the Magician problem, a magician is presented with a series of boxes one by one. He has k magic wands that can be used to open the boxes. On each box is written a probability q_i. If a wand is used on a box, it opens, but with at most probability q_i the wand breaks. Given $\sum_i q_i \le k$ and any $\gamma \le 1 - \frac{1}{\sqrt{k+3}}$, a γ-conservative magician guarantees that each box is opened with an ex-ante expected probability at least γ.*

Using above lemma, we describe our mechanism for multi-dimensional double auction problem. Recall that in the classical auction setting, all items are sold by the auctioneer. However, in the double auction setting, items are sold by different sellers and more efforts should be taken to handle the truthfulness issue of sellers. We extend Alaei's rounding mechanism from one-dimension (considering buyers one by one) to two-dimension (considering each pair of buyer and seller sequentially) as follows.

Mechanism (Modified γ-Pre-Rounding Mechanism)

(I) Solve the following convex program and let $\bar{k}_{ij}$ denote an optimal assignment for it.

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{i\in[n],j\in[m]} \bar{R}_{ij}(x_{ij}) && && \text{(CP)}\\
\text{Subject to:} \quad & \sum_{j\in[m]} x_{ij} \le d_i && \text{for all } i \in [n] &&\\
& \sum_{i\in[n]} x_{ij} \le k_j && \text{for all } j \in [m] &&\\
& x_{ij} \ge 0 && \text{for all } i \in [n],\ j \in [m] &&
\end{aligned}$$

(II) For each buyer i, create an instance of γ-conservative magician with d_i wands (this will be referred to as the buyer i's magician). For each item j create an instance of γ-conservative magician with k_j wands (this will be referred to as the seller j's magician).

(III) For each pair of buyer and seller (i, j):
(a) Write $\bar{k}_{ij}$ on a box and present it to the buyer i's magician and the seller j's magician.
(b) If both of them open the box, run $M_{ij}(\bar{k}_{ij})$ on buyer i and seller j otherwise consider next pair.
(c) If the mechanism buys an item from seller j and sells it to buyer i, then break the wands of buyer i's magician and seller j's magician.

Theorem 2 (Modified γ-Pre-Rounding Mechanism). *Suppose for each buyer and seller pair (i, j), we have an α-approximate primary mechanism M_{ij} and a corresponding primary benchmark $\bar{R}_{ij}$. Then for any $\gamma \in [0, 1 - \frac{1}{\sqrt{k^*+3}}]$ where $k^* = min_{i,j}\{d_i, k_j\}$, the Modified γ-Pre-Rounding Mechanism is a $\gamma^2 \cdot \alpha$-approximation mechanism.*

Proof. The proof is similar to the one in [1]. First, we prove that the expected revenue of any mechanism is upper bounded by $\sum_i \sum_j \bar{R}_{ij}(\bar{k}_{ij})$. For any mechanism $M = (x, p, y, q)$, let $k_{ij} = E_{v,w} x_{ij}(v, w)$. Due to the feasibility of M, k_{ij} must be a feasible solution of the convex programming (CP). So we have,

$$R(M) = \sum_i \sum_j R_{ij}(k_{ij}) \le \sum_i \sum_j \bar{R}_{ij}(k_{ij}) \le \sum_i \sum_j \bar{R}_{ij}(\bar{k}_{ij})$$

Then it suffices to show that for each pair (i, j), our mechanism can gain the revenue $\bar{R}_{ij}(\bar{k}_{ij})$ with probability at least $\gamma^2 \cdot \alpha$, i.e. each box will be opened with probability at least γ^2. This can be deduced from Lemma 1 easily. □

Then we consider the multi-dimensional double auction design problem and present a constant approximate mechanism. For each buyer and seller pair i, j, we use the mechanism in Section 3 for one-dimensional cases to be the primary mechanism M_{ij} and the expected revenue of M_{ij} to be the primary benchmark $\bar{R}_{ij}$.

Theorem 3. *Assume that all bidders' bids are drawn from continuous distributions. A* $1/4$ *approximate double auction for the multi-dimensional setting can be found and implemented in polynomial time.*

For the discrete distribution case, the optimal mechanism for single buyer and single seller can be computed by Linear Programming. So we have the similar result.

Theorem 4. *Assume that all bidders' bids are drawn from discrete distributions. A* $1/4$ *approximate double auction for the multi-dimensional setting can be found and implemented in polynomial time.*

5 Optimal Mechanism for Discrete Distributions

In this section, we consider the multi-dimensional double auction when all the bidders' value distributions are discrete. Unlike Section 4, we consider two special cases of the problem. One is the case where all buyers have unlimited demand, i.e., $d_i = +\infty$ for all buyer i and the other one is the case where all sellers have unlimited supply, i.e. $k_j = +\infty$ for all seller j. In this section, we focus on the previous case. The mechanism and the proof of the latter case are similar.

