Strategy Improvement for Parity Games: A combinatorial perspective

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Overview

Theorem

The strategy improvement algorithm for parity games is a bottom-antipodal sink-finding algorithm on an acyclic unique sink orientation of the strategy hypercube.

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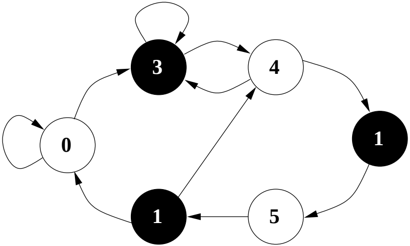
Parity Games

Ubiquitous in formal verification

Polynomially equivalent to modal μ-calculus model checking

Parity Games: Example



Played by two players: Player 0 (∃ve) and Player 1 (∀dam) on a (finite) directed graph with priorities on vertices.

Players take turns moving a token around the graph and the winner is determined by the parity of the largest priority seen infinitely often.

Parity Games: Definitions

A parity game is a tuple (V, V_0, V_1, E, χ) where:

- V_0 and V_1 partition V,
- (V, E) is a directed graph,
- $\chi : V \rightarrow \mathrm{P} \subset \omega$ is a priority function

A play is a (possibly infinite) path in (V, E)

$$\pi = v_1 v_2 \cdots$$

A play is winning for Player i iff $\limsup \chi(v_n) = i \mod 2$

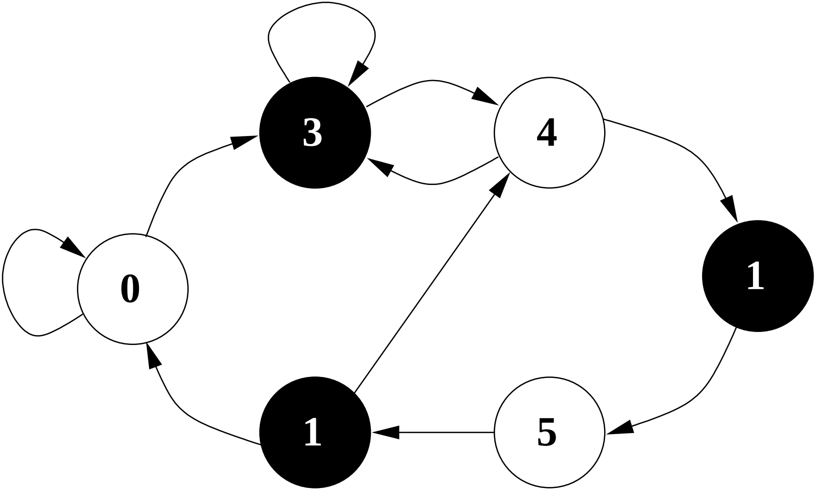
Parity Games: Strategies

A strategy for Player i is a function $\sigma : V^* V_i \rightarrow V$.

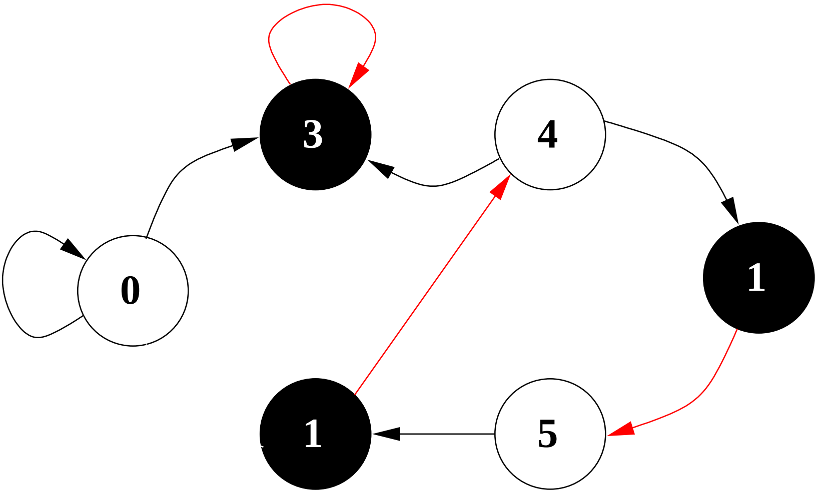
A positional (or memoryless) strategy for Player i is a function $\sigma : V_i \rightarrow V$

A strategy is winning if all plays consistent with that strategy are winning

Parity Games: Strategies



Parity Games: Strategies



Parity Games: Properties

Theorem (Memoryless determinacy)

From every vertex, one player has a memoryless winning strategy

Open Problem

Are parity games in P?

