Robot Games of Degree Two

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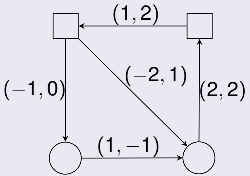
Reachability Games

- Played on labeled directed (finite or infinite) graph $G = (V, E)$ with edges labeled by $x \in X$.
- Two players, Attacker (∀dam), Defender (∃ve).
- Configuration $[v, x] \in V \times X$.
- Successor configuration is $[v', x * x']$, where $[v, x', v'] \in E$.
- Play is an infinite sequence of successive configurations.

Game with $X = \mathbb{Z}^2$ and '$*$'='$+$'



Reachability Games

Different semantics for available edges

Such as VASS (edge is disabled if after applying it counter is negative), NBVASS (negative values get truncated to zero).

Different winning conditions

- Reachability
  - Attacker's goal is to reach some configuration $[v, x]$.
- Energy
  - Upper and lower bounds on counter that ensure victory for one of the players .
- Parity
  - Each vertex has colour $\{1, \ldots, k\}$. In winning play the smallest/largest colour appearing infinitely often is even.

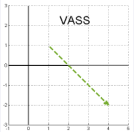
Reachability Games

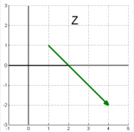
Given graph $(V_\forall \cup V_\exists, E)$, initial and target configurations $(v, \mathbf{x}), (v', \mathbf{x}')$.
Decision Problem: Does there exist a winning strategy for Attacker for reaching $(v', \mathbf{x}')$ from $(v, \mathbf{x})$?

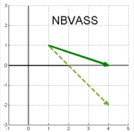
Known results, dimension 2

semantics	vectors in	complexity
VASS	$\{-1, 0, 1\}^2$	undecidable [Brázdil, Jančar, Kučera, ICALP 2010]
$\mathbb{Z}$	$\{-1, 0, 1\}^2$	undecidable [Reichert, RP 2013]
NBVASS	$\{-1, 0, 1\}^2$	undecidable [Reichert, RP 2013]

Differences between semantics





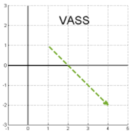


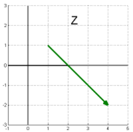
Reachability Games (dimension 1)

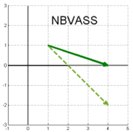
Known results, dimension 1

semantics	vectors in	complexity
VASS	$\{-1, 0, 1\}$	**PSPACE**-complete [Brázdil, Jančar, Kučera, ICALP 2010]
VASS	$\mathbb{Z}$	**EXPTIME**-hard, **EXPSPACE**
$\mathbb{Z}$	$\{-1, 0, 1\}$	**PSPACE**-complete [Reichert, RP 2013]
$\mathbb{Z}$	$\mathbb{Z}$	**EXPTIME**-hard, **EXPSPACE**
NBVASS	$\{-1, 0, 1\}$	**PSPACE**-complete [Reichert, RP 2013]
NBVASS	$\mathbb{Z}$	**EXPTIME**-hard, **EXPSPACE**

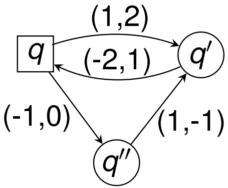
Differences between semantics







Counter Reachability Games

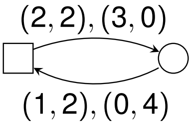


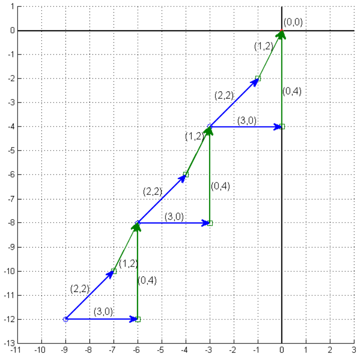
Other results:

