Comparing Operational Models of Name-Passing Process Calculi ⋆

Marcelo Fiore [1] and Sam Staton

Computer Laboratory
University of Cambridge

Abstract

We study three operational models of name-passing process calculi: coalgebras on (pre)sheaves, indexed labelled transition systems, and history dependent automata.

The coalgebraic model is considered both for *presheaves* over the category of finite sets and injections, and for its subcategory of atomic sheaves known as the *Schanuel topos*. Each coalgebra induces an *indexed labelled transition system*. Such transition systems are characterised, relating the coalgebraic approach to an existing model of name-passing. Further, we consider *internal* labelled transition systems within the sheaf topos, and axiomatise a class that is in precise correspondence with the coalgebraic and the indexed labelled transition system models. By establishing and exploiting the equivalence of the Schanuel topos with a category of *named-sets*, these internal labelled transition systems are also related to the theory of *history dependent automata*.

Operational models of concurrent computation typically describe processes in terms of a state space together with its possible evolution by performing atomic actions. Transition systems have proved useful in modelling the kinds of processes involved in static networks, like those described by CCS and related calculi. In these situations, processes evolve by communicating along named channels. Modern systems, though, often contain an element of mobility and reconfiguration. In languages such as the π-calculus, this dynamic structure is described in terms of the communication of the channel names themselves: *name-passing*. This allows, for instance, one process to advise another process to begin communicating on a particular channel. Not surprisingly, techniques and models relevant to static networks are inadequate in the name-passing context. Thus operational models of name-passing process calculi have been investigated.

⋆ This paper supersedes the extended abstract with the same title that appeared in the Proceedings of CMCS'04 [5].

[1] Research supported by an EPSRC Advanced Research Fellowship.

Cattani and Sewell [1] have observed that labelled transition systems are too generous a model for name-passing systems. They have thus constrained the labelled transition systems that they consider in two ways. The first is that the state space must be *indexed*, meaning that the names available to each state are explicit and a notion of renaming of states is built-in. Secondly, the labelled transition systems under consideration are required to satisfy certain axioms that are theorems of the transition systems induced by π-calculus processes. For instance, input actions must occur in a particular uniform way and transitions must be invariant under injective renamings.

The theory of coalgebras has arisen as a general abstract theory of systems. Coalgebras provide a general way of describing the stepwise evolution of a system, together with an abstract notion of bisimulation. Thus, for instance, it is possible to reformulate and generalise familiar notions such as rule formats and modal logics. Fiore and Turi [6] have developed coalgebraic models for name-passing, modelling the early and late kinds of bisimulation that arise. Coalgebras are considered in a presheaf category; thus a renaming structure is imposed on states and the naturality of morphisms ensures that evolution is invariant under these renamings.

Neither the transition system model nor the coalgebraic model is immediately suitable for implementation because of the cardinality of the state spaces involved once all the renamings are considered. Montanari and Pistore [8] have introduced several notions of *named-sets* which often provide finite descriptions of state spaces by recording certain features of canonical states. By reformulating the theory of automata in this setting they have been able to implement tools for the verification of name-passing processes.

The theme of this paper is to compare and relate the above developments.

In Section 1, we recall the coalgebraic model of Fiore and Turi for early bisimulation. The carriers of the coalgebras are presheaves over **I**, the category of finite sets of names and *injections* between them, reflecting the idea that bisimilarity is invariant under injective renaming. We recall how a labelled transition system arises from such a coalgebra and axiomatise the transition systems that arise in this way. Subsequently, in Section 2, we relate these axiomatised transition systems with those suggested by Cattani and Sewell. A major difference is that Cattani and Sewell were concerned with a form of open bisimilarity, which is invariant under *arbitrary* renaming. Thus it is necessary to consider presheaves over **F**, the category of finite sets of names and all functions between them. In Section 3 we revisit the presheaves of Section 1 to consider the implications of a sheaf condition. This leads us to the Schanuel topos, and we explain how this topos is the Kleisli category of a monad on the category of presheaves on **B**, the category of finite sets of names and *bijections* between them. Recasting Section 1 in this light we are able to simplify our axioms on transition systems. Lastly, in Section 4, we introduce a new notion of *internal labelled transition system*, that is, a labelled transition system internal to the Schanuel topos satisfying certain conditions phrased in its internal language. By exhibiting the Schanuel topos as equivalent to a category of named-sets we connect these internal transition systems with the history dependent automata of Montanari and Pistore.

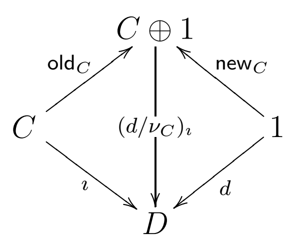
1 Coalgebraic models over presheaves

Presheaves for name-passing process calculi. A key component of the fully abstract models of the π-calculus of Fiore, Moggi, and Sangiorgi [4], and of Stark [10] is the use of presheaves to index the domains of processes/states by the names that they may use.

Fixing an infinite universe of names $\mathcal{N}$, a suitable indexing category $\mathbf{I}$ is the category of all finite subsets of $\mathcal{N}$ and injections between them. Indeed, $\mathbf{I}$ is equivalent to the free symmetric monoidal category with an initial unit on one generator, and as such has the appropriate structure for modelling name generation. Accordingly, thus, we will consider $\mathbf{I}$ in this vein, denoting the generator (a singleton) as 1, the initial unit (the empty set) as $\emptyset$, and the tensor product (a chosen disjoint union) by $\oplus$. Importantly, it follows that every finite name-set $C \subseteq_{\mathrm{f}} \mathcal{N}$ comes equipped with canonical maps

$$\mathsf{old}_C : C \to (C \oplus 1) \leftarrow 1 : \mathsf{new}_C$$

given by $\mathsf{old}_C = \big(C \cong (C \oplus \emptyset) \to (C \oplus 1)\big)$ and $\mathsf{new}_C = \big(1 \cong (\emptyset \oplus 1) \to (C \oplus 1)\big)$. These maps induce a notion of injective renaming as follows: for an injection $\imath : C \to D$ and for $d \in D \setminus \mathrm{im}(\imath)$, we let $(d/\nu_C)_\imath : (C \oplus 1) \to D$ be the unique injective function making the following diagram



commute. Further, whenever $(d/\nu_C)_\imath$ is a bijection we write $(\nu_C/d)_\imath$ for its inverse. Finally, as a notational convention, we drop the subindex whenever $\imath$ is an inclusion.

A *presheaf* (*i.e.*, a set-valued functor) $P : \mathbf{I} \to \mathbf{Set}$ can be thought of as mapping each name-set $C \subseteq_{\mathrm{f}} \mathcal{N}$ to a set of processes PC that use (some of) the names in C, and mapping each injective renaming function $\imath : C \to D$ to a renaming function $P\imath : PC \to PD$ on processes. We write $[\imath]p$ for $P\imath(p)$ when it is clear which presheaf we are referring to.

Coalgebras for early bisimulation. The work of Fiore and Turi [6] provides a model of name-passing using coalgebras in $\mathbf{Set}^{\mathbf{I}}$, the category of presheaves over $\mathbf{I}$ and natural transformations. Early and late bisimulation are captured in terms of coalgebraic bisimulation for particular behaviour functors.

We recall the relevant type constructors on presheaves.

- A type of names N — the inclusion functor $\mathbf{I} \to \mathbf{Set}$.
- A dynamic allocation operator δP, given by $(\delta P)C = P(C \oplus 1)$. An injection $\imath : C \to D$ maps $p \in (\delta P)C$ to $P(\imath \oplus 1)(p)$.
- Non-empty covariant powerset $\wp^+$, acting pointwise as the non-empty covariant powerset functor in $\mathbf{Set}$.
- The unit type 1 — the constantly 1 presheaf (terminal in $\mathbf{Set}^{\mathbf{I}}$).
- Product and sum, defined pointwise in the standard fashion.
- The exponential P^Q with $P^Q C$ given (via the Yoneda lemma) by the set of natural transformations $\mathbf{I}(C, -) \times Q \dot{\to} P$.

  When $Q = N$, a finitary description is permitted, namely

  $$P^N C = (PC)^C \times P(C \oplus 1)$$

  since a natural transformation $\alpha : \mathbf{I}(C, -) \times N \dot{\to} P$ is completely determined by the components

  $$\alpha_C(\mathrm{id}_C, -) : C \to PC \qquad \text{and} \qquad \alpha_{(C \oplus 1)}\big(\mathsf{old}_C, \mathsf{new}_C()\big) \in P(C \oplus 1)$$

  that is, by its action on 'known' names and its action on a generic new name. An injection $\imath : C \to D$ acts on a pair $(f, p) \in P^N C$ to produce the pair $(f', p') \in P^N D$ given by

  $$f'\,c' = \begin{cases} [\imath](f\,c) & \text{, if } c' = \imath c \\ [(c'/\nu_C)_\imath]\,p & \text{, otherwise} \end{cases} \qquad\qquad p' = [\imath \oplus 1]p \qquad . \tag{1}$$

- Pointwise partial exponentials $N \rightharpoonup P$ and $1 \rightharpoonup P$ given as follows.

  $(N \rightharpoonup P)C$ is the set $C \rightharpoonup PC$ of partial functions from C to PC. For any injection $\imath : C \to D$ and partial function $f : C \rightharpoonup PC$, the partial function $(N \rightharpoonup P)\imath f : D \rightharpoonup PD$ is defined as the composite

  $$D \overset{\imath^R}{\rightharpoonup} C \overset{f}{\rightharpoonup} PC \overset{P\imath}{\to} PD$$

  where $\imath^R$ denotes the partial function defined at $d \in D$ iff $d = \imath c$ for some (necessarily unique) $c \in C$, in which case $\imath^R(d) = c$.

Analogously, $(1 \rightharpoonup P)C = (1 \rightharpoonup PC)$ with $(1 \rightharpoonup P)\imath f = (P\imath) \circ f : 1 \rightharpoonup PD$ for all $\imath : C \to D$ in $\mathbf{I}$ and $f : 1 \rightharpoonup PC$.

A suitable behaviour endofunctor B_{e} on $\mathbf{Set}^{\mathbf{I}}$ for early bisimulation is given by

$$\begin{aligned} B_{\mathrm{e}}P = {} & N \rightharpoonup ((\wp^+ P)^N) && \textit{input} \\ & \times N \rightharpoonup (\wp^+(N \times P)) && \textit{output} \\ & \times N \rightharpoonup (\wp^+ \delta P) && \textit{bound output} \\ & \times 1 \rightharpoonup (\wp^+ P) && \textit{silent action.} \end{aligned}$$

A B_{e}-coalgebra is given by a presheaf $P \in \mathbf{Set}^{\mathbf{I}}$ together with a natural transformation $h : P \dot{\to} B_{\mathrm{e}}P$ in $\mathbf{Set}^{\mathbf{I}}$. A component h_C ($C \in \mathbf{I}$) of such a natural transformation maps a *process* in PC to a *behaviour* in $B_{\mathrm{e}}P(C)$; that is, a tuple in

$$\begin{aligned} & C \rightharpoonup \big((\wp^+ PC)^C \times \wp^+ P(C \oplus 1)\big) \\ & \times C \rightharpoonup \big(\wp^+(C \times PC)\big) \\ & \times C \rightharpoonup \big(\wp^+ P(C \oplus 1)\big) \\ & \times 1 \rightharpoonup (\wp^+ PC) \end{aligned}$$

indicating the capabilities of the process. For example, for $p \in PC$, if $h_C(p) = (i, o, b, t)$, then i is a partial function to be interpreted as follows. For some channel name $c \in C$, i is defined at c if p is able to input on the channel c, in which case $i(c)$ is a pair $(\phi, \psi) \in (\wp^+ PC)^C \times \big(\wp^+ P(C \oplus 1)\big)$. Now suppose a known name $d \in C$ was to be input, then p would continue as one of the processes in the non-empty set $\phi(d)$. For a fresh name $d \notin C$, we use ψ as a set of templates for the resultant process, continuing as $[d/\nu_C]p' \in P(C \cup \{d\})$ for some $p' \in \psi$.

Just as a coalgebra $X \to \wp(\mathit{Lab} \times X)$ in $\mathbf{Set}$ induces a transition relation over the state space given by X, a coalgebra $P \dot{\to} B_{\mathrm{e}}P$ in $\mathbf{Set}^{\mathbf{I}}$ induces a transition relation with state space given by the *elements* of P, *i.e.* the set $\int P = \sum_{C \in \mathbf{I}} PC$. We write $C \vdash p$ for an element $(C, p) \in \int P$.

The labels on the transitions are taken from the set

$$\mathit{Lab} = (\mathcal{N} \times \mathcal{N}) + (\mathcal{N} \times \mathcal{N}) + 1 \qquad ,$$

with input (written $c?d$), output (written $c!d$), and silent (written τ) actions respectively. Each label ℓ has associated with it some channels $\mathsf{ch}(\ell)$ and data $\mathsf{dat}(\ell)$, which for convenience we will consider as sets; here they will have at most one element, as follows.

ℓ	$c?d$	$c!d$	τ
$\mathsf{ch}(\ell)$	$\{c\}$	$\{c\}$	$\emptyset$
$\mathsf{dat}(\ell)$	$\{d\}$	$\{d\}$	$\emptyset$

For a label ℓ and a function f between subsets of $\mathcal{N}$ we write $[f]\ell$ for the obvious renaming.

Given a coalgebra $h : P \dot{\to} B_e P$, a transition relation $\longrightarrow_h \subseteq \int P \times \mathit{Lab} \times \int P$ is induced as follows:

$$\begin{aligned}
C \vdash p \xrightarrow{c?d}_h C \vdash p' &\iff p' \in \pi_1(\pi_1(h_C p)c)d \\
C \vdash p \xrightarrow{c?z}_h C \cup \{z\} \vdash [z/\nu_C]q &\iff q \in \pi_2(\pi_1(h_C p)c) \\
C \vdash p \xrightarrow{c!d}_h C \vdash p' &\iff (d, p') \in \pi_2(h_C p)c \\
C \vdash p \xrightarrow{c!z}_h C \cup \{z\} \vdash [z/\nu_C]q &\iff q \in \pi_3(h_C p)c \\
C \vdash p \xrightarrow{\tau}_h C \vdash p' &\iff p' \in \pi_4(h_C p)()
\end{aligned} \tag{2}$$

where $c, d \in C$ and $z \notin C$.