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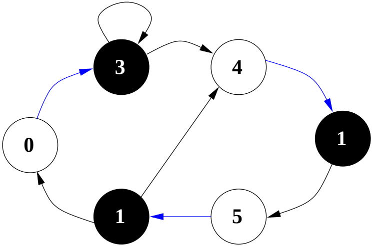
Strategy Improvement: Overview

Introduced by Vöge and Jurdziński in 2000.

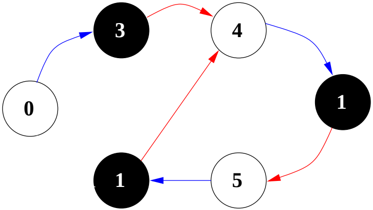
Basic principle:

1. Define a measure for strategies
2. Start with a random strategy
3. Compute the measure for the current strategy
4. Make improvements to the strategy based on the measure
5. Repeat steps 3 and 4 until no improvements can be made
6. Compute winning sets based on optimal strategies



Given a strategy for Player 0 and a vertex, a valuation represents the "best" counter-strategy for Player 1 from that vertex.

Strategy Improvement: The details



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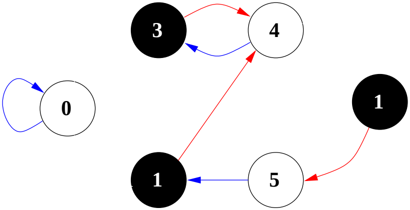

Player 0 then chooses “best” successor from each vertex

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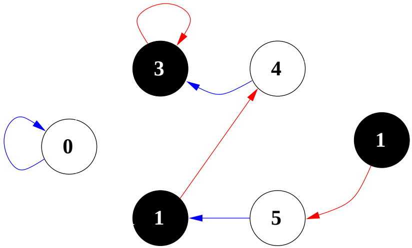

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The strategy changes are local and independent

The valuation is easy to compute $(O(|V||E|))$

Problem

How many iterations does this algorithm perform?

Strategy Improvement: Observations

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Theorem

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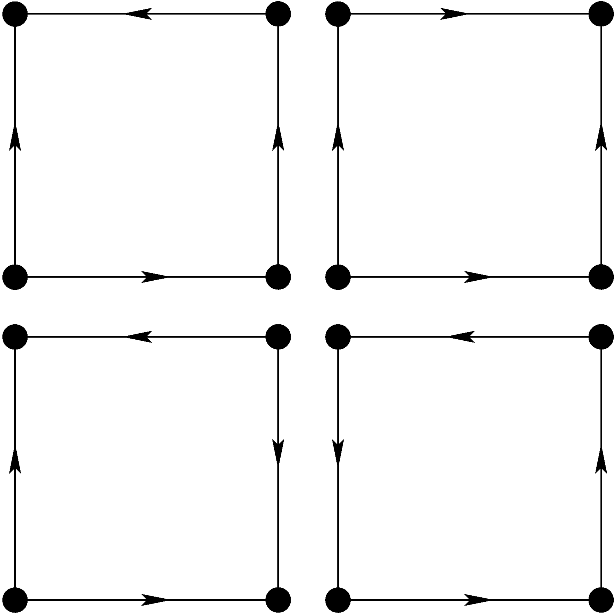
AUSOs

An orientation of a hypercube is an acyclic unique sink orientation (AUSO) if

- it is acyclic
- every sub-hypercube has a unique sink

AUSOs are also known as completely unimodal pseudo-boolean functions

AUSOs: Examples (2 dimensions)