VASS: undecidable, dim. 2
CRG: **EXPTIME**-hard, dim. 1
CRG: undecidable, dim. 2

- Directed graph $G = (V, E)$ with $E \subseteq V \times \mathbb{Z}^n \times V$.
- Two players (Defender and Attacker) with sets V_1, V_2.
- Configuration: $[v, \mathbf{x}] \in V \times \mathbb{Z}^n$.
- Play: $[v_1, \mathbf{x}_1], [v_2, \mathbf{x}_1 + \mathbf{x}_2], \ldots$, where $(v_i, \mathbf{x}_{i+1}, v_{i+1}) \in E$ for all i.
- **Target**: a configuration $[v, (0, \ldots, 0)]$ for some $v \in V$.
- **Decision Problem:** Does Attacker have a winning strategy starting from $[v_0, \mathbf{x}_0]$?

Robot Games





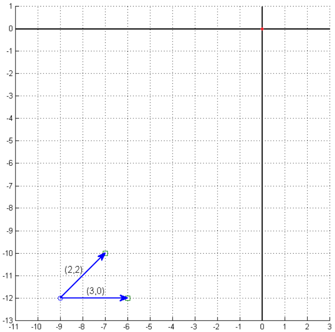
- Special case of CRG with very restricted graph.
- $|V| = 2$ and each player has one vertex.
- Target: Defender's vertex with counters at zero.

Robot Games

Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

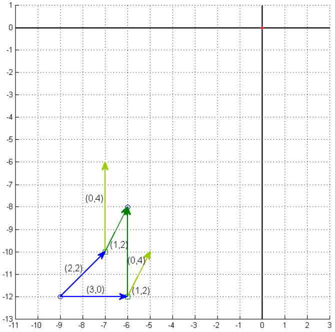
Configuration after Defender's 1st turn: $(-7,-10)$ or $(-6,-12)$



Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

Configuration after Defender's 1st turn: $(-7,-10)$ or $(-6,-12)$
Configuration after Attacker's 1st turn: $(-7,-6)$, $(-6,-8)$ or $(-5,-10)$

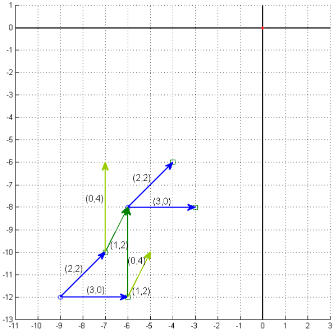


Robot Games

Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

Configuration after Attacker's 1st turn: $(-7,-6)$, $(-6,-8)$ or $(-5,-10)$
Configuration after Defender's 2nd turn: $(-4,-6)$ or $(-3,-8)$

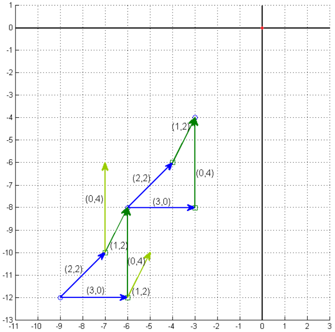


Robot Games

Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

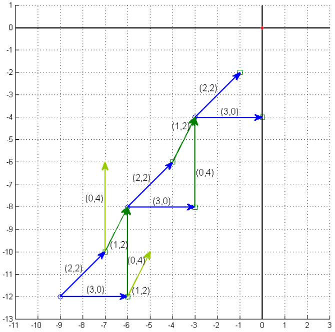
Configuration after Defender's 2nd turn: $(-4,-6)$ or $(-3,-8)$
Configuration after Attacker's 2nd turn: $(-3,-4)$



Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

Configuration after Attacker's 2nd turn: $(-3,-4)$
Configuration after Defender's 3rd turn: $(-1,-2)$ or $(0,-4)$