We define early bisimulation for transition relations such as these.

Definition 1 *Consider two transition relations, $\longrightarrow_1 \subseteq \int P_1 \times \mathit{Lab} \times \int P_1$ and $\longrightarrow_2 \subseteq \int P_2 \times \mathit{Lab} \times \int P_2$, for presheaves $P_1, P_2 \in \mathbf{Set}^{\mathbf{I}}$.*

A relation $R \subseteq \int P_1 \times \int P_2$ is an early bisimulation *between $\longrightarrow_1$ and $\longrightarrow_2$ if whenever $(C_1 \vdash p_1)\ R\ (C_2 \vdash p_2)$ then the following conditions hold:*

(1) $\forall \ell \in \mathit{Lab}, (C_1' \vdash p_1') \in \int P_1.$
$$\Big(C_1 \vdash p_1 \xrightarrow{\ell}_1 C_1' \vdash p_1' \implies \exists (C_2' \vdash p_2') \in \textstyle\int P_2.\ C_2 \vdash p_2 \xrightarrow{\ell}_2 C_2' \vdash p_2' \text{ and } (C_1' \vdash p_1')\ R\ (C_2' \vdash p_2')\Big)$$

(2) $\forall \ell \in \mathit{Lab}, (C_2' \vdash p_2') \in \int P_2.$
$$\Big(C_2 \vdash p_2 \xrightarrow{\ell}_2 C_2' \vdash p_2' \implies \exists (C_1' \vdash p_1') \in \textstyle\int P_1.\ C_1 \vdash p_1 \xrightarrow{\ell}_1 C_1' \vdash p_1' \text{ and } (C_1' \vdash p_1')\ R\ (C_2' \vdash p_2')\Big) \quad .$$

An early bisimulation R that further satisfies

(3) $(C_1 \vdash p_1)\ R\ (C_2 \vdash p_2) \implies C_1 = C_2$

(4) $(C \vdash p_1)\ R\ (C \vdash p_2) \implies \forall \imath : C \to C'$ *in* $\mathbf{I}$. $(C' \vdash [\imath]p_1)\ R\ (C' \vdash [\imath]p_2)$

is called an **I**-indexed early bisimulation.

Fiore and Turi [6] show that B_e-coalgebraic bisimulations between coalgebras (P_1, h_1), (P_2, h_2) correspond to **I**-indexed early bisimulations for the induced transition relations $(\int P_1, \longrightarrow_{h_1})$, $(\int P_2, \longrightarrow_{h_2})$. We remark that for the π-calculus, early bisimilarity and **I**-indexed early bisimilarity coincide.

I-*indexed labelled transition systems*

It is certainly not the case that every transition relation is induced by a B_e-coalgebra. In order to understand this coalgebraic model we characterise the transition relations that are induced.

Definition 2 *An* **I**-indexed labelled transition system (**I**-LTS) *is a presheaf* $P \in \mathbf{Set}^\mathbf{I}$ *together with a transition relation* $\longrightarrow \subseteq \int P \times Lab \times \int P$ *satisfying Conditions* **I**1–**I**6 *of Figure 1.*

Notation. For a function $f : C \to D$ and $C' \subseteq C$ the notation $f|_{C'}$ stands for the surjection $C' \twoheadrightarrow f(C')$ given by restricting the domain to C' and the codomain to the corresponding image $f(C') \subseteq D$.

Conditions **I**5 and **I**6 together capture a dichotomy in the induced transition systems: a transition is either allowed or disallowed. Transitions cannot depend upon extraneous names.

The introduced notion of **I**-LTS is justified by the following result.

Theorem 3 *The mapping (2) associating a transition relation to a* B_e*-coalgebra yields a bijective correspondence between* B_e*-coalgebras in* $\mathbf{Set}^\mathbf{I}$ *and* **I**-*LTSs over presheaves in* $\mathbf{Set}^\mathbf{I}$.

An **I**-LTS $(P, \longrightarrow)$ induces a B_e-coalgebra $\vec{h} : P \to B_\mathrm{e}P$ whose components are given according to the following definition: for $C \in \mathbf{I}$, $p \in PC$ and $c \in C$,

$$\begin{aligned}
\pi_1(\vec{h}_C p)c\downarrow &\iff \exists\, d \in \mathcal{N}, D \in \mathbf{I}, p' \in PD.\ C \vdash p \xrightarrow{c?d} D \vdash p' \\
\pi_2(\vec{h}_C p)c\downarrow &\iff \exists\, d \in C, D \in \mathbf{I}, p' \in PD.\ C \vdash p \xrightarrow{c!d} D \vdash p' \\
\pi_3(\vec{h}_C p)c\downarrow &\iff \exists\, z \in \mathcal{N} \setminus C, D \in \mathbf{I}, p' \in PD.\ C \vdash p \xrightarrow{c!z} D \vdash p' \\
\pi_4(\vec{h}_C p)()\downarrow &\iff \exists\, D \in \mathbf{I}, p' \in PD.\ C \vdash p \xrightarrow{\tau} D \vdash p'
\end{aligned} \tag{3}$$

I1. Channel is known and at most transmitted data is learnt:

$$C \vdash p \xrightarrow{\ell} C' \vdash p' \implies \mathsf{ch}(\ell) \subseteq C \;\wedge\; C' = C \cup \mathsf{dat}(\ell)$$

I2. If one name can be input, then so can any other: for all $z \in \mathcal{N}$:

$$C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash p' \\ \implies \exists p'' \in P(C \cup \{z\}).\quad C \vdash p \xrightarrow{c?z} C \cup \{z\} \vdash p''$$

I3. Bijective maps preserve transitions: for all $D \in \mathbf{I}$, with $C \cup C' \overset{\beta}{\cong} D$:

$$C \vdash p \xrightarrow{\ell} C' \vdash p' \implies \beta(C) \vdash [\beta|_C]p \xrightarrow{[\beta]\ell} \beta(C') \vdash [\beta|_{C'}]p'$$

I4a. Knowing/forgetting input data preserves transitions:

$$C \vdash p \xrightarrow{c?z} C \cup \{z\} \vdash p' \\ \iff C \cup \{z\} \vdash [C \hookrightarrow C \cup \{z\}]p \xrightarrow{c?z} C \cup \{z\} \vdash p'$$

I4b. Known output data must really be known:

$$C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \xrightarrow{c!d} C \cup \{d\} \vdash p' \implies d \in C$$

I5. Inclusion maps preserve transitions:

$$C \vdash p \xrightarrow{\ell} C' \vdash p' \;\wedge\; D \cap \mathsf{dat}(\ell) = \emptyset \\ \implies C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{\ell} C' \cup D \vdash [C' \hookrightarrow C' \cup D]p'$$

I6. Inclusion maps reflect transitions:

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{\ell} C' \cup D \vdash p' \;\wedge\; D \cap \mathsf{dat}(\ell) = \emptyset \\ \implies \exists p'' \in P(C \cup \mathsf{dat}(\ell)). \\ [C \cup \mathsf{dat}(\ell) \hookrightarrow C' \cup D]p'' = p' \;\wedge\; C \vdash p \xrightarrow{\ell} C \cup \mathsf{dat}(\ell) \vdash p''$$

Fig. 1. Requirements on an **I**-indexed labelled transition system.

with

$$\begin{aligned}
\pi_1(\vec{h}_C p)c &= \Big(\lambda d \in C.\ \Big\{p' \in PC \mid C \vdash p \xrightarrow{c?d} C \vdash p'\Big\}\ , \\
&\qquad \Big\{p' \in P(C \oplus 1) \mid C \vdash p \xrightarrow{c?z} C \cup \{z\} \vdash [z/\nu_C]p' \text{ for } z \notin C\Big\} \Big) \\
\pi_2(\vec{h}_C p)c &= \Big\{(d, p') \in C \times PC \mid C \vdash p \xrightarrow{c!d} C \vdash p'\Big\} \\
\pi_3(\vec{h}_C p)c &= \Big\{p' \in P(C \oplus 1) \mid C \vdash p \xrightarrow{c!z} C \cup \{z\} \vdash [z/\nu_C]p' \text{ for } z \notin C\Big\} \\
\pi_4(\vec{h}_C p)() &= \Big\{p' \in PC \mid C \vdash p \xrightarrow{\tau} C \vdash p'\Big\} \quad .
\end{aligned} \tag{4}$$

In Appendix A we show that the transition relation $(P, \longrightarrow_h)$ of (2) induced by a coalgebra $h : P \to B_e P$ is an **I**-LTS and, conversely, that the definition of (3) and (4) yields a natural family of maps $\vec{h}_C : PC \to B_e PC$ ($C \in \mathbf{I}$). Theorem 3 follows because these transformations are inverses of each other.

In brief, Conditions **I**1 and **I**2 correspond to the well-formedness of the induced family of maps $\vec{h}$. Conditions **I**3–**I**6 correspond to the naturality of the induced family of maps, with Condition **I**4a accounting for the action of the exponential that is used to model input, and Condition **I**4b enforcing the separation between output and bound output. (This axiom system corrects an oversight in that of [5].)

2 F-indexed labelled transition systems

The model considered above is concerned with describing early bisimulation. We now turn to a finer notion. To formulate this, we require a different indexing category: let $\mathbf{F}$ be the category of finite subsets of $\mathcal{N}$ and *all* functions between them. Precomposition with the inclusion functor $\mathbf{I} \hookrightarrow \mathbf{F}$ gives a forgetful functor $|-| : \mathbf{Set}^{\mathbf{F}} \to \mathbf{Set}^{\mathbf{I}}$. Since the sets $\int X = \sum_{C \in \mathbf{F}} X(C)$ and $\int |X|$ are equal, a transition relation $\twoheadrightarrow \subseteq \int X \times Lab \times \int X$ is also a transition relation $\twoheadrightarrow \subseteq \int |X| \times Lab \times \int |X|$, and vice versa.

Definition 4 *A relation $R \subseteq \int X_1 \times \int X_2$ is an* $\mathbf{F}$-indexed early bisimulation *between transition relations $(\int X_1, \twoheadrightarrow_1)$ and $(\int X_2, \twoheadrightarrow_2)$ if it is an early bisimulation in the sense of Definition 1 and satisfies the following additional conditions.*

(3') $(C_1 \vdash p_1)\ R\ (C_2 \vdash p_2) \implies C_1 = C_2$

(4') $(C \vdash p_1)\ R\ (C \vdash p_2)$

$$\implies \forall f : C \to C' \text{ in } \mathbf{F}.\ (C' \vdash [f]p_1)\ R\ (C' \vdash [f]p_2)$$

Note that this definition slightly differs from Sangiorgi's notion of open bisimulation [9], in which distinctions are used to exempt names introduced by bound output transitions from being joined by the renamings that are considered. However, we think that the above definition is still of interest as it arises from the general theory of Fiore and Turi [6] and because, for the π-calculus, $\mathbf{F}$-indexed early bisimilarity is the greatest congruence that is an early bisimulation.

Cattani and Sewell have developed a model of name-passing that is based on the above notion of $\mathbf{F}$-indexed early bisimulation. They consider a class of indexed labelled transition systems that are required to satisfy certain axioms. These axioms are suggested according to experience and intuition, but are not induced from mathematical structure as in the case of Conditions **I**1–**I**6 for

I-LTSs (Figure 1). However, our axioms essentially match up with theirs. The main difference highlights the relationship between **I**-indexed and **F**-indexed early bisimulation.

Definition 5 (Cattani and Sewell) *An* **F**-indexed labelled transition system (**F**-LTS) *is a presheaf* $X \in \mathbf{Set}^{\mathbf{F}}$ *together with a transition relation* $\twoheadrightarrow \subseteq \int X \times Lab \times \int X$ *satisfying Conditions* **F**1–**F**4 *of Figure 2.*

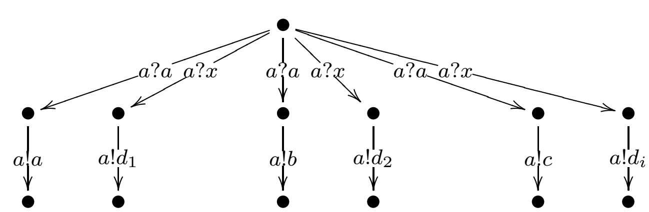
Conditions **F**1–**F**4 are Conditions 1–4 of Cattani and Sewell rewritten in our notation. Furthermore, in Condition **F**4 we have only considered inclusion maps, while Cattani and Sewell consider all injections in their Condition 4; in the presence of the other conditions these two conditions are equivalent.

Notation. For $A \subseteq \mathcal{N}$ and $a, z \in \mathcal{N}$, we let $[z/a] : A \cup \{a\} \to A \cup \{z\}$ be the function given by $[z/a](x) = x$ for all $x \neq a$, and $[z/a](a) = z$. Further, for functions $f_i : A_i \to B_i$ $(i = 1, 2)$ with A_1 and A_2 disjoint and also B_1 and B_2 disjoint, we let $f_1 + f_2 : A_1 \cup A_2 \to B_1 \cup B_2$ be the function given by $(f_1 + f_2)(x) = f_i(x)$ for all $x \in A_i$ $(i = 1, 2)$.