Recall that, in the multi-dimensional setting, the auctioneer collects each buyer's bid, denoted by a vector $v_i = (v_{i1}, \ldots, v_{im})$ drawn from a public known distribution V_i and seller's bid denoted by w_j drawn from W_j. Throughout this section, V_i and W_j are discrete distributions and we use f_i and g_j to denote their probability mass function, i.e. $f_i(t) = \Pr[v_i = t]$ and $g_j(t) = \Pr[w_j = t]$. It should be emphasized that, unlike Section 4, we do not need to assume that the buyer's bids for each item should be independent, i.e. v_{ij} and $v_{ij'}$ can be correlated in this section. We also add a dummy buyer 0 with only one type v_0 for buyers and seller 0 with w_0 for sellers.

Our approach is motivated by the recent results of Cai et al. [4] and Aleai et al. [2] which require a reduced form of x, y, p, q denoted by $\bar{x}$, $\bar{y}$, $\bar{p}$ and $\bar{q}$ respectively, defined as follows:

$$\begin{array}{ll} \bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[x_{ij}(v, w)] & \bar{y}_j(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[y_j(v, w)] \\ \bar{p}_i(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[p_i(v, w)] & \bar{q}_j(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[q_j(v, w)] \end{array}$$

Now we are ready to convert an optimization problem of x, p, y, q to a problem of $\bar{x}, \bar{p}, \bar{y}, \bar{q}$ which can be represented by a Linear Program with polynomial size in T, n and m where T is the maximum among all $|V_i|$ and $|W_j|$.

Then BIC constraints (1) and IR constraints (2) can be rewritten as

$$\begin{array}{ll} \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i, w_j)v_{ij} - \bar{p}_i(v_i, w_j)] & \geq \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i', w_j)v_{ij} - \bar{p}_i(v_i', w_j)] \\ \mathrm{E}_{v_i}[\bar{q}_j(v_i, w_j) - \bar{y}_j(v_i, w_j)w_j)] & \geq \mathrm{E}_{v_i}[\bar{q}_j(v_i, w_j') - \bar{y}_j(v_i, w_j')w_j] \\ \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i, w_j)v_{ij} - \bar{p}_i(v_i, w_j)] & \geq 0 \\ \mathrm{E}_{v_i}[\bar{q}_j(v_i, w_j) - \bar{y}_j(v_i, w_j)w_j)] & \geq 0 \end{array} \quad (3)$$

Finally, all mechanism should satisfy the supply constraints, i.e., for each item j and profiles $v, w, y_j(v, w) = \sum_i x_{ij}(v, w) \leq k_j$. Note that there is no demand constraint on buyers. With loss of generality, we assume that $k_j = 1$ for all j. Otherwise, we can normalize x by setting $x'_{ij}(v, w) = x_{ij}(v, w)/k_j$ and refine v, w by setting $v'_{ij} = k_j v_{ij}$ and $w'_j = k_j w_j$ such that $k'_j = 1$ for all item j.

For the single-item setting of classical auction, i.e. $m = 1$ and seller's value for his item is always 0, Alaei et al. [2] prove a sufficient and necessary condition for the supply constraint. We generalize their result to a multi-dimensional double auction setting.

Lemma 2. *Given a reduced form $\bar{x}$, there exists an ex-post implementation x such that $x_{ij}(v, w) \geq 0, \sum_i x_{ij}(v, w) \leq 1$ and $\bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[x_{ij}(v, w)]$ iff there exists (s, z) such that, for each seller j and $w_j \in W_j$*

$$\begin{array}{ll}
s_0^{(j)}(v_0, w_j, 0) = 1 & \\
s_i^{(j)}(v_i, w_j, i) = \sum_{k=0}^{i-1} \sum_{v_k \in V_k} z_{ki}^{(j)}(v_k, v_i, w_j) & \forall i, v_i \in V_i \\
s_k^{(j)}(v_k, w_j, i) = s_k^{(j)}(v_k, w_j, i-1) - \sum_{v_i \in V_i} z_{ki}^{(j)}(v_k, v_i, w_j) & \forall i, k < i, v_k \in V_k \\
z_{ki}^{(j)}(v_k, v_i, w_j) \leq s_k^{(j)}(v_k, w_j, i-1) f_i(v_i) & \forall i, k < i, v_i \in V_i, v_k \in V_k \\
\bar{x}_{ij}(v_i, w_j) f_i(v_i) = s_i^{(j)}(v_i, w_j, n) & \forall i, v_i \in V_i
\end{array} \tag{4}$$

Moreover, given any feasible reduced allocation rule $\bar{x}$, the ex-post of $\bar{x}$ can be found efficiently.