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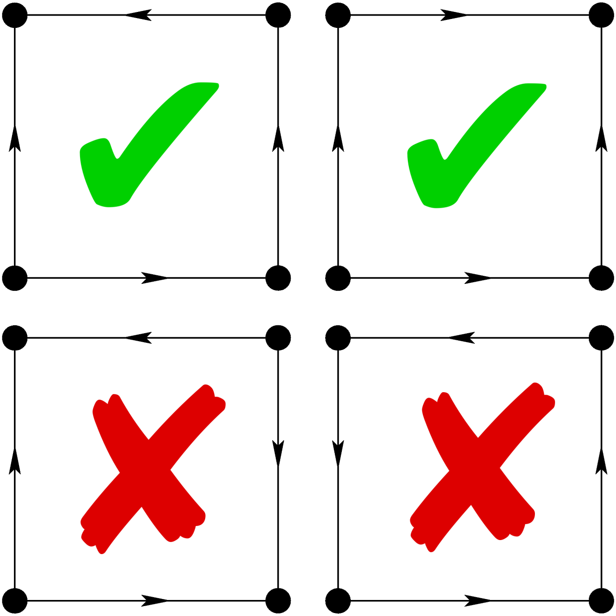

AUSOs: Properties

Common combinatorial structure with elegant properties

Theorem (Williamson Hoke 88)

1. *An acyclic orientation is an AUSO iff every 2-dimensional subcube is an AUSO*
2. *AUSOs satisfy the Hirsch conjecture*
3. *The vector of improving directions of any AUSO is a bijection*

AUSOs: Polynomial Local Search

Common problem for AUSOs: *Find the (global) sink by making local queries*

Cornerstone problem of the complexity class PLS

Open problem

Can we find the sink of an AUSO with polynomially many queries?

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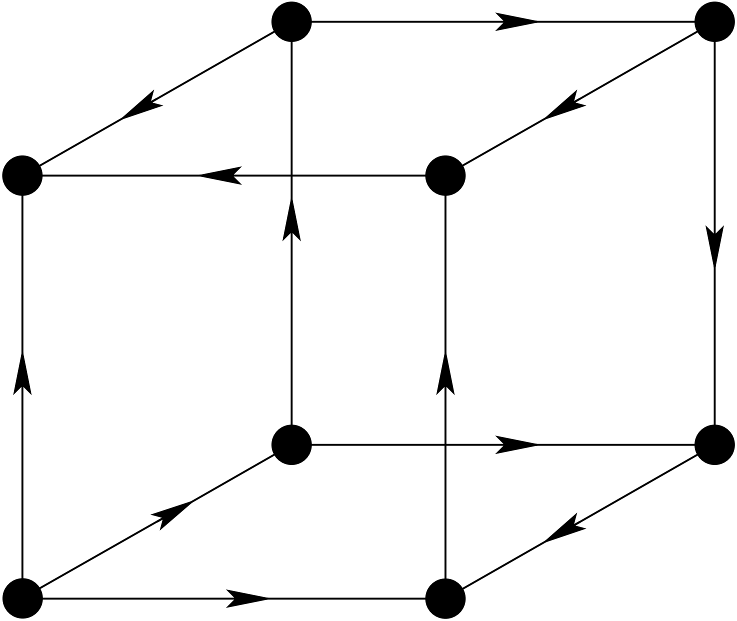
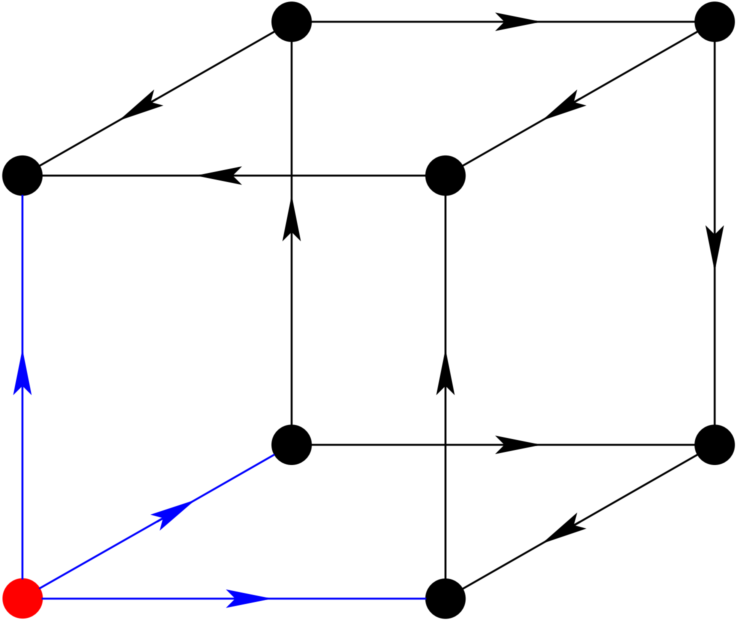
AUSOs: Finding the sink