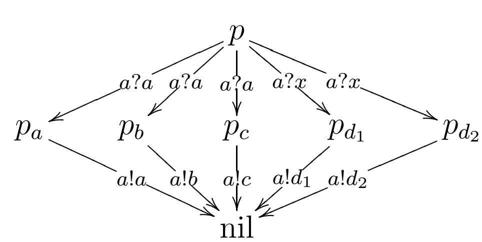
Conditions **F**2a and **F**2b are not entirely relevant in the context of early bisimilarity. For instance, consider the processes

$$\begin{aligned} p_i = {} & a(x).\,\mathrm{if}\ x = a\ \mathrm{then}\ \bar{a}\langle a\rangle\ \mathrm{else}\ \bar{a}\langle d_1\rangle \\ & + a(x).\,\mathrm{if}\ x = a\ \mathrm{then}\ \bar{a}\langle b\rangle\ \mathrm{else}\ \bar{a}\langle d_2\rangle \qquad (i = 1, 2) \\ & + a(x).\,\mathrm{if}\ x = a\ \mathrm{then}\ \bar{a}\langle c\rangle\ \mathrm{else}\ \bar{a}\langle d_i\rangle \end{aligned}$$

where we write '$a(x).p$' for 'input a name on channel a, binding it to x in p'; '$\bar{a}\langle x\rangle$' for 'output the name x on channel a'; and '+' for nondeterministic sum. The state graphs of the p_i $(i = 1, 2)$, with the transition $a?x$ representing all transitions for which $x \neq a$, are given by



which, up to early bisimilarity, minimise to the following one



F1. Channel is known and at most transmitted data is learnt:

$$C \vdash p \overset{\ell}{\rightharpoonup} C' \vdash p' \implies \mathsf{ch}(\ell) \subseteq C \;\wedge\; C' = C \cup \mathsf{dat}(\ell)$$

F2a. Input of new names induces input of old names: for all $z \in \mathcal{N} \setminus C,\ d \in C$:

$$C \vdash p \overset{c?z}{\rightharpoonup} C \cup \{z\} \vdash p' \implies C \vdash p \overset{c?d}{\rightharpoonup} C \vdash [d/z]p'$$

F2b. Input of old names induces input of new names: for all $z \in \mathcal{N} \setminus C$:

$$C \vdash p \overset{c?d}{\rightharpoonup} C \vdash p'$$
$$\implies \exists p'' \in X(C \cup \{z\}).\ C \vdash p \overset{c?z}{\rightharpoonup} C \cup \{z\} \vdash p'' \;\wedge\; [d/z]p'' = p'$$

F3a. Injective renaming: for all $\imath : C \to D,\ \beta : \mathsf{dat}(\ell) \setminus C \xrightarrow{\sim} D'$, with $D \cap D' = \emptyset$:

$$C \vdash p \overset{\ell}{\rightharpoonup} C \cup \mathsf{dat}(\ell) \vdash p'$$
$$\implies D \vdash [\imath]p \overset{[\imath+\beta]\ell}{\rightharpoonup} D \cup D' \vdash [\imath + \beta]p'$$

F3b. Knowing fresh input data preserves transitions: for all $z \in \mathcal{N} \setminus C$:

$$C \vdash p \overset{c?z}{\rightharpoonup} C \cup \{z\} \vdash p'$$
$$\implies C \cup \{z\} \vdash [C \hookrightarrow C \cup \{z\}]p \overset{c?z}{\rightharpoonup} C \cup \{z\} \vdash p'$$

F4. Inclusion maps reflect transitions:

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \overset{\ell}{\rightharpoonup} C' \vdash p'$$
$$\implies \quad \mathsf{dat}(\ell) \cap (D \setminus C) = \emptyset$$
$$\wedge\ \exists p'' \in X(C \cup \mathsf{dat}(\ell)).$$
$$C \vdash p \overset{\ell}{\rightharpoonup} C \cup \mathsf{dat}(\ell) \vdash p'' \;\wedge\; [C \cup \mathsf{dat}(\ell) \hookrightarrow C']p'' = p'$$

OR

$$\exists c \in C, z \in D \setminus C,\ p'' \in X(C \cup \{z\}).$$
$$\ell = c?z \;\wedge\; C \vdash p \overset{c?z}{\rightharpoonup} C \cup \{z\} \vdash p''$$
$$\wedge\ [C \cup \{z\} \hookrightarrow C']p'' = p'$$

Fig. 2. Requirements on an **F**-indexed labelled transition system.

yielding an **I**-LTS (according to Definition 2) but not an **F**-LTS (according to Definition 5) as it does not satisfy Condition **F**2b. If it did satisfy the condition then, for $C = \{a, b, c, d_1, d_2\}$ and $z \notin C$, we must have some $p'_a = [z/a]p_a$, $p'_b = [z/a]p_b$, and $p'_c = [z/a]p_c$, with $C \vdash p \xrightarrow{a?z} C \cup \{z\} \vdash p'_a$, $C \vdash p \xrightarrow{a?z} C \cup \{z\} \vdash p'_b$, and $C \vdash p \xrightarrow{a?z} C \cup \{z\} \vdash p'_c$. The only possibilities are $(p'_a = p_{d_i},\ p'_b = p_{d_j},\ p'_c = p_{d_k})$ for $i, j, k \in \{1, 2\}$. Recall that **F**-LTSs admit renaming of states by all functions. In particular, we can then consider the retraction $[a/z] : C \cup \{z\} \twoheadrightarrow C$. Now, we have that

$$[a/z]p_{d_i} = p_a \qquad [a/z]p_{d_j} = p_b \qquad [a/z]p_{d_k} = p_c \qquad .$$

Since $i, j, k \in \{1, 2\}$ it follows that two of the states in $\{p_a, p_b, p_c\}$ are equal, which is not the case.

In summary, p_1 is early bisimilar to p_2, but, considering the context $a(d_1).[-]$, we have that $a(d_1).p_1$ is not early bisimilar to $a(d_1).p_2$, so p_1 and p_2 are not related by any early bisimulation congruence.

Similar considerations apply to Condition **F**2a.

Conditions **F**2a and **F**2b serve to strengthen Condition **I**2. They not only require that 'if one name can be input then so can any other', but also ensure that the input behaviour is parametric in the input data. In addition, Conditions **F**1, **F**3, and **F**4 do not mention non-injective renamings, and moreover are together equivalent to Conditions **I**1 and **I**3–**I**6. Thus we have the following results.

Proposition 6 *(1) An* $\mathbf{F}$*-LTS over* $X \in \mathbf{Set}^{\mathbf{F}}$ *is an* $\mathbf{I}$*-LTS over* $|X| \in \mathbf{Set}^{\mathbf{I}}$*.*
(2) For $X \in \mathbf{Set}^{\mathbf{F}}$*, an* $\mathbf{I}$*-LTS over* $|X| \in \mathbf{Set}^{\mathbf{I}}$ *that satisfies Conditions* **F**2*a and* **F**2*b is an* $\mathbf{F}$*-LTS over* $X \in \mathbf{Set}^{\mathbf{F}}$*.*

In the journal version of their paper, Cattani and Sewell have introduced a class $\mathcal{N}_{\text{inj}}$-LTS of indexed labelled transition systems for presheaves over **I**. Conditions **F**1, **F**3, and **F**4 can be reconsidered as conditions on such systems, and indeed an $\mathcal{N}_{\text{inj}}$-LTS is a system that satisfies these axioms.

Proposition 7 *An indexed labelled transition system on a presheaf in* $\mathbf{Set}^{\mathbf{I}}$ *is an* $\mathcal{N}_{\text{inj}}$*-LTS if and only if it satisfies Conditions* **I**1*,* **I**3*–***I**6*.*

3 From presheaves to sheaves: Refining the model

We now return to the model of Section 1 based on injective renamings. We describe how the state space can be refined by imposing a *sheaf* condition.

The Schanuel topos. Consider a presheaf $P \in \mathbf{Set}^{\mathbf{I}}$. For $p \in P(D)$ and an inclusion $D \subseteq D'$, we have $[D \hookrightarrow D']p \in P(D')$. We have assumed that it does no harm to suppose that a process uses more names than it actually does. Furthermore, it may be that D itself contains more names than p actually uses, that is to say, perhaps there exists $C \subseteq D$ and $p' \in P(C)$ with $[C \hookrightarrow D]p' = p$.

We can also identify the names that $p \in P(D)$ uses by observing how the injections act on it. For instance, if every automorphism of D that fixes (*i.e.* does not move) all of $C \subseteq D$ also fixes p, then we expect that p only uses the names in C. More generally, we have the following notion of support.

Definition 8 *For a presheaf P in $\mathbf{Set}^{\mathbf{I}}$ we say that a name-set $C \subseteq D$* supports *an element $p \in PD$ if and only if, for all $\imath, \jmath : D \to E$ in $\mathbf{I}$, whenever $\imath|_C = \jmath|_C$ then $[\imath]p = [\jmath]p$.*

Given the intuitions discussed earlier, one would expect that if C supports $p \in PD$, then p would exist uniquely in PC. This is precisely the sheaf condition for the *atomic topology*:

> (Sheaf condition) Whenever $C \subseteq D$ supports $p \in PD$, there exists a unique $q \in PC$ with $[C \hookrightarrow D]q = p$.

That is, the statement "C supports p" defines a *compatible family* and the sheaf condition requires that it has a *unique gluing* at C. For our purposes, this is a sensible condition to impose. The full subcategory $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ of presheaves satisfying this condition is known as the *Schanuel topos*.

We briefly recall the analysis of the Schanuel topos given by Fiore [3]. Let $\mathbf{B}$ be the category of all finite name-sets and *bijections*; *i.e.*, the groupoid underlying $\mathbf{I}$. For $P \in \mathbf{Set}^{\mathbf{I}}$, define a presheaf $\langle P \rangle \in \mathbf{Set}^{\mathbf{B}}$ with

$$\langle P \rangle C \;=\; \left\{ p \in PC \;\middle|\; \begin{array}{l} \forall\, C_0 \subseteq C.\ \forall\, p_0 \in P(C_0). \\ \qquad [C_0 \hookrightarrow C]p_0 = p \implies C_0 = C \end{array} \right\}$$

and, conversely, from $Q \in \mathbf{Set}^{\mathbf{B}}$ generate a presheaf $Q_! \in \mathbf{Set}^{\mathbf{I}}$ by freely acting on the canonical inclusion maps as follows:

$$Q_! C \;=\; \sum_{C' \subseteq C} Q(C') \quad , \qquad Q_! \imath(C', q) \;=\; (\imath(C'), Q(\imath|_{C'})q) \qquad .$$

For every $Q \in \mathbf{Set}^{\mathbf{B}}$, we have that $Q_!$ is actually a sheaf in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ and there is a canonical natural isomorphism

$$Q \cong \langle Q_! \rangle \quad \text{in } \mathbf{Set}^{\mathbf{B}}$$

mapping $q \in Q(C_0)$ to $(C_0, q) \in \langle Q_! \rangle C_0$. Also, for every $P \in \mathbf{Set}^{\mathbf{I}}$, we have a canonical natural epimorphism

$$\varphi_P : \langle P \rangle_! \twoheadrightarrow P \quad \text{in } \mathbf{Set}^{\mathbf{I}}$$

given by

$$\Big(C_0 \subseteq C, p \in \langle P\rangle C_0\Big) \in \langle P\rangle_! C \ \mapsto P(C_0 \hookrightarrow C)p \in PC \qquad .$$

Moreover, a presheaf P in $\mathbf{Set}^{\mathbf{I}}$ is a sheaf in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ if and only if φ_P is a monomorphism, and hence an isomorphism.

For a sheaf P in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ and $p \in PC$, we let $\mathsf{supp}(p) \subseteq C$ (the *support* of p) and $\mathsf{seed}(p) \in \langle P\rangle(\mathsf{supp}\ p)$ (the *seed* of p) determine the unique $(\mathsf{supp}\ p, \mathsf{seed}\ p) \in \langle P\rangle_! C$ such that $(\varphi_P)_C(\mathsf{supp}\ p, \mathsf{seed}\ p) = p$. We note that $\mathsf{supp}(p)$ is the least support of p, and that

$$\mathsf{supp}([\imath]p) = \imath(\mathsf{supp}\ p)$$

for all $p \in PC$ and $\imath : C \to D$ in $\mathbf{I}$.

The construction $(-)_!$ extends to a functor $\mathbf{Set}^{\mathbf{B}} \to \mathbf{Set}^{\mathbf{I}}$, left adjoint to the forgetful functor $|-| : \mathbf{Set}^{\mathbf{I}} \to \mathbf{Set}^{\mathbf{B}}$; the Schanuel topos is (equivalent to) the Kleisli category arising from this adjunction. Thus, the sheaves in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ can be equivalently considered as presheaves in $\mathbf{Set}^{\mathbf{B}}$. Further, the maps $P \to P'$ in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ are in bijective correspondence with the maps $\langle P\rangle \to |\langle P'\rangle_!|$ in $\mathbf{Set}^{\mathbf{B}}$; hence, in addition to acting naturally on bijections, they are permitted to reduce the support.

B-*indexed labelled transition systems*

The early behaviour endofunctor B_{e} on $\mathbf{Set}^{\mathbf{I}}$ restricts to an endofunctor on $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ and it thus makes sense to discuss B_{e}-coalgebras in this full subcategory. In particular, we now ask which transition systems over presheaves in $\mathbf{Set}^{\mathbf{B}}$ should be considered.

Definition 9 *A* **B**-indexed labelled transition system (**B**-LTS) *is a presheaf* $Q \in \mathbf{Set}^{\mathbf{B}}$ *together with a transition relation* $\dashrightarrow \subseteq \textstyle\int Q \times \mathit{Lab} \times \int Q$*, where* $\int Q = \sum_{C\in\mathbf{B}} QC$*, satisfying Conditions* **B**1–**B**3 *of Figure 3.*

We have the following result relating the notions of indexed labelled transition systems introduced.

Theorem 10 *For sheaves* $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$*,* **B**-*LTSs over* $\langle P\rangle$ *and* **I**-*LTSs over* P *are in bijective correspondence.*

Details of the proof of Theorem 10 are given in Appendix B where we show that $\dashrightarrow_!$ and $\langle\longrightarrow\rangle$ as defined below are respectively an **I**-LTS and a **B**-LTS, and that $\langle\dashrightarrow_!\rangle = \dashrightarrow$ and $\langle\longrightarrow\rangle_! = \longrightarrow$.

B1. Channel is known and at most the transmitted data is learnt:

$$C \vdash p \overset{\ell}{\dashrightarrow} C' \vdash p' \implies \mathsf{ch}(\ell) \subseteq C \ \wedge \ C' \subseteq C \cup \mathsf{dat}(\ell)$$

B2. If one name can be input, then so can any other: for all $z \in \mathcal{N}$:

$$C \vdash p \overset{c?d}{\dashrightarrow} C' \vdash p' \implies \exists C'' \in \mathbf{B},\ p'' \in Q(C'').\ C \vdash p \overset{c?z}{\dashrightarrow} C'' \vdash p''$$

B3. Bijective maps preserve transitions: for all $D \in \mathbf{B}$ and $\beta : C \cup C' \cup \mathsf{dat}(\ell) \xrightarrow{\sim} D$:

$$C \vdash p \overset{\ell}{\dashrightarrow} C' \vdash p' \implies \beta C \vdash [\beta|_C]p \overset{[\beta]\ell}{\dashrightarrow} \beta C' \vdash [\beta|_{C'}]p'$$

Fig. 3. Requirements on a **B**-indexed labelled transition system.