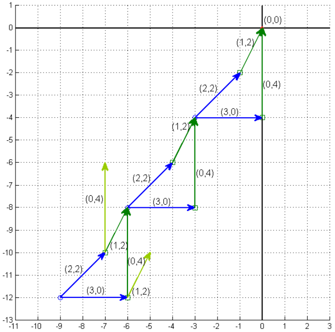


Robot Games

Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

Configuration after Attacker's 3rd turn:
$(-1,-2)+(1,2) = (0,-4)+(0,4)+(0,0)$

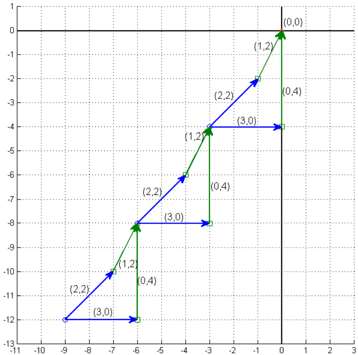


Robot Games

Example

Let $U = \{(1,2),(0,4)\}$ be Attacker's vector set and $V = \{(2,2),(3,0)\}$ Defender's and initial point $\mathbf{a} = (-9,-12)$.

Attacker has a winning strategy in this game.



Robot Games

Decision Problem:

Given vector sets $U, V \subseteq \mathbb{Z}^n$ for Attacker and Defender, initial point $\mathbf{a}$. Does Attacker have a winning strategy for reaching the origin from $\mathbf{a}$?

Known results:

- Arul, Reichert (2013): In dimension one is **EXPTIME**-complete.
- Doyen, Rabinovich refer to personal communications with Velner that the problem is undecidable for dimensions ≥ 9.

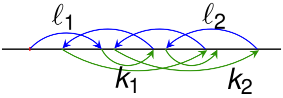
Games of degree two:

Attacker and Defender have 2 vectors, i.e. $U = \{\mathbf{u}_1, \mathbf{u}_2\}$, $V = \{\mathbf{v}_1, \mathbf{v}_2\}$

Main Result:

Checking for existance of winning strategy in Robot Game of degree 2 in dimension n is in **P**.

Dimension One



Attacker: k_1, k_2 Defender: ℓ_1, ℓ_2 Initial point: a

- x – $\#$ of k_1's played
- y – $\#$ of k_2's played
- z – $\#$ of ℓ_1's played
- w – $\#$ of ℓ_2's played

Goal

Define winning conditions for Attacker

Winning configuration for Attacker

$$\begin{cases} xk_1 + yk_2 + z\ell_1 + w\ell_2 + a & = 0 \\ x + y - z - w & = 0. \end{cases}$$

Dimension One

Winning configuration for Attacker

$$\begin{cases} xk_1 + yk_2 + z\ell_1 + w\ell_2 + a & = 0 \\ x + y - z - w & = 0. \end{cases}$$

Solving x, y

$$x = \frac{(k_2 + \ell_1)z + (k_2 + \ell_2)w + a}{k_2 - k_1},$$

$$y = \frac{(-k_1 - \ell_1)z + (-k_1 - \ell_2)w - a}{k_2 - k_1}.$$

- x – # of k_1's played
- y – # of k_2's played
- z – # of ℓ_1's played
- w – # of ℓ_2's played

Dimension One

Solving x, y

$$x = \frac{(k_2 + \ell_1)z + (k_2 + \ell_2)w + a}{k_2 - k_1},$$
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- x – # of k_1's played
- y – # of k_2's played
- z – # of ℓ_1's played
- w – # of ℓ_2's played

Game as a sequence

Consider a game as a sequence where $x + y$ and $z + w$ increase by one after each turn.