AUSOs: Finding the sink – Bottom-Antipodal



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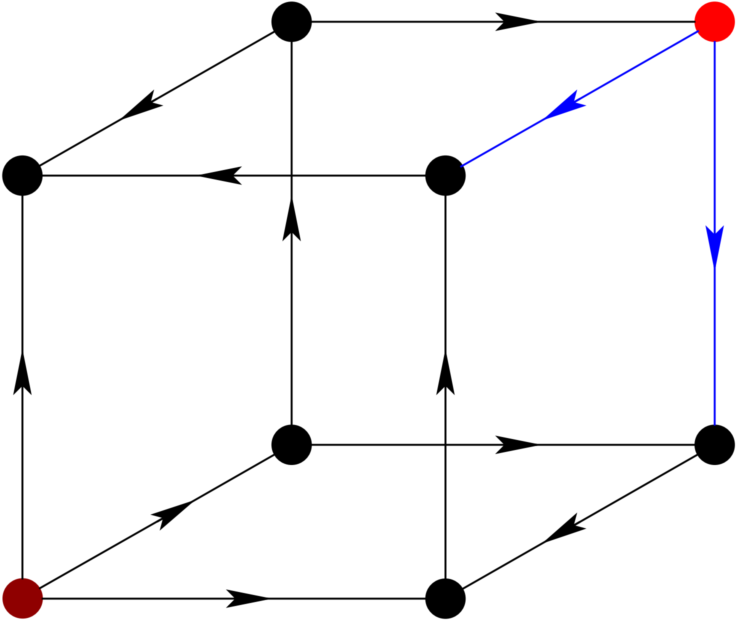

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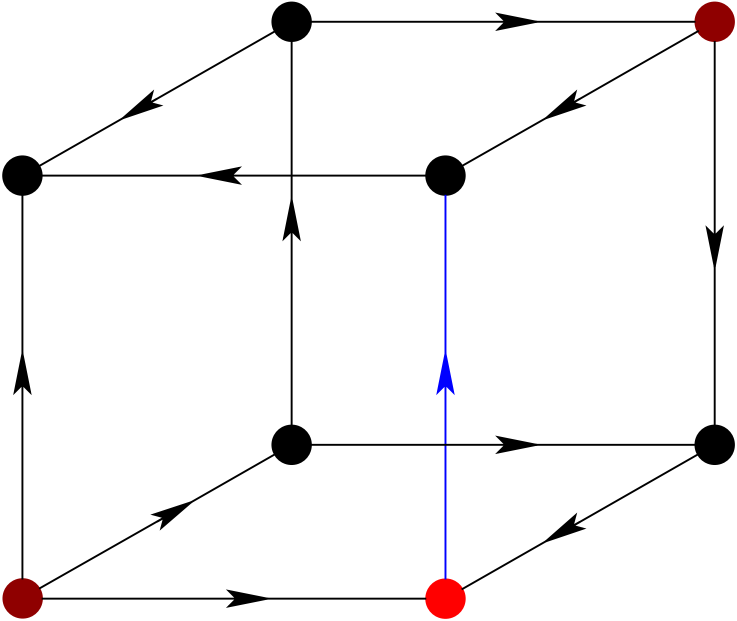