The **I**-*LTS induced by a* **B**-*LTS:*

Let $\dashrightarrow \subseteq \textstyle\int\langle P\rangle \times Lab \times \int\langle P\rangle$ be a **B**-LTS with $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$. We define

$$\dashrightarrow_! \subseteq \textstyle\int P \times Lab \times \int P$$

to be the least indexed transition relation satisfying the following.

If $C_0 \vdash p \overset{c?d}{\dashrightarrow} C_0' \vdash p'$ and $C_0 \subseteq C$ and $C_0' \subseteq C \cup \{d\}$,
then $C \vdash [C_0 \hookrightarrow C]p \overset{c?d}{\dashrightarrow}_! C \cup \{d\} \vdash [C_0' \hookrightarrow C \cup \{d\}]p'$.
If $C_0 \vdash p \overset{c!d}{\dashrightarrow} C_0' \vdash p'$, and $d \in C_0 \subseteq C$ and $C_0' \subseteq C$,
then $C \vdash [C_0 \hookrightarrow C]p \overset{c!d}{\dashrightarrow}_! C \vdash [C_0' \hookrightarrow C]p'$.
If $C_0 \vdash p \overset{c!d}{\dashrightarrow} C_0' \vdash p'$, $C_0 \subseteq C$ and $C_0' \subseteq C \cup \{d\}$, and $d \notin C$
then $C \vdash [C_0 \hookrightarrow C]p \overset{c!d}{\dashrightarrow}_! C \cup \{d\} \vdash [C_0' \hookrightarrow C \cup \{d\}]p'$.
If $C_0 \vdash p \overset{\tau}{\dashrightarrow} C_0' \vdash p'$, and $C_0 \subseteq C$ and $C_0' \subseteq C$,
then $C \vdash [C_0 \hookrightarrow C]p \overset{\tau}{\dashrightarrow}_! C \vdash [C_0' \hookrightarrow C]p'$.

The **B**-*LTS induced by an* **I**-*LTS:*

Let $\longrightarrow \subseteq \int P \times Lab \times \int P$ be an **I**-LTS with $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$. We define

$$\langle\longrightarrow\rangle \ \subseteq \ \textstyle\int\langle P\rangle \times Lab \times \int\langle P\rangle$$

to be the least indexed transition relation such that:

If $C \vdash p \xrightarrow{\ell} C' \vdash p'$
then $\mathsf{supp}(p) \vdash \mathsf{seed}(p) \ \langle\xrightarrow{\ell}\rangle\ \mathsf{supp}(p') \vdash \mathsf{seed}(p')$.

4 Internal transition systems

So far, we have been concerned with relating coalgebras on variable sets (in $\mathbf{Set}^{\mathbf{I}}$ and $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$) with indexed transition systems ($\mathbf{I}$-LTSs, $\mathbf{F}$-LTSs, and $\mathbf{B}$-LTSs). Our motivation for studying such transition systems was to understand the nature of the B_{e}-coalgebras from a traditional point of view. Having done this, then, it is possible to consider the classes of $\mathbf{I}$-, $\mathbf{F}$-, $\mathbf{B}$-LTSs themselves as models of name-passing. This is the direction pursued by Cattani and Sewell [1]. Another approach is to work with *internal* transition systems — that is to say, transition relations taken as subobjects of $P \times L \times P$, for an object of states P and a distinguished object of labels L. This is very much the approach taken by Montanari and Pistore in their *History Dependent Automata (HDA)* [8].

We now introduce a notion of internal labelled transition system, relating it to the other models that we have studied, and to HDA. We fix a sheaf of labels $L = (N \times N) + (N \times N) + 1$ in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$, respectively considering the components (and naming the injections) as input (in), output (out), and silent action (tau). Since we will be using the internal language of $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ we need structure particular to this topos, namely, the map $\mathsf{up}_P : P \to \delta P$ given by $(\mathsf{up}_P)_C(p) = P(\mathsf{old}_C)p$.

Definition 11 *An* internal labelled transition system (i-LTS) *is a sheaf* P *together with a relation* $\rightsquigarrow \subseteq P \times L \times P$ *in* $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ *satisfying Conditions* $\mathsf{i}1$–$\mathsf{i}3$ *of Figure 4.*

$\mathsf{i}1$. The channel is known:

$$\forall\, x : P,\ \ell : \delta L,\ y : \delta P.\ (\mathsf{up}_P(x), \ell, y) \in \delta(\rightsquigarrow) \implies \exists\, c : N, d : \delta N.\ \ell = \delta\mathsf{in}(\mathsf{up}_N(c), d) \ \vee\ \ell = \delta\mathsf{out}(\mathsf{up}_N(c), d) \ \vee\ \ell = \delta\mathsf{tau}$$

$\mathsf{i}2$. The map up reflects transition derivatives:

$$\forall\, x : P,\ \ell : L,\ y : \delta P.\ (\mathsf{up}_P(x), \mathsf{up}_L(\ell), y) \in \delta(\rightsquigarrow) \implies \exists\, y' : P.\ \mathsf{up}_P(y') = y$$

$\mathsf{i}3$. If one name can be input, then so can any other:

$$\forall\, x : P,\ c, d : N,\ y : P.\ (x, \mathsf{in}(c, d), y) \in \rightsquigarrow \implies \forall\, e : N.\ \exists y' : P.\ (x, \mathsf{in}(c, e), y') \in \rightsquigarrow$$

Fig. 4. Requirements on an i-indexed labelled transition system, expressed in the internal logic of $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$.

Proposition 12 *Collectively, the conditions of Figure 4 can be equivalently presented in elementary terms as follows, where we write* $C \vdash p \overset{\ell}{\rightsquigarrow} p'$ *for* $(p, \ell, p') \in {\rightsquigarrow}(C)$.

i1. *The channel is known:*

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \overset{\ell}{\rightsquigarrow} p' \implies \mathsf{ch}(\ell) \subseteq C$$

i2. *Inclusion maps reflect transition derivatives:*

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \overset{[C \hookrightarrow C \cup D]\ell}{\rightsquigarrow} p' \implies \exists p'' \in PC.\ [C \hookrightarrow C \cup D]p'' = p'$$

i3. *If one name can be input, then so can any other:*

$$C \vdash p \overset{\mathsf{in}(c,d)}{\rightsquigarrow} p' \implies \forall e \in \mathcal{N}.\ \exists p'' \in P(C \cup \{e\}).$$
$$C \cup \{e\} \vdash [C \hookrightarrow C \cup \{e\}]p \overset{\mathsf{in}(c,e)}{\rightsquigarrow} p''$$

Separately, each of Conditions i1 and i2 of Figure 4 and Proposition 12 are equivalent; Conditions i2 and i3 of Figure 4 imply Condition i3 of Proposition 12, which in turn implies Condition i3 of Figure 4.

We can now relate the internal structures of this section with those already studied.

Theorem 13 i*-LTSs and* **I***-LTSs over sheaves in* $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ *are in bijective correspondence.*

Let P be a sheaf. Given an i-LTS $\rightsquigarrow$ on P, let

$$\rightsquigarrow_{\mathbf{I}} \subseteq \textstyle\int P \times \mathit{Lab} \times \int P$$

be the least indexed transition relation satisfying the following.

If $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \overset{\mathsf{in}(c,d)}{\rightsquigarrow} p'$,

then $C \vdash p \overset{c?d}{\rightsquigarrow_{\mathbf{I}}} C \cup \{d\} \vdash p'$.

If $C \vdash p \overset{\mathsf{out}(c,d)}{\rightsquigarrow} p'$ and $d \in \mathsf{supp}(p)$,

then $C \vdash p \overset{c!d}{\rightsquigarrow_{\mathbf{I}}} C \vdash p'$.

If $C \vdash p \overset{\mathsf{out}(c,d)}{\rightsquigarrow} p'$ and $d \notin \mathsf{supp}(p)$,

then $C \setminus \{d\} \vdash p_0 \overset{c!d}{\rightsquigarrow_{\mathbf{I}}} C \vdash p'$,

where $p_0 \in P(C \setminus \{d\})$ is the unique such with $[C \setminus \{d\} \hookrightarrow C]p_0 = p$,

existing since P is a sheaf and $(C \setminus \{d\})$ supports p.

If $C \vdash p \overset{\mathsf{tau}}{\rightsquigarrow} p'$,

then $C \vdash p \overset{\tau}{\rightsquigarrow_{\mathbf{I}}} C \vdash p'$.

Conversely, given an **I**-LTS $\longrightarrow$ on P, let

$$C \vdash - \xrightarrow{-}_{\mathsf{i}} - \;\subseteq\; PC \times LC \times PC \qquad\qquad (C \in \mathbf{I})$$

be the least family of transition relations such that:

If $C \vdash p \xrightarrow{\ell} C' \vdash p'$
then $C \cup C' \vdash [C \hookrightarrow C \cup C']p \xrightarrow{\ell}_{\mathsf{i}} [C' \hookrightarrow C \cup C']p'$.

eliding the obvious translation of labels (which makes sense as a result of Condition **I**1).

The verification that this correspondence is bijective and actually yields i-LTSs and **I**-LTSs is deferred to Appendix C.

Named-sets with symmetries. The idea of interpreting the notion of transition system inside the Schanuel topos is similar in spirit to the idea of interpreting the notion of automaton inside a category of named-sets — that is, the idea of *History Dependent Automata* due to Montanari and Pistore [8]. In fact, the two notions are essentially the same, since as we show below the category of *finitely supported named-sets with symmetries* is equivalent to the Schanuel topos.

A variety of categories of named-sets have been proposed; see, *e.g.*, [2,8]. Here, we consider named-sets with symmetries as introduced by Pistore in his thesis [8, Chapter 7].

Definition 14 (Pistore) *A named-set with symmetries* $(X, H = \{H_x\}_{x \in X})$ *is given by a set* X*, with each element* $x \in X$ *equipped with a subgroup* $H_x \subseteq \mathsf{Sym}(\mathcal{N})$ *of the symmetric group* $\mathsf{Sym}(\mathcal{N})$ *on the infinite set of names* $\mathcal{N}$*.*

For each $x \in X$, H_x is to be thought of as the group of permutations that fix x. This can be made more precise, as follows. Recall that a *left action* of $\mathsf{Sym}(\mathcal{N})$ on a set A is a function $\alpha : \mathsf{Sym}(\mathcal{N}) \times A \to A$ that respects the group structure (*i.e.*, satisfies $\alpha(\mathrm{id}, a) = a$ and $\alpha(\tau\sigma, a) = \alpha(\tau, \alpha(\sigma, a))$). The *stabiliser* of $a \in A$ is the subgroup $\mathsf{Stab}(a) = \{\sigma \in \mathsf{Sym}(\mathcal{N}) \mid \alpha(\sigma, a) = a\}$ of all the permutations that fix a, and the *orbit-stabiliser theorem* exhibits a bijection between the orbit of each a, $\mathsf{Orb}(a) = \{a' \mid \exists \sigma \in \mathsf{Sym}(\mathcal{N}).\ \alpha(\sigma, a) = a'\}$, and the set of left cosets of the stabiliser $\mathsf{Stab}(a)$. Thus, for a section of the quotient map $A \twoheadrightarrow \{\mathsf{Orb}(a) \mid a \in A\}$ with image $O \subseteq A$, we have a bijection $A \cong \sum_{o \in O}\{\sigma\,\mathsf{Stab}(o) \mid \sigma \in \mathsf{Sym}(\mathcal{N})\}$. In this vein, a named-set (X, H) can be thought of as a representation of an action: X provides canonical members of the orbits and H_x describes the stabiliser of each $x \in X$. This intuition will guide us in what follows.

From a named-set (X, H) one can recover a notion of support. Following Definition 8, we say that a finite set $C \subseteq \mathcal{N}$ *supports* $x \in X$ if whenever two permutations $\sigma, \sigma' \in \mathsf{Sym}(\mathcal{N})$ agree on C then they induce the same left cosets of H_x. That is, C *supports* x in (X, H) if

$$\forall\, \sigma, \sigma' \in \mathsf{Sym}(\mathcal{N}).\ \sigma|_C = \sigma'|_C \implies \sigma H_x = \sigma' H_x \tag{5}$$

where, as above, we write $\sigma|_C$ for the bijection given by restricting the domain of σ to C.

Note that $\sigma H_x = \sigma' H_x$ if and only if $\sigma^{-1}\sigma' \in H_x$. Thus, an equivalent formulation of (5) is

$$\forall\, \sigma \in \mathsf{Sym}(\mathcal{N}).\ \sigma|_C = \mathrm{id}_C \implies \sigma \in H_x \qquad .$$

We restrict attention to those named-sets in which each element is supported by a finite set. In this case, every element $x \in X$ admits a (necessarily finite) set

$$\mathsf{supp}_H(x) = \bigcap_{C' \subseteq C} \{\, C' \mid C' \text{ supports } x \text{ in } (X, H)\ \}$$

, where C is a finite set supporting x

which is least among all finite sets supporting x in (X, H). (To see this show that the finite supporting sets of an element are closed under intersection.)