Dimension One

Solving x, y

$$x = \frac{(k_2 + \ell_1)z + (k_2 + \ell_2)w + a}{k_2 - k_1},$$
$$y = \frac{(-k_1 - \ell_1)z + (-k_1 - \ell_2)w - a}{k_2 - k_1}.$$

- x – $\#$ of k_1's played
- y – $\#$ of k_2's played
- z – $\#$ of ℓ_1's played
- w – $\#$ of ℓ_2's played

Cases to consider

- x, y are rational
- all factors of z and w are positive
- some factors of z and w are negative

Dimension One (x, y are rational)

$$x = \frac{(k_2+\ell_1)z + (k_2+\ell_2)w + a}{k_2 - k_1},$$
$$y = \frac{(-k_1-\ell_1)z + (-k_1-\ell_2)w - a}{k_2 - k_1}.$$

- x – # of k_1's played
- y – # of k_2's played
- z – # of ℓ_1's played
- w – # of ℓ_2's played

Corollary

Defender can spoil all games not satisfying

1. $\ell_1 \equiv \ell_2 \pmod{k_2 - k_1}$, *and*
2. $j(k_2+\ell_1) \equiv a \pmod{k_2 - k_1}$ *and* $j(-k_1-\ell_1) \equiv -a \pmod{k_2 - k_1}$ *for some* $j \geq 0$, $j \in \mathbb{N}$.

From now on we assume that the conditions of Corollary hold.

Dimension One (all factors of z, w are positive)

$$x = \frac{(k_2+\ell_1)z + (k_2+\ell_2)w + a}{k_2 - k_1},$$
$$y = \frac{(-k_1-\ell_1)z + (-k_1-\ell_2)w - a}{k_2 - k_1}.$$

- x – # of k_1's played
- y – # of k_2's played
- z – # of ℓ_1's played
- w – # of ℓ_2's played

- If all factors are non-negative, then by previous Corollary either $\ell_1 = \ell_2$ or $-\ell_1 = k_1$ and $-\ell_2 = k_2$.

Dimension One (all factors of z, w are positive)

$$x = \frac{(k_2+\ell_1)z + (k_2+\ell_2)w + a}{k_2 - k_1},$$
$$y = \frac{(-k_1-\ell_1)z + (-k_1-\ell_2)w - a}{k_2 - k_1}.$$

- x – # of k_1's played
- y – # of k_2's played
- z – # of ℓ_1's played
- w – # of ℓ_2's played

- If all factors are non-negative, then by previous Corollary either $\ell_1 = \ell_2$ or $-\ell_1 = k_1$ and $-\ell_2 = k_2$.
- Defender has no input. Attacker has a winning strategy if $x(k_1+\ell_1) + y(k_2+\ell_1) + a = 0$ has a solution.

Dimension One (all factors of z, w are positive)

$$x = \frac{(k_2+\ell_1)z + (k_2+\ell_2)w + a}{k_2-k_1},$$
$$y = \frac{(-k_1-\ell_1)z + (-k_1-\ell_2)w - a}{k_2-k_1}.$$

- x – # of k_1's played
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- z – # of ℓ_1's played
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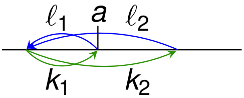
- If all factors are non-negative, then by previous Corollary either $\ell_1 = \ell_2$ or $-\ell_1 = k_1$ and $-\ell_2 = k_2$.
- After the first turn, Defender can counter whichever integer Attacker plays.



Dimension One (some factors of z, w are negative)

Changing one w to z

$$\frac{k_2 + \ell_1 - k_2 - \ell_2}{k_2 - k_1} = \frac{\ell_1 - \ell_2}{k_2 - k_1} = -d$$

in equation of x.

- If $|d| \geq 2$, then Defender can choose correct vector during the last turn to keep the game from reaching 0. Attacker cannot counter as he needs d moves to correct the course of the game.
- If $d = \pm 1$, then $k_i + \ell_i = m$ for $i = 1, 2$ and m is added to the counter after each turn. Attacker has to force the game into $-tm$ for some $t \in \mathbb{N}$. This can be done only during the first turn.

From this case analysis we get:

Theorem

Deciding winner in one-dimensional Robot Game of degree 2 is in **P**.