Finally, we convert the problem of multi-dimensional double auction design problem to a Linear Program with reduced form which can be solved in polynomial time in m, n, T.

Theorem 5. *Assume all bidders' bids are drawn from discrete distributions and all bidders are without demand constraints. An optimal double auction for multi-dimensional setting can be found and implemented in polynomial time.*

Theorem 6. *Assume that all bidders' bids are drawn from discrete distributions and all sellers are without supply constraints. An optimal double auction for multi-dimensional setting can be found and implemented in polynomial time.*

6 Conclusion

In this paper, we present several optimal or approximately-optimal auctions for a double auction market. Double auction platforms have started to gain importance in electronic commerce. One possible example is the ad exchange market proposed to bring advertisers and web publishers together [12]. There is other potential in setting up electronic platforms for sellers and buyers of other types of resources in the context of cloud computing.

Our results on the one hand show the power of recent significant progress in one-sided markets, and on the other hand raise new challenges in the development of mathematical and algorithmic tools for market design.

References

1. Alaei, S.: Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. In: Procs. of 52nd IEEE FOCS Symposium, pp. 512–521 (2011)
2. Alaei, S., Fu, H., Haghpanah, N., Hartline, J., Malekian, A.: Bayesian optimal auctions via multi- to single-agent reduction. In: Proceedings of the 14th ACM Conference on Electronic Commerce, pp. 17–17 (2012)
3. Armstrong, M.: Competition in two-sided markets. The RAND Journal of Economics 37(3), 668–691 (2006)
4. Cai, Y., Daskalakis, C., Weinberg, S.M.: An algorithmic characterization of multi-dimensional mechanisms. In: Procs. of the 44th Annual ACM STOC Symposium, pp. 459–478 (2012)
5. Chawla, S., Hartline, J.D., Malec, D.L., Ivan, B.S.: Multi-parameter mechanism design and sequential posted pricing. In: Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, New York, NY, USA, pp. 311–320 (2010)
6. Daskalakis, C., Weinberg, S.M.: Symmetries and optimal multi-dimensional mechanism design. In: Proceedings of the 13th ACM Conference on Electronic Commerce, EC 2012, pp. 370–387. ACM, New York (2012)
7. Deshmukh, K., Goldberg, A.V., Hartline, J.D., Karlin, A.R.: Truthful and Competitive Double Auctions. In: Möhring, R.H., Raman, R. (eds.) ESA 2002. LNCS, vol. 2461, pp. 361–373. Springer, Heidelberg (2002)
8. Dobzinski, S., Fu, H., Kleinberg, R.D.: Optimal auctions with correlated bidders are easy. In: Proceedings of the 43rd Annual ACM Symposium on Theory of Computing, STOC 2011, pp. 129–138. ACM, New York (2011)
9. Friedman, D.: The double auction market institution: A survey. The Double Auction Market Institutions Theories and Evidence 14, 3–25 (1993)
10. Johnson, D.M., Dulmage, A.L., Mendelsohn, N.S.: On an algorithm of G. Birkhoff concerning doubly stochastic matrices. Canadian Mathematical Bulletin (1960)
11. Klemperer, P.: The economic theory of auctions. Edward Elgar Publishing (2000)
12. Muthukrishnan, S.: Ad Exchanges: Research Issues. In: Leonardi, S. (ed.) WINE 2009. LNCS, vol. 5929, pp. 1–12. Springer, Heidelberg (2009)
13. Myerson, R.: Optimal auction design. Mathematics of Operations Research 6(1), 58–73 (1981)
14. Myerson, R.B., Satterthwaite, M.A.: Efficient mechanisms for bilateral trading. Journal of Economic Theory 29(2), 265–281 (1983)
15. Papadimitriou, C.H., Pierrakos, G.: On optimal single-item auctions. In: Procs. of the 43rd Annual ACM STOC Symposium, pp. 119–128 (2011)
16. Rochet, J.-C., Tirole, J.: Platform competition in two-sided markets. Journal of the European Economic Association 1(4), 990–1029 (2003)
17. Ronen, A.: On approximating optimal auctions. In: Procs. of the 3rd ACM Conference on Electronic Commerce, EC 2001, pp. 11–17 (2001)