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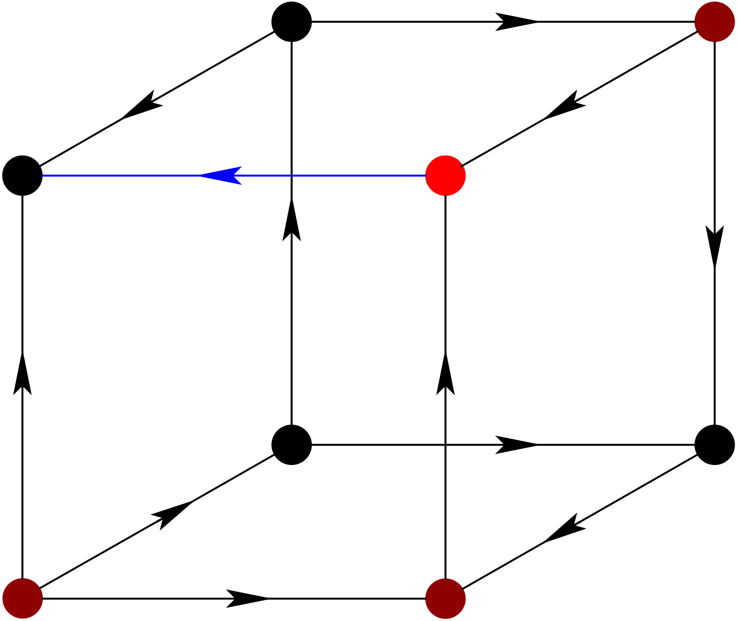

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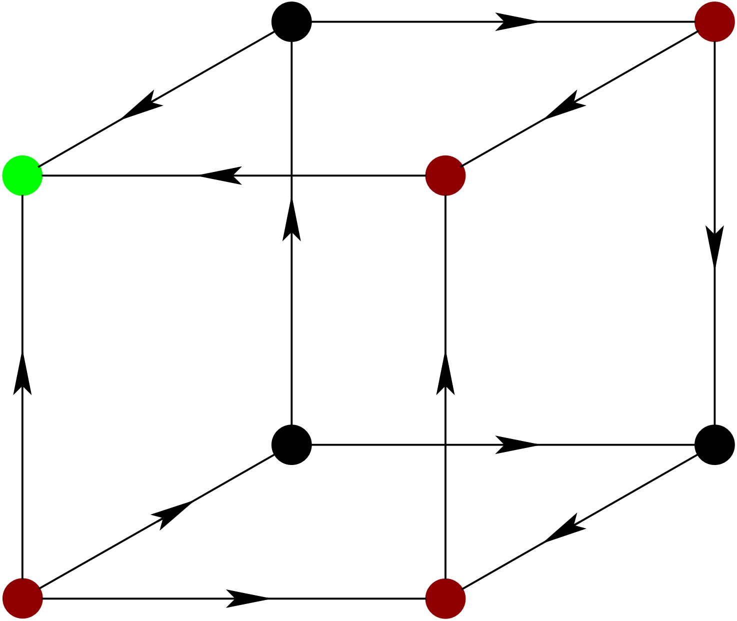

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Tying it all together: The strategy hypercube

Can assume every vertex in the arena of a parity game has out-degree 2

Positional strategies for Player 0 are then vectors in $\{0,1\}^n$ where $n = |V_0|$, that is, vertices of a hypercube

Edges of the hypercube connect strategies which differ at only one vertex

Tying it all together: The orientation

The orientation is defined by the local valuation: an edge from σ to σ' if it is "better" for Player 0 to switch at the specified vertex

Theorem (Vöge and Jurdziński 2000)

The local valuation induces an AUSO on the strategy hypercube

Corollary

The strategy improvement algorithm for parity games is a bottom-antipodal sink-finding algorithm on an acyclic unique sink orientation of the strategy hypercube.

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What does this tell us?

Improve bounds:

Mansour and Singh (1999): Bottom-antipodal takes at most $2^n/n$ steps

Szabó and Welzl (2001): Fibonacci see-saw takes at most $O(1.61^n)$ steps

Schurr and Szabó (2005): There exist AUSOs such that bottom-antipodal takes $O(2^{n/2})$ steps

Further work

Question

Is every AUSO realizable as the oriented strategy hypercube of a parity game?

Further work

Improve bounds on the number of iterations necessary for the strategy improvement algorithm

Classify AUSOs defined by parity games and restricted classes of parity games