Definition 15 *The category* **fsNSet** *has as objects finitely-supported named-sets with symmetries, and morphisms*

$$(m, K = \{K_x\}_{x \in X}) : (X, H) \to (X', H')$$

given by a function $m : X \to X'$ *together with, for each* $x \in X$*, a left coset* $K_x = \sigma_x H'_{m\,x}$ *such that* $H_x \subseteq \sigma_x H'_{m\,x} {\sigma_x}^{-1}$*. (Note that this makes sense, for if* $\sigma H'_{m\,x} = \sigma' H'_{m\,x}$ *then also* $\sigma H'_{m\,x} \sigma^{-1} = \sigma' H'_{m\,x} \sigma'^{-1}$*.)*

The identity morphism on (X, H) *is* $(\mathrm{id}_X, H = \{H_x\}_{x \in X})$*, and the composition of*

$$(X, H) \xrightarrow{(m, \{\sigma_x H'_{m\,x}\}_{x \in X})} (X', H') \xrightarrow{(m', \{\sigma'_{x'} H''_{m'\,x'}\}_{x' \in X'})} (X'', H'')$$

is $(m' \circ m, \{\sigma_x \sigma'_{m\,x} H''_{m'(m\,x)}\}_{x \in X})$*. (Note that the definition is independent of the descriptions of cosets used, in the sense that if* $\sigma_x H'_{m\,x} = \tau_x H'_{m\,x}$ *and* $\sigma'_{m\,x} H''_{m'(m\,x)} = \tau'_{m\,x} H''_{m'(m\,x)}$*, then also* $\sigma_x \sigma'_{m\,x} H''_{m'(m\,x)} = \tau_x \tau'_{m\,x} H''_{m'(m\,x)}$*.)*

It is important to note that, for a morphism $(m, \{\sigma_x H'_{m\,x}\}_{x \in X}) : (X, H) \to (X', H')$ in **fsNSet**, if C supports x in (X, H) then ${\sigma_x}^{-1}(C)$ supports $m\,x$ in (X', H'). Indeed, if $\tau|_{{\sigma_x}^{-1}(C)} = \mathrm{id}_{{\sigma_x}^{-1}(C)}$ then $\tau {\sigma_x}^{-1}|_C = {\sigma_x}^{-1}|_C$ and, assuming that C supports x, we have $\sigma_x \tau {\sigma_x}^{-1} \in H_x$ from which it follows that $\tau \in H'_{m\,x}$ as required.

The morphisms that we use are based on the informal discussion in Pistore's thesis (although the formal definition here is slightly different). The second component K of each morphism describes, for each $x \in X$, how the permutations in H_x correspond to the permutations in $H'_{m\,x}$. Every permutation $\sigma_x \in \mathsf{Sym}(\mathcal{N})$ defines a homomorphism by conjugation $H_x \to \mathsf{Sym}(\mathcal{N})$ given by $\tau \mapsto {\sigma_x}^{-1}\tau\sigma_x$. The condition $H_x \subseteq \sigma_x H'_{m\,x}{\sigma_x}^{-1}$ ensures that the image of this homomorphism lies within $H'_{m\,x}$. Pistore remarks that some of these homomorphisms should be equated; we have used cosets to achieve this.

These morphisms also have an interpretation in terms of the intuition of named-sets as representing group actions. Let (X, H) and (X', H') be named-sets regarded as $\mathsf{Sym}(\mathcal{N})$-actions in the manner outlined after Definition 14. Recall that a homomorphism of actions $(A, \alpha) \to (A', \alpha')$ is a function $f : A \to A'$ that respects the actions (*i.e.*, satisfying $f(\alpha(\sigma, a)) = \alpha'(\sigma, fa)$). Consider a morphism of named-sets $(m, K) : (X, H) \to (X', H')$. From the viewpoint of actions, the first component m is to be thought of as providing a mapping between canonical members of orbits. However, since homomorphisms of group actions need not preserve canonical representatives of orbits, the second component K of a morphism of named-sets provides a permutation to rectify this. That is, if $K_x = \sigma_x H'_{m\,x}$ then the named-set morphism is to be thought of as mapping $x \in X$ to the result of the action of σ_x on $m\,x$.

The following result is the main step towards relating i-LTSs and HDA.

Theorem 16 *The category* **fsNSet** *of finitely-supported named-sets with symmetries is equivalent to the Schanuel topos* $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$.

For a named-set $(X, H) \in \mathbf{fsNSet}$ we define the presheaf $\Sigma(X, H) \in \mathbf{Set}^{\mathbf{B}}$ as follows:

- For $C \in \mathbf{B}$,

$$\Sigma(X, H)C = \left\{ (x, \sigma H_x) \;\middle|\; \begin{array}{l} \mathsf{supp}_H(x) = \sigma^{-1}(C) \\ \text{with } x \in X \text{ and } \sigma \in \mathsf{Sym}(\mathcal{N}) \end{array} \right\}$$

  It is interesting to note that if $\sigma^{-1}(C)$ supports x then $\sigma H_x = \tau H_x$ implies that $\tau^{-1}(C)$ also supports x.
- For $\beta : C \xrightarrow{\sim} D$ in $\mathbf{B}$ and $(x, J) \in \Sigma(X, H)C$,

$$\Sigma(X, H)\beta(x, J) = (x, \sigma J) \in \Sigma(X, H)D$$

  where $\sigma \in \mathsf{Sym}(\mathcal{N})$ is an extension of β (*i.e.*, $\sigma|_C = \beta$).

  Observe that if $\sigma, \sigma' \in \mathsf{Sym}(\mathcal{N})$ are extensions of β then $\sigma J = \sigma' J$. Indeed, since $\sigma\tau|_{\tau^{-1}(C)} = \sigma'\tau|_{\tau^{-1}(C)}$, then for $J = \tau H_x$ we have, as $\tau^{-1}(C)$ supports x, that $\sigma\tau H_x = \sigma'\tau H_x$ as required.

Note also that for any extension $\sigma \in \mathsf{Sym}(\mathcal{N})$ of β and any permutation $\tau \in \mathsf{Sym}(\mathcal{N})$ for which $J = \tau H_x$, we have that $(\sigma\tau)^{-1}(D) = \tau^{-1}\big(\sigma^{-1}(D)\big) = \tau^{-1}(C)$ is least among the finite supports of x in (X, H). Thus the image of $\Sigma(X, H)\beta$ is within the codomain.

The above construction induces a functor $\Sigma_! : \mathbf{fsNSet} \to \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ given on objects (X, H) as

$$\Sigma_!(X,H)C = \left\{ (x, \sigma H_x) \;\middle|\; \begin{array}{c} \sigma^{-1}(C) \text{ supports } x \in X \\ \text{with } x \in X \text{ and } \sigma \in \mathsf{Sym}(\mathcal{N}) \end{array} \right\}$$

for $C \in \mathbf{I}$, and

$$\Sigma_!(X,H)\imath(x, \sigma H_x) = (x, \tau\sigma H_x)$$

$$\text{, where } \tau \text{ is an extension of } \imath|_{\sigma(\mathsf{supp}_H x)}$$

for $\imath : C \to D$ in $\mathbf{I}$. That is, recalling the functor $(-)_! : \mathbf{Set}^{\mathbf{B}} \to \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ from Section 3,

$$\Sigma_!(X,H) \cong \big(\Sigma(X,H)\big)_! \qquad .$$

To each morphism

$$(m, K) : (X, H) \to (X', H') \quad \text{in } \mathbf{fsNSet}$$

we associate a natural family of functions

$$\{\, \Sigma_!(m,K)_C : \Sigma_!(X,H)C \to \Sigma_!(X',H')C \,\}_{C \in \mathbf{I}}$$

defined by $\Sigma_!(m,K)_C(x, \sigma H_x) = (m\,x, \sigma K_x)$.

We must now show (i) that this definition is independent of the choice of σ; (ii) that the image falls within the codomain; (iii) that the family is natural. To proceed, for each $x \in X$ we suppose that $K_x = \sigma_x H'_{m\,x}$, in accordance with the definition of morphism in $\mathbf{fsNSet}$.

(i) The definition is independent of the choice of σ. If $\sigma' H_x = \sigma'' H_x$, then ${\sigma'}^{-1}\sigma'' \in H_x$. Since (m, K) is a morphism in $\mathbf{fsNSet}$, we know that $\sigma_x^{-1}{\sigma'}^{-1}\sigma''\sigma_x \in H'_{m\,x}$. That is, $\sigma' K_x = \sigma'\sigma_x H'_{m\,x} = \sigma''\sigma_x H'_{m\,x} = \sigma'' K_x$.

(ii) The image of each function $\Sigma_!(m,K)_C$ is within the codomain $\Sigma_!(X', H')C$. Because if $\sigma^{-1}(C)$ supports x in (X, H), then $(\sigma\sigma_x)^{-1}(C) = {\sigma_x}^{-1}\big(\sigma^{-1}(C)\big)$ supports $m(x)$ in (X', H'), as observed after Definition 15.

(iii) The family is natural. Let $\imath : C \to D$ in $\mathbf{I}$. On the one hand, we have that $\Sigma_!(m,K)\big([\imath](x, \sigma H_x)\big) = \Sigma_!(m,K)(x, \tau\sigma H_x) = (m\,x, \tau\sigma K_x) = (m\,x, \tau\sigma\sigma_x H'_{m\,x})$, where τ is an extension of $\imath|_{\sigma(\mathsf{supp}_H x)}$. On the other hand,

$[\imath]\Big(\Sigma_!(m,K)(x,\sigma H_x)\Big) = [\imath](m\,x,\sigma K_x) = [\imath](m\,x,\sigma\sigma_x H'_{m\,x}) = (m\,x,\rho\sigma\sigma_x H'_{m\,x})$, where ρ is an extension of $\imath|_{(\sigma\sigma_x)(\mathsf{supp}_{H'}(m\,x))}$.

Since, by the observation after Definition 15, we have that $\sigma_x\Big(\mathsf{supp}_{H'}(m\,x)\Big) \subseteq \mathsf{supp}_H(x)$, every extension τ of $\imath|_{\sigma(\mathsf{supp}_H x)}$ is also an extension of $\imath|_{\sigma(\sigma_x(\mathsf{supp}_{H'}(m\,x)))} = \imath|_{(\sigma\sigma_x)(\mathsf{supp}_{H'}(m\,x))}$ and hence $[\imath]\Big(\Sigma_!(m,K)(x,\sigma H_x)\Big) = (m\,x,\tau\sigma\sigma_x H'_{m\,x}) = \Sigma_!(m,K)\Big([\imath](x,\sigma H_x)\Big)$. Thus, the family $\Sigma_!(m,K)$ is natural.

Finally, we must verify that the construction $\Sigma_!$ is functorial. The identity morphism $(\mathrm{id}_X, H = \{H_x\}_{x\in X})$ is mapped to $\Sigma_!(\mathrm{id}_X,H)$ and, for each $x \in X$ and $\sigma \in \mathsf{Sym}(\mathcal{N})$, we have $\Sigma_!(\mathrm{id}_X,H)(x,\sigma H_x) = (x,\sigma H_x)$. Thus the identity of named-sets is mapped to the identity of sheaves.

Consider the composite $(m' \circ m, (K' \circ K) = \{\sigma_x\sigma'_{m\,x} H''_{m'(m\,x)}\}_{x\in X})$ of named-set morphisms $(m, K = \{\sigma_x H'_{m\,x}\}_{x\in X})$ and $(m', K' = \{\sigma_{x'} H''_{m'\,x'}\}_{x'\in X'})$. In $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$, for some $x \in X$ and $\sigma \in \mathsf{Sym}(\mathcal{N})$, we have $\Sigma_!(m,K)_C(x,\sigma H_x) = (m\,x,\sigma\sigma_x H'_{m\,x})$; so that $\Sigma_!(m',K')_C(\Sigma_!(m,K)_C(x,\sigma H_x)) = (m'(m\,x),\sigma\sigma_x\sigma'_{m\,x}H''_{m'(m\,x)})$. This is precisely the value of $\Sigma_!(m'\circ m, K'\circ K)_C(x,\sigma H_x)$. Thus composition is preserved and we have a functor $\Sigma_! : \mathbf{fsNSet} \to \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$.

We show in Appendix D that this functor is essentially surjective (*i.e.*, that for every $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ there exists $(X,H) \in \mathbf{fsNSet}$ such that $\Sigma_!(X,H) \cong P$) and full and faithful. Thus Theorem 16 is proved.

History dependent automata. We remarked in Section 3 that the operators on $\mathbf{Set}^{\mathbf{I}}$ introduced in Section 1 restrict to $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$; it is now routine to translate them into operators on **fsNSet**. It is also straightforward to interpret the Conditions i1–i3 in **fsNSet**, and, in this way, obtain a class of HDA that correspond to B_{e}-coalgebras. An example of such an interpretation was provided in the Proceedings of CMCS'04 [5].

5 Concluding remarks

Rule formats. Throughout the present work we have not considered how the coalgebra, transition system, or automaton is initially defined. In practice, transition relations are often defined over terms using structural induction over rules. There are various rule formats for calculi such as CCS that guarantee bisimilarity to be a congruence for the induced transition system. It is well-known (and was recalled in Section 2) that early bisimilarity is typically not a congruence for name-passing calculi. In this context, though, we have developed a format for rules inducing **F**-LTSs. Within this format, **F**-indexed early bisimilarity is seen to be a congruence.

Minimisation. An application of final coalgebra semantics is the use of minimisation techniques to determine, for instance, whether processes are bisimilar. We have a framework for understanding partition refinement techniques in a coalgebraic setting. It can be shown that the partition refinement procedure will terminate if performed on a coalgebra whose state space is a finitely presentable sheaf in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$. This latter condition on the state space translates to the requirement that the first component of the named-set representing the sheaf is finite. The second component of such a named-set is thus a finite family of infinite permutation groups. The finite support requirement, however, ensures that each of these groups has a finite description. Thus the framework of named-sets is convenient from a practical point of view. Indeed, problems of minimisation for name-passing systems have already been investigated as related to history dependent automata [2].

Further related work. Gadducci, Miculan, and Montanari [7] have obtained a result analogous to our Theorem 16 for a variant of named-sets similar to that considered by Ferrari, Montanari, and Pistore [2]. One important difference in this variant of named-sets is that finite descriptions of support and groups, as mentioned in the previous paragraph, are explicitly given.

Acknowledgements. We thank Peter Sewell for useful discussions, especially about Section 2.

References

[1] Cattani, G. L. and P. Sewell, *Models for name-passing processes: Interleaving and causal*, Information and Computation **190** (2004), pp. 136–178. (Extended abstract in *Proc. LICS'00.*)

[2] Ferrari, G., U. Montanari and M. Pistore, *Minimising transition systems for name passing calculi: A co-algebraic formulation*, in: *Proc. FOSSACS'02*, LNCS **2302** (2002), pp. 129–158.

[3] Fiore, M. P., *Notes on combinatorial functors*, Draft available on-line (January 2001).

[4] Fiore, M. P., E. Moggi and D. Sangiorgi, *A fully abstract model for the π-calculus*, Information and Computation **179** (2002), pp. 76–117. (Extended abstract in *Proc. LICS'96.*)

[5] Fiore, M. P. and S. Staton, *Comparing operational models of name-passing process calculi*, in: *Proc. CMCS'04*, ENTCS **106** (2004), pp. 91–104.