- Case of rational x, y is in **P**
- Case of positive factors of z, w is in **P**
- Case of mixed factors of z, w is in **P**

- Attacker's set: $U = \{\mathbf{u}_1, \mathbf{u}_2\}$.
- Defender's set: $V = \{\mathbf{v}_1, \mathbf{v}_2\}$.
- Initial vector: $\mathbf{a}$.

Instead of considering two one-dimensional games that have to be won simultaneously, we simplify the game by using a simple substitution.

Vector $\mathbf{u}_2$ is played by default

- Attacker's set: $U' = \{\mathbf{u}_1 - \mathbf{u}_2, (0,0)\} = \{\mathbf{u}', (0,0)\}$.
- Defender's set: $V' = \{\mathbf{v}_1 + \mathbf{u}_2, \mathbf{v}_2 + \mathbf{u}_2\} = \{\mathbf{v}_1', \mathbf{v}_2'\}$.

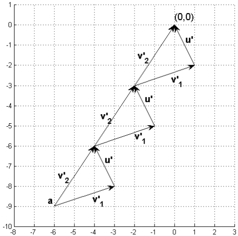
Dimension Two

Lemma

Attacker can win a game if and only if $\mathbf{v}'_1 + \mathbf{u}' = \mathbf{v}'_2$ *and* $\mathbf{a} = -k\mathbf{v}'_2$ *or* $\mathbf{v}'_2 + \mathbf{u}' = \mathbf{v}'_1$ *and* $\mathbf{a} = -k\mathbf{v}'_1$ *for some* $k \in \mathbb{N}$.

Recall:

$$\mathbf{u}' = \mathbf{u}_1 - \mathbf{u}_2$$
$$\mathbf{v}'_1 = \mathbf{v}_1 + \mathbf{u}_2$$
$$\mathbf{v}'_2 = \mathbf{v}_2 + \mathbf{u}_2$$



Theorem

Deciding winner in two-dimensional Robot Game of degree 2 is in **P**.

Dimension Three or Higher

- Attacker's set: $U = \{(\alpha_1, \alpha_2, \dots, \alpha_n), (\beta_1, \beta_2, \dots, \beta_n)\}$.
- Defender's set: $V = \{(\gamma_1, \gamma_2, \dots, \gamma_n), (\delta_1, \delta_2, \dots, \delta_n)\}$.
- Initial vector: $\mathbf{a} = (a_1, \dots, a_n)$.

$$\begin{cases} x\alpha_1 + y\beta_1 + z\gamma_1 + w\delta_1 + a_1 & = 0 \\ \vdots & \\ x\alpha_n + y\beta_n + z\gamma_n + w\delta_n + a_n & = 0 \end{cases} \quad \text{and}$$

$$x + y - z - w = 0$$

under constrain $x, y, z, w \in \mathbb{N}$.

Number of linearly independent equations

- There are at least 5 linearly independent equations.
- There are 4 linearly independent equations.
- There are 3 linearly independent equations.
- There are 2 linearly independent equations.
- There is 1 linearly independent equation.

Dimension Three or Higher

Number of linearly independent equations

- There are at least 5 linearly independent equations.
  - There is no solution to the system of equations. Attacker cannot win.
- There are 4 linearly independent equations.
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- There are 3 linearly independent equations.
  - We have two-dimensional game. Attacker's winning conditions have been classified previously.
- There are 2 linearly independent equations.
- There is 1 linearly independent equation.

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- There are 2 linearly independent equations.
  - We have one-dimenisional game. Attacker's winning conditions have been classified previously.
- There is 1 linearly independent equation.
  - Attacker always wins after the first turn.

Summary

Theorem

Deciding winner in n-dimensional Robot Game of degree 2 is in **P***.*

Theorem (Reichert (2012))

Checking for winner in Counter Reachability Game in dimension two is undecidable.

Corollary

Checking for winner in Counter Reachability Game in dimension two of degree two is undecidable.

Open Question

Deciding winner in n-dimensional Robot Game of degree $3, 4, \ldots$.

THANK YOU!