[6] Fiore, M. P. and D. Turi, *Semantics of name and value passing*, in: *Proc. LICS'01* (2001), pp. 93–104.

[7] Gadducci, F., M. Miculan and U. Montanari, *About permutation algebras, (pre)sheaves and named sets*, Private communication (July 2005). (Preliminary version as *Some characterization results for permutation algebras*, in *Proc. COMETA*, 2004.)

[8] Pistore, M., "History Dependent Automata", Ph.D. thesis, University of Pisa (1999).

[9] Sangiorgi, D., *A theory of bisimulation for the π-calculus*, Acta Informatica **33** (1996), pp. 69–97. (Extended abstract in *Proc. CONCUR'93*.)

[10] Stark, I., *A fully abstract domain model for the π-calculus*, in: *Proc. LICS'96* (1996), pp. 36–42.

A Proof of Theorem 3

A.1 The **I***-LTS induced by a* B_e*-coalgebra*

We explain how the transition relation $\longrightarrow_h$ induced by a B_e-coalgebra h satisfies the conditions in Figure 1.

Condition **I**1 is guaranteed by definition. For example, if $C \vdash p \xrightarrow{c?d}_h C' \vdash p'$ is induced by $p' \in \pi_1(\pi_1(h_C p)c)d$ then $C' = C$. Since $\pi_1(h_C p)$ is a partial function $C \rightharpoonup ((\wp^+ PC)^C \times \wp^+ P(C \oplus 1))$, we have $c \in C$ — the channel is known. We have $d \in C$ since $\pi_1(\pi_1(h_C p)c)$ is a function $(C \to \wp^+ PC)$. So certainly $C' = C \cup \{d\}$ — the data is learnt. Again, if $C \vdash p \xrightarrow{c?z}_h C' \vdash p'$ is induced by $q \in \pi_2(\pi_1(h_C p)c)$ then we know $z \notin C$, and we have $C' = C \cup \{z\}$ and $p' = [d/\nu]q$. We have $c \in C$ since $\pi_1(h_C p)$ is a partial function $C \rightharpoonup ((\wp^+ PC)^C \times \wp^+ P(C \oplus 1))$.

Condition **I**2 is guaranteed by the careful use of partial exponentials and non-empty powersets, as follows. Suppose we are concerned with the behaviour of $p \in PC$. Recall that the input component is of type $N \rightharpoonup (\wp^+ P)^N$, so, at stage $C \in \mathbf{I}$, we have an element i of type $C \rightharpoonup (\wp^+ PC)^C \times \wp^+ P(C \oplus 1)$. That is, on each channel $c \in C$ there must be either *no* input communication (the partial function i is undefined at c) or input of *every* name: on inputting a known name $d \in C$ we proceed as a state in the non-empty set $\pi_1(ic)d \in \wp^+ PC$, and on inputting a fresh name $z \notin C$ we proceed as a state in the non-empty set $(\wp^+ P)[z/\nu](\pi_2(ic)) \in \wp^+ P(C \cup \{z\})$.

Condition **I**3 captures the naturality of h with respect to bijective renamings. Indeed, suppose that $C \vdash p \xrightarrow{c!d}_h C' \vdash p'$ is induced by $(d, p') \in \pi_2(h_C\, p)c$ (so $C = C'$). Consider a bijection $\beta : C \xrightarrow{\sim} D$. Since h is natural, we have $h_D([\beta]p) = [\beta](h_C\, p)$. In particular, from the definitions of the various type constructors, $\pi_2\Big(h_D([\beta]p)\Big)(\beta c) = [\beta](\pi_2(h_C\, p)c)$. So $(\beta d, [\beta]p') \in \pi_2\Big(h_D([\beta]p)\Big)(\beta c)$, inducing $D \vdash [\beta]p \xrightarrow{\beta c!\beta d}_h D \vdash [\beta]p'$. The other kinds of transition behave in a similar manner. Thus Condition **I**3 is satisfied by the induced transition system.

Condition **I**4a is essentially a result of the structure of the exponential. Recall (1) that for $(\phi, \psi) \in (\wp^+ P)^N C$ and $z \in \mathcal{N} \setminus C$ we have that

$$\begin{aligned} &\pi_1\Big((\wp^+ P)^N (C \hookrightarrow C \cup \{z\})(\phi, \psi)\Big)z \\ &\quad = (\wp^+ P)[z/\nu](\psi) \\ &\quad = \Big\{P[z/\nu]q \in P(C \cup \{z\}) \mid q \in \psi\Big\} \qquad . \end{aligned} \tag{A.1}$$

We will prove the left to right direction of Condition **I**4a; the opposite direction is proved by following the same steps in reverse. Suppose that $C \vdash p \xrightarrow{c?z}_h C \cup \{z\} \vdash p'$ is induced, for $z \notin C$ (otherwise the result is trivial). This must have been induced by $P[\nu/z]p' \in \pi_2(\pi_1(h_C p)c)$. By (A.1) above, we thus have that

$$p' = [z/\nu][\nu/z]p' \in \pi_1\Big((\wp^+ P)^N (C \hookrightarrow C \cup \{z\})(\pi_1(h_C p)c)\Big)z \qquad .$$

Further, since h is natural, we also have that

$$(\wp^+ P)^N (C \hookrightarrow C \cup \{z\})(\pi_1(h_C p)c) = \pi_1\Big(h_{C \cup \{z\}}(P(C \hookrightarrow C \cup \{z\})p)\Big)c \qquad .$$

Hence

$$p' \in \pi_1\bigg(\pi_1\Big(h_{C \cup \{z\}}(P(C \hookrightarrow C \cup \{z\})p)\Big)c\bigg)z$$

and Condition **I**4a is satisfied by the induced transition.

Condition **I**4b is a result of the naturality of h. For if

$$C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \xrightarrow{c!d}_h C \cup \{d\} \vdash p'$$

is induced, we must have $(d, p') \in \pi_2(h_{C \cup \{d\}}([C \hookrightarrow C \cup \{d\}]p))c$. Since h is natural,

$$\begin{aligned} &(d, p') \in \pi_2\Big(h_{C \cup \{d\}}(P(C \hookrightarrow C \cup \{d\})p)\Big)c \\ &\qquad = \Big(\wp^+(N \times P)\Big)(C \hookrightarrow C \cup \{d\})(\pi_2(h_C p)c) \\ &\qquad = \Big\{\ (e, P(C \hookrightarrow C \cup \{d\})q) \in C \times P(C \cup \{d\}) \mid (e, q) \in \pi_2(h_C p)c\ \Big\} \qquad . \end{aligned}$$

So $d \in C$, as required by Condition **I**4b.

Just as Condition **I**3 captures the naturality of h with respect to bijective renamings, Condition **I**5 captures the naturality with respect to inclusion maps.

For instance, if $C \vdash p \xrightarrow{\tau}_h C \vdash p'$ is induced, it must be by $p' \in \pi_4(h_C\, p)()$. Since h is natural, we have that

$$\begin{aligned} P(C \hookrightarrow C \cup D)p' &\in (\wp^+ P)(C \hookrightarrow C \cup D)\Big(\pi_4(h_C\, p)()\Big) \\ &= \pi_4\Big(h_{C \cup D}(P(C \hookrightarrow C \cup D)p)\Big)() \end{aligned}$$

and thus $C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{\tau}_h C \cup D \vdash [C \hookrightarrow C \cup D]p'$ is induced. The other kinds of transition are similar; thus Condition **I**5 is satisfied by the induced transition system.

Condition **I**6 also results from the naturality of h.

(1) Suppose for instance that

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c!z}_h C' \cup D \vdash p'$$

is induced, with $z \notin C \cup D$. Then $C' \cup D = C \cup D \cup \{z\}$ and

$$[\nu_{C \cup D}/z]p' \in \pi_3\Big(h_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c \qquad .$$

Since h is natural, we have that

$$\begin{aligned} &\pi_3\Big(h_{C \cup D}(P(C \hookrightarrow C \cup D)p)\Big) \\ &\quad = (N \rightharpoonup \wp^+ \delta P)(C \hookrightarrow C \cup D)\Big(\pi_3(h_C\, p)\Big) \\ &\quad = (\wp^+ P)\Big((C \hookrightarrow C \cup D) \oplus 1\Big) \circ \pi_3(h_C p) \circ (C \hookrightarrow C \cup D)^R \end{aligned}$$

and, since this partial function is defined at c, it follows that $c \in C$. Moreover,

$$\begin{aligned} [\nu_{C \cup D}/z]p' \in \pi_3\Big(h_{C \cup D}(P(C \hookrightarrow C \cup D)p)\Big)c & \\ = (\wp^+ P)\Big((C \hookrightarrow C \cup D) \oplus 1\Big)(\pi_3(h_C p)c) & \\ = \Big\{\ [(C \hookrightarrow C \cup D) \oplus 1]q \ \mid\ q \in \pi_3(h_C p)c\ \Big\} & \qquad . \end{aligned}$$

So there exists $q \in \pi_3(h_C p)c \in \wp^+ P(C \oplus 1)$ with

$$[(C \hookrightarrow C \cup D) \oplus 1]q = [\nu_{C \cup D}/z]p' \qquad .$$

Finally, considering the diagram

$$\begin{array}{ccc} C \oplus 1 & \xrightarrow[\sim]{[z/\nu_C]} & C \cup \{z\} \\ {\scriptstyle (C \hookrightarrow C \cup D) \oplus 1}\big\downarrow & & \big\downarrow \\ (C \cup D) \oplus 1 & \xrightarrow[{[z/\nu_{C \cup D}]}]{\sim} & C \cup D \cup \{z\} \end{array}$$

in $\mathbf{I}$, it follows that $p'' = [z/\nu_C]q \in P(C \cup \{z\})$ satisfies

$$[C \cup \{z\} \hookrightarrow C \cup \{z\} \cup D]p'' = p' \quad \text{and} \quad C \vdash p \xrightarrow{c!z}_h C \cup \{z\} \vdash p'' \quad .$$

(2) Now, suppose that

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c!d}_h C \cup D \vdash p'$$

this time with $d \in C \setminus D$. Then $(d, p') \in \pi_2\Big(h_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c$. Since h is natural, we have that

$$\begin{aligned}(d, p') \in \pi_2\Big(h_{C \cup D}(P(C \hookrightarrow C \cup D)p)\Big)c \\ &= \Big(\wp^+(N \times P)\Big)(C \hookrightarrow C \cup D)(\pi_2(h_C p)c) \\ &= \Big\{\ (e, P(C \hookrightarrow C \cup D)q) \in C \times P(C \cup D)\ \mid\ (e, q) \in \pi_2(h_C p)c\ \Big\} \quad .\end{aligned}$$

Thus we have $p'' \in PC$ with $[C \hookrightarrow C \cup D]p'' = p'$ and $(d, p'') \in \pi_2(h_C p)c$. So $C \vdash p \xrightarrow{c!d}_h C \vdash p''$ is induced by the coalgebra.

Condition **I**6 is proved similarly for the other kinds of transition.

A.2 The B_{e}-coalgebra induced by an **I***-LTS*

We show that the definition (3–4) inducing a B_{e}-coalgebra $\vec{h}$ from an **I**-LTS $\longrightarrow$ makes sense and yields a natural transformation.

Of principle concern for well-definedness is the input component, where the function space and non-empty powersets are used in a particularly intricate manner. For $C \in \mathbf{I}$, $p \in PC$, and $c \in C$, suppose that $\pi_1(\vec{h}_C p)c$ is defined. Then, by definition and using Condition **I**1, there is some $d \in \mathcal{N}$ and $p' \in P(C \cup \{d\})$ with $C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash p'$. So, by Condition **I**2, for any $d' \in C$, we have $p'' \in PC$ with $C \vdash p \xrightarrow{c?d'} C \vdash p''$. So $\pi_1(\pi_1(\vec{h}_C\, p)c)$ as described is indeed a total function $C \to \wp^+ PC$. On the other hand, also by Condition **I**2, for any $z \notin C$ we have $p'' \in P(C \cup \{z\})$ with $C \vdash p \xrightarrow{c?z} C \cup \{z\} \vdash p''$. So $[\nu/z]p'' \in \pi_2(\pi_1(\vec{h}_C\, p)c)$ and thus the induced $\pi_2(\pi_1(\vec{h}_C\, p)c)$ is indeed a non-empty subset of $P(C \oplus 1)$. So $\pi_1(\vec{h}_C p)$ is a partial function of type $C \rightharpoonup ((\wp^+ PC)^C \times \wp^+ P(C \oplus 1))$. Analogously, one establishes that $\pi_2(\vec{h}_C p)$, $\pi_3(\vec{h}_C p)$, and $\pi_4(\vec{h}_C p)$ are respectively partial functions of type $C \rightharpoonup \wp^+(C \times PC)$, $C \rightharpoonup \wp^+ P(C \oplus 1)$, and $1 \rightharpoonup \wp^+ PC$. Thus each map $\vec{h}_C : PC \to B_{\mathrm{e}} PC$ $(C \in \mathbf{I})$ is well-defined.

We now proceed to establish the naturality condition for the family of maps $\vec{h}_C : PC \to B_{\mathrm{e}} PC$ $(C \in \mathbf{I})$. For all $\imath : C \to D$ in $\mathbf{I}$ and $p \in PC$, we need to establish the following identities.

$$\pi_1\Big(\vec{h}_D(P\imath p)\Big) = (\wp^+ P)^N \imath \circ \pi_1(\vec{h}_C p) \circ \imath^R \tag{A.2}$$
$$\pi_2\Big(\vec{h}_D(P\imath p)\Big) = \Big(\wp^+ (N \times P)\Big)\imath \circ \pi_2(\vec{h}_C p) \circ \imath^R \tag{A.3}$$
$$\pi_3\Big(\vec{h}_D(P\imath p)\Big) = (\wp^+ \delta P)\imath \circ \pi_3(\vec{h}_C p) \circ \imath^R \tag{A.4}$$
$$\pi_4\Big(\vec{h}_D(P\imath p)\Big) = (\wp^+ P)\imath \circ \pi_4(\vec{h}_C p) \tag{A.5}$$

Let us first consider (A.4) for $\imath$ a *bijection* $C \xrightarrow{\sim} D$. We need to establish that

$$\begin{aligned} \pi_3\Big(\vec{h}_D(P\imath p)\Big)(\imath c) &= (\wp^+ \delta P)\imath(\pi_3(\vec{h}_C p)c) \\ &= \Big\{P(\imath \oplus 1)p' \mid p' \in \pi_3(\vec{h}_C p)c\Big\} \end{aligned}$$

for all $c \in C$. We show each inclusion in turn:

($\supseteq$) Let $p' \in \pi_3(\vec{h}_C\, p)c \in \wp^+ P(C \oplus 1)$. Then, there is a transition

$$C \vdash p \xrightarrow{c!z} C \cup \{z\} \vdash [z/\nu_C]p'$$

for some $z \notin C$. Applying Condition **I**3 to this transition with respect to the bijection $(z'/\nu_D) \circ (\nu_D/z)_{\imath^{-1}} : C \cup \{z\} \xrightarrow{\sim} D \cup \{z'\}$ for some $z' \notin D$, we obtain the transition

$$D \vdash [\imath]p \xrightarrow{(\imath c)!z'} D \cup \{z'\} \vdash [z'/\nu_D][(\nu_D/z)_{\imath^{-1}}][z/\nu_C]p' \quad .$$

Since $(\nu_D/z)_{\imath^{-1}} \circ (z/\nu_C) = \imath \oplus 1$ the above transition amounts to the following one

$$D \vdash [\imath]p \xrightarrow{(\imath c)!z'} D \cup \{z'\} \vdash [z'/\nu_D][\imath \oplus 1]p'$$

showing that $[\imath \oplus 1]p' \in \pi_3\Big(\vec{h}_D([\imath]p)\Big)(\imath c)$ as required.

Note that if $(\wp^+ \delta P)\imath(\pi_3(\vec{h}_C p)c)$ is defined then so is $\pi_3(\vec{h}_C p)c$ and, consequently, also $\pi_3\Big(\vec{h}_D(P\imath p)\Big)(\imath c)$ is defined.

($\subseteq$) Let $q \in \pi_3\Big(\vec{h}_D([\imath]p)\Big)(\imath c) \in \wp^+ P(D \oplus 1)$. Then, there is a transition

$$D \vdash [\imath]p \xrightarrow{(\imath c)!z} D \cup \{z\} \vdash [z/\nu_D]q$$

for some $z \notin D$. Applying Condition **I**3 to this transition with respect to the bijection $(z'/\nu_C) \circ (\nu_C/z)_\imath : D \cup \{z\} \xrightarrow{\sim} C \cup \{z'\}$ for some $z' \notin C$, we obtain the transition

$$C \vdash [\imath^{-1}][\imath]p \xrightarrow{(\imath^{-1}\imath\, c)!z'} C \cup \{z'\} \vdash [z'/\nu_C][(\nu_C/z)_\imath][z/\nu_D]q$$

Since $\imath^{-1} \circ \imath = \mathrm{id}_C$ and $(\nu_C/z)_\imath \circ (z/\nu_D) = \imath^{-1} \oplus 1$ the above transition amounts to the following one

$$C \vdash p \xrightarrow{c!z'} C \cup \{z'\} \vdash [z'/\nu_C][\imath^{-1} \oplus 1]q$$

showing that $[\imath^{-1} \oplus 1]q \in \pi_3(\vec{h}_C p)c$. As $q = [\imath \oplus 1][\imath^{-1} \oplus 1]q$, we are done.

Note that if $\pi_3\Big(\vec{h}_D(P\imath p)\Big)(\imath c)$ is defined then so is $\pi_3(\vec{h}_C p)c$ and, consequently, also $(\wp^+\delta P)\imath(\pi_3(\vec{h}_C p)c)$ is defined.

One establishes (A.2), (A.3), and (A.5) with respect to bijections in a similar manner: Condition **I**3 is the key to naturality with respect to bijections.

Finally, we consider naturality with respect to inclusions. The case of input (A.2) is the most complex. Recall the action of the exponential (1): $(f', q') = Q^N(C \hookrightarrow C \cup D)(f, q) \in \Big(Q(C \cup D)\Big)^{C \cup D} \times Q\Big((C \cup D) \oplus 1\Big)$ where

$$f'(d) = \begin{cases} Q(C \hookrightarrow C \cup D)(f(d)) & \text{, if } d \in C \\ Q\Big((C \cup \{d\} \hookrightarrow C \cup D) \circ (d/\nu_C)\Big)q & \text{, otherwise} \end{cases}$$

and

$$q' = Q\Big((C \hookrightarrow C \cup D) \oplus 1\Big)q \quad .$$

For *disjoint* $C, D \in \mathbf{I}$ and $p \in PC$, we must show that:

(1) For $c \in C \cup D$, $\pi_1\Big(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c$ is defined iff c is in C and $\pi_1(\vec{h}_C p)c$ is defined.

(2) For $c, d \in C$ we have $p' \in \pi_1(\pi_1\Big(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c)d$ iff there exists $p'' \in \pi_1(\pi_1(\vec{h}_C\, p)c)d \in \wp^+ PC$ such that $[C \hookrightarrow C \cup D]p'' = p'$.

(3) For $c \in C$ and $d \in D$ we have

$$p' \in \pi_1(\pi_1\Big(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c)d \in \wp^+ P(C \cup D)$$

iff there exists $p'' \in \pi_2(\pi_1(\vec{h}_C\, p)c) \in \wp^+ P(C \oplus 1)$ such that

$$[C \cup \{d\} \hookrightarrow C \cup D][d/\nu_C]p'' = p' \quad .$$

(4) For $c \in C$, we have

$$p' \in \pi_2(\pi_1\Big(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c) \in \wp^+ P\Big((C \cup D) \oplus 1\Big)$$

iff there exists $p'' \in \pi_2(\pi_1(\vec{h}_C p)c) \in \wp^+ P(C \oplus 1)$ such that

$$[(C \hookrightarrow C \cup D) \oplus 1]p'' = p' \quad .$$

First, we show (1). Suppose that $\pi_1\Big(\vec{h}_{C\cup D}([C \hookrightarrow C \cup D]p)\Big)c$ is defined for $c \in C \cup D$. Then, by definition and Condition **I**1, there is a transition of the form

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c?d} C \cup D \cup \{d\} \vdash p' \qquad .$$

Since $C \cup D = C \cup (D \cap \{d\}) \cup (D \setminus \{d\})$, Condition **I**6 ensures that there exists $p'' \in P(C \cup \{d\})$ such that $[C \cup \{d\} \hookrightarrow C \cup \{d\} \cup D]p'' = p'$ and

$$C \cup (D \cap \{d\}) \vdash [C \hookrightarrow C \cup (D \cap \{d\})]p \xrightarrow{c?d} C \cup \{d\} \vdash p'' \qquad .$$

If $d \notin D$, or otherwise by Condition **I**4a, $C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash p''$. Thus, by Condition **I**1, we have $c \in C$ and, by definition, $\pi_1(\vec{h}_C p)c$ is defined. Conversely, suppose that $\pi_1(\vec{h}_C p)c$ is defined for $c \in C$. Then, by definition and Condition **I**1, there is a transition of the form

$$C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash p'$$

where, because of Condition **I**2, we can assume that $d = c$ without loss of generality. Thus, by Condition **I**5, the above transition induces the following one

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c?c} C \cup D \vdash [C \hookrightarrow C \cup D]p'$$

from which it follows by definition that $\pi_1\Big(\vec{h}_{C\cup D}([C \hookrightarrow C \cup D]p)\Big)c$ is defined.

We now show (2). Suppose we have $p' \in \pi_1(\pi_1\Big(\vec{h}_{C\cup D}([C \hookrightarrow C \cup D]p)\Big)c)d$ for $c, d \in C$. As above, there is a transition $C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c?d} C \cup D \vdash p'$ and, by Condition **I**6, we have $p'' \in PC$ such that $[C \hookrightarrow C \cup D]p'' = p'$ and $C \vdash p \xrightarrow{c?d} C \vdash p''$. Thus, we obtain $p'' \in \pi_1(\pi_1(\vec{h}_C\, p)\, c)\, d$ as required. Conversely, suppose we have $p'' \in \pi_1(\pi_1(\vec{h}_C p)c)d$ for $c, d \in C$. Again as above, there is a transition $C \vdash p \xrightarrow{c?d} C \vdash p''$ and, by Condition **I**5, we have the transition $C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c?d} C \cup D \vdash [C \hookrightarrow C \cup D]p''$ showing that $[C \hookrightarrow C \cup D]p'' \in \pi_1(\pi_1\Big(\vec{h}_{C\cup D}([C \hookrightarrow C \cup D]p)\Big)c)d$ as required.

Case (4) can be shown in a similar manner.

Case (3) is particularly specific to the input behaviour. Suppose we have

$$p' \in \pi_1(\pi_1\Big(\vec{h}_{C\cup D}([C \hookrightarrow C \cup D]p)\Big)c)d$$

for $c \in C$ and $d \in D$. This must have been induced by a transition

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c?d} C \cup D \vdash p' \qquad .$$

Now, $D = (D \setminus \{d\}) \cup \{d\}$, so by Condition **I**6, we have $p'' \in P(C \cup \{d\})$ with $[C \cup \{d\} \hookrightarrow C \cup D]p'' = p'$ and $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \xrightarrow{c?d} C \cup \{d\} \vdash p''$. By

Condition **I**4a, we have $C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash p''$. This yields $[\nu_C/d]p'' \in \pi_2(\pi_1(\vec{h}_C p)c)$ with

$$[C \cup \{d\} \hookrightarrow C \cup D][d/\nu_C][\nu_C/d]p'' = [C \cup \{d\} \hookrightarrow C \cup D]p'' = p'$$

as required. Conversely, let $p'' \in \pi_2(\pi_1(\vec{h}_C p)c)$ for $c \in C$. This must have been induced by a transition

$$C \vdash p \xrightarrow{c?z} C \cup \{z\} \vdash [z/\nu_C]p''$$

with $z \notin C$. For any $d \in D$, applying Condition **I**3 with respect to the bijection $(d/\nu_C) \circ (\nu_C/z) : C \cup \{z\} \xrightarrow{\sim} C \cup \{d\}$, we have $C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash [d/\nu_C]p''$. By Condition **I**4a, we can know the fresh name d and the transition will still occur. That is, $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \xrightarrow{c?d} C \cup \{d\} \vdash [d/\nu_C]p''$. Finally, by Condition **I**5, we have

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c?d} C \cup D \vdash [C \cup \{d\} \hookrightarrow C \cup D][d/\nu_C]p'' \quad .$$

This yields

$$[C \cup \{d\} \hookrightarrow C \cup D][d/\nu_C]p'' \in \pi_1(\pi_1\Big(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p)\Big)c)d$$

as required.

Turning now to output transitions, we will prove (A.3) for inclusions $(C \hookrightarrow C \cup D)$, where C and D are disjoint. For $c \in C$, suppose that

$$\Big(\wp^+(N \times P)\Big)(C \hookrightarrow C \cup D)(\pi_2(\vec{h}_C p)c)$$

is defined and let (d, p') be in it. Then, also $\pi_2(\vec{h}_C p)c$ is defined, and $p' = [C \hookrightarrow C \cup D]p''$ for $(d, p'') \in \pi_2(\vec{h}_C p)c$. It follows that $C \vdash p \xrightarrow{c!d} C \vdash p''$. By Condition **I**1, $d \in C$, so that $d \notin D$. Further Condition **I**5 gives

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c!d} C \cup D \vdash [C \hookrightarrow C \cup D]p'' \quad .$$

Hence $\pi_2(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p))c$ is defined and $(d, p') = (d, [C \hookrightarrow C \cup D]p'')$ is in it as required by (A.3). Conversely, for $c \in C \cup D$, suppose that

$$\pi_2(\vec{h}_{C \cup D}([C \hookrightarrow C \cup D]p))c$$

is defined with (d, p') in it. This must be because

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{c!d} C \cup D \vdash p' \quad .$$

By Condition **I**6 we have $p'' \in P(C \cup \{d\})$ with $[C \cup \{d\} \hookrightarrow C \cup D]p'' = p'$ and $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \xrightarrow{c!d} C \cup \{d\} \vdash p''$. By Condition **I**4b, then, $d \in C$.

So $\pi_2(\vec{h}_C p)c$ is defined with (d, p'') in it. Thus, also

$$\Big(\wp^+(N \times P)\Big)(C \hookrightarrow C \cup D)(\pi_2(\vec{h}_C p)c)$$

is defined with $(d, p') = (N \times P)(C \hookrightarrow C \cup D)(d, p'')$ in it as required by (A.3).

One establishes (A.4) and (A.5) with respect to inclusions in a similar manner.

Thus, our definition yields a B_e-coalgebra $\vec{h} : P \dot{\to} B_e P$ in $\mathbf{Set}^{\mathbf{I}}$.

B Proof of Theorem 10

B.1 The **I**-*LTS induced by a* **B**-*LTS*

For a **B**-LTS $\dashrightarrow$ over a sheaf $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$, we note that $\dashrightarrow_! \subseteq \smallint P \times \mathit{Lab} \times \smallint P$ satisfies Conditions **I**1–**I**6 of Figure 1.

Condition **B**1 ensures that the channel is known beforehand, and the definition ensures that exactly the data is learnt. Thus Condition **I**1 is satisfied. Condition **B**2 guarantees that a resumption exists for every input if it exists for one, and Condition **B**1 ensures that the resumption is at the correct stage. Thus Condition **I**2 is satisfied. Condition **B**3 implies Condition **I**3.

Conditions **I**4–**I**6 are guaranteed by the use of minimal supports in the definition in conjunction with Condition **B**1, as we now illustrate.

Suppose that $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \overset{c!d}{\dashrightarrow_!} C \cup \{d\} \vdash p'$, as in the premise of Condition **I**4b. This must have been because $C_0 \vdash p_0 \overset{c!d}{\dashrightarrow} C_0' \vdash p_0{}'$ for some C_0, C_0' and $p_0 \in \langle P \rangle(C_0)$, $p_0{}' \in \langle P \rangle(C_0')$ with $d \in C_0 \subseteq C \cup \{d\} \supseteq C_0'$ and $[C_0 \hookrightarrow C \cup \{d\}]p_0 = [C \hookrightarrow C \cup \{d\}]p$, $[C_0' \hookrightarrow C \cup \{d\}]p_0{}' = p'$. By definition of $\langle P \rangle$, $C_0 = \mathsf{supp}(p_0)$. Also, $\mathsf{supp}(p_0) = \mathsf{supp}([C_0 \hookrightarrow C \cup \{d\}]p_0)$. Now C supports $[C_0 \hookrightarrow C \cup \{d\}]p_0$, but C_0 is the minimal such and so $C_0 \subseteq C$. Thus $d \in C$, and Condition **I**4b is satisfied.

Suppose that $C \cup D \vdash [C \hookrightarrow C \cup D]p \overset{\ell}{\dashrightarrow_!} C' \cup D \vdash p'$, as in the premise of Condition **I**6. Suppose that $\ell = c!d$ and that $d \notin C \cup D$. Then we must have $C' \cup D = C \cup D \cup \{d\}$, and there must exist C_0, C_0' and $p_0 \in \langle P \rangle(C_0)$, $p_0{}' \in \langle P \rangle(C_0')$ with $C_0 \subseteq C \cup D$, and $C_0' \subseteq C \cup D \cup \{d\}$, and

$$[C_0 \hookrightarrow C \cup D]p_0 = [C \hookrightarrow C \cup D]p, \quad [C_0' \hookrightarrow C \cup D \cup \{d\}]p_0{}' = p'$$

and such that $C_0 \vdash p_0 \overset{c!d}{\dashrightarrow} C_0' \vdash p_0{}'$. By definition of $\langle P \rangle$, $C_0 = \mathsf{supp}(p_0) = \mathsf{supp}([C_0 \hookrightarrow C \cup D]p_0)$. We know C supports $[C_0 \hookrightarrow C \cup D]p_0$; C_0 is the minimal such and so $C_0 \subseteq C$. So, by Condition **B**1, $C_0' \subseteq C \cup \{d\}$. Since $d \notin C$, we have, by definition and the fact that $[C_0 \hookrightarrow C]p_0 = p$, that

$$C \vdash p \overset{c!d}{\dashrightarrow}_! C \cup \{d\} \vdash [C_0' \hookrightarrow C \cup \{d\}]p_0' \quad .$$

Thus Condition **I**6 is satisfied in this case.

B.2 The **B***-LTS induced by an* **I***-LTS*

For an **I**-LTS $\longrightarrow$ over a sheaf $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$, we note that $\langle\longrightarrow\rangle \subseteq \textstyle\int\langle P\rangle \times \mathit{Lab} \times \textstyle\int\langle P\rangle$ satisfies Conditions **B**1–**B**3 of Figure 3.

Suppose that $C_0 \vdash p_0 \langle\overset{\ell}{\longrightarrow}\rangle C_0' \vdash p_0'$. Then there must exist C, C' and $p \in P(C)$, $p' \in P(C')$ with $C_0 = \mathsf{supp}(p)$, $C_0' = \mathsf{supp}(p')$, $p_0 = \mathsf{seed}(p)$, $p_0' = \mathsf{seed}(p')$, and such that $C \vdash p \overset{\ell}{\longrightarrow} C' \vdash p'$. Let $C'' = C_0 \cup (C \cap \mathsf{dat}(\ell))$ and $D = C \setminus C''$. So $D \cap \mathsf{dat}(\ell) = \emptyset$ and $C = C'' \cup D$. By Condition **I**1, $C' = C \cup \mathsf{dat}(\ell) = C'' \cup D \cup \mathsf{dat}(\ell)$. By Condition **I**6, we have $p'' \in P(C'' \cup \mathsf{dat}(\ell))$ with

$$C'' \vdash [C_0 \hookrightarrow C'']p_0 \overset{\ell}{\longrightarrow} C'' \cup \mathsf{dat}(\ell) \vdash p''$$

and $[C'' \cup \mathsf{dat}(\ell) \hookrightarrow C']p'' = [C_0' \hookrightarrow C']p_0'$. So $C'' \cup \mathsf{dat}(\ell)$ supports p', but C_0' is the least support, so $C_0' \subseteq C'' \cup \mathsf{dat}(\ell)$. Since P is a sheaf we have $p'' = [C_0' \hookrightarrow C'' \cup \mathsf{dat}(\ell)]p_0'$. In summary, we have

$$C'' \vdash [C_0 \hookrightarrow C'']p_0 \overset{\ell}{\longrightarrow} C'' \cup \mathsf{dat}(\ell) \vdash [C_0' \hookrightarrow C'' \cup \mathsf{dat}(\ell)]p_0' \quad .$$

Clearly $C_0' \subseteq C'' \cup \mathsf{dat}(\ell) = C_0 \cup (C \cap \mathsf{dat}(\ell)) \cup \mathsf{dat}(\ell) = C_0 \cup \mathsf{dat}(\ell)$. So it remains for us to show that $\mathsf{ch}(\ell) \subseteq C_0$. By Condition **I**1, $\mathsf{ch}(\ell) \subseteq C''$, *i.e.*, $\mathsf{ch}(\ell) \subseteq C_0 \cup (C \cap \mathsf{dat}(\ell))$. If $\mathsf{ch}(\ell) \neq \mathsf{dat}(\ell)$, then $\mathsf{ch}(\ell) \subseteq C_0$ as required. Suppose that $\mathsf{ch}(\ell) = \mathsf{dat}(\ell)$; our reasoning will depend on the form of ℓ. If $\ell = \tau$, then the result is trivial. If $\ell = d?d$, then, by Condition **I**4a, $C_0 \vdash p_0 \overset{\ell}{\longrightarrow} C_0 \cup \mathsf{dat}(\ell) \vdash [C_0' \hookrightarrow C_0 \cup \mathsf{dat}(\ell)]p'$; so by Condition **I**1, $\mathsf{ch}(\ell) \subseteq C_0$. If $\ell = d!d$, then, by Condition **I**4b, $d \in C_0$; so $\mathsf{ch}(\ell) = \{d\} \subseteq C_0$. Thus Condition **B**1 holds.

Condition **B**2 follows from Condition **I**2. Condition **B**3 follows from Condition **I**3.

B.3 The bijective correspondence between **B***-LTSs and* **I***-LTSs*

First, we show that $\longrightarrow = \langle\longrightarrow\rangle_!$. To see that $\longrightarrow \subseteq \langle\longrightarrow\rangle_!$, suppose that $C \vdash p \overset{\ell}{\longrightarrow} C' \vdash p'$. We will concentrate on the case $\ell = c!d$. Then, by definition, $\mathsf{supp}(p) \vdash \mathsf{seed}(p) \ \langle \xrightarrow{c!d} \rangle \ \mathsf{supp}(p') \vdash \mathsf{seed}(p')$. If $d \in \mathsf{supp}(p)$, then, since $\mathsf{supp}(p) \subseteq C$, we have $d \in C$ and, by Condition **I**1, $C' = C$; so $C \vdash p \ \langle \xrightarrow{c!d} \rangle_! \ C \vdash p'$. Otherwise, if $d \notin \mathsf{supp}(p)$, in order to check that $C \vdash p \ \langle \xrightarrow{c!d} \rangle_! \ C' \vdash p'$ is induced we must show that $d \notin C$. Suppose that $d \in C$; then by Condition **I**1,

$$C \vdash [C \setminus \{d\} \hookrightarrow C][\mathsf{supp}(p) \hookrightarrow C \setminus \{d\}](\mathsf{seed}(p)) \xrightarrow{c!d} C \cup \{d\} \vdash p'$$

so by Condition **I**4b, $d \in C \setminus \{d\}$, which is absurd; so $d \notin C$. The cases for input and silent actions are proved in a similar manner; thus $\longrightarrow \subseteq \langle \longrightarrow \rangle_!$.

Conversely, to see that $\longrightarrow \supseteq \langle \longrightarrow \rangle_!$, suppose that $C \vdash p \ \langle \xrightarrow{\ell} \rangle_! \ C' \vdash p'$. By definition of $\langle \longrightarrow \rangle_!$ and of $\langle P \rangle$, no matter what form ℓ takes, we must have $C' = C \cup \mathsf{dat}(\ell)$ and

$$\mathsf{supp}(p) \vdash \mathsf{seed}(p) \ \langle \xrightarrow{\ell} \rangle \ \mathsf{supp}(p') \vdash \mathsf{seed}(p') \qquad .$$

This, in turn, must have been induced by some D, D', $q \in P(D)$, $q' \in P(D')$ with $D \vdash q \xrightarrow{\ell} D' \vdash q'$ and

$$\mathsf{supp}(p) = \mathsf{supp}(q), \ \ \mathsf{seed}(p) = \mathsf{seed}(q)$$
$$\mathsf{supp}(p') = \mathsf{supp}(q'), \ \ \mathsf{seed}(p') = \mathsf{seed}(q') \qquad .$$

We consider the case $\ell = c!d$ for $d \notin C$, so that $d \notin \mathsf{supp}(p) \subseteq C$. By Condition **I**4b, $d \notin D$. So, by Condition **I**6, and since P is a sheaf,

$$\mathsf{supp}(p) \vdash \mathsf{seed}(p) \xrightarrow{c!d} \mathsf{supp}(p) \cup \{d\} \vdash [\mathsf{supp}(p') \hookrightarrow \mathsf{supp}(p) \cup \{d\}]\mathsf{seed}(p') \qquad .$$

By Condition **I**5, considering $d \notin C$, we have $C \vdash p \xrightarrow{c!d} C \cup \{d\} \vdash p'$. The cases for $\ell = \tau$ and $\ell = c!d$ with $d \in C$ are proved in a similar way. The case for $\ell = c?d$ requires separate attention, however. In this case, since $d \notin (D \setminus (D \cap \{d\}))$ and P is a sheaf, Condition **I**6 gives

$$\begin{aligned} &\mathsf{supp}(p) \cup (D \cap \{d\}) \vdash [\mathsf{supp}(p) \hookrightarrow \mathsf{supp}(p) \cup (D \cap \{d\})]\mathsf{seed}(p) \\ &\qquad \xrightarrow{c?d} \ \mathsf{supp}(p) \cup \{d\} \vdash [\mathsf{supp}(p') \hookrightarrow \mathsf{supp}(p) \cup \{d\}]\mathsf{seed}(p') \quad . \end{aligned}$$

Now, either $d \in D$ or Condition **I**4a applies; either way,

$$\begin{aligned} &\mathsf{supp}(p) \cup \{d\} \vdash [\mathsf{supp}(p) \hookrightarrow \mathsf{supp}(p) \cup \{d\}]\mathsf{seed}(p) \\ &\qquad \xrightarrow{c?d} \mathsf{supp}(p) \cup \{d\} \vdash [\mathsf{supp}(p') \hookrightarrow \mathsf{supp}(p) \cup \{d\}]\mathsf{seed}(p') \qquad . \end{aligned}$$

Whether $d \in C$ or $d \notin C$, Condition **I**5 gives, by considering $C \setminus \{d\}$, $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}][\mathsf{supp}(p) \hookrightarrow C]\mathsf{seed}(p) \xrightarrow{c?d} C \cup \{d\} \vdash p'$. Hence, by Condition **I**4a, $C \vdash p \xrightarrow{c?d} C \cup \{d\} \vdash p'$ as required.

Finally, we note that $\dashrightarrow = \langle \dashrightarrow_! \rangle$. Indeed, suppose that $C_0 \vdash p \overset{\ell}{\dashrightarrow} C_0' \vdash p'$. Then, by definition, no matter what form ℓ takes,

$$C_0 \vdash p \overset{\ell}{\dashrightarrow}_! C_0 \cup \mathsf{dat}(\ell) \vdash [C_0' \hookrightarrow C_0 \cup \mathsf{dat}(\ell)]p' \qquad .$$

Thus, we have $C_0 \vdash p \langle \overset{\ell}{\dashrightarrow}_! \rangle C_0' \vdash p'$. The converse is equally simple to prove.

C Proof of Theorem 13

Throughout this appendix we use the formulation of Conditions i1–I3 presented in Proposition 12.

C.1 The **I**-*LTS induced by an* i-*LTS*

First, we show Condition **I**1. We will focus on output transitions; Condition **I**1 is shown for other modes of communication in a very similar manner. Suppose that $C \vdash p \overset{c!d}{\rightsquigarrow}_{\mathbf{I}} C' \vdash p'$. If $d \in C$, then we have $C' = C$ and $C \vdash p \overset{\mathsf{out}(c,d)}{\rightsquigarrow} p'$. Thus, as $\mathsf{out}(c,d) \in L(C)$, we have that $c, d \in C$, as required. On the other hand, if $d \notin C$, then we have $C = C' \setminus \{d\}$ and $C' \vdash [C \hookrightarrow C']p \overset{\mathsf{out}(c,d)}{\rightsquigarrow} p'$. Thus, as $\mathsf{out}(c,d) \in L(C)$, we have $c, d \in C'$ and so $C' = C \cup \{d\}$. Moreover, by Condition i1, $c \in C$ as required.

Condition **I**2 follows from Condition i3 in a straightforward manner.

Conditions **I**3 and **I**5 are satisfied because $\rightsquigarrow$ is a subfunctor.

Suppose $C \vdash p \overset{c?z}{\rightsquigarrow}_{\mathbf{I}} C \cup \{z\} \vdash p'$, as in the left hand side of Condition **I**4a. This must have been induced by $C \cup \{z\} \vdash [C \hookrightarrow C \cup \{z\}]p \overset{\mathsf{in}(c,z)}{\rightsquigarrow} p'$. Let $C' = C \cup \{z\}$. Then we have

$$C' \cup \{z\} \vdash [C' \hookrightarrow C' \cup \{z\}][C \hookrightarrow C']p \overset{\mathsf{in}(c,z)}{\rightsquigarrow} p' \qquad .$$

So $C \cup \{z\} \vdash [C \hookrightarrow C \cup \{z\}]p \overset{c?z}{\rightsquigarrow}_{\mathbf{I}} C \cup \{z\} \vdash p'$ is induced. The other direction of Condition **I**4a is also a result of the definition; thus Condition **I**4a is satisfied. Similarly, suppose that

$$C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \overset{c!d}{\rightsquigarrow}_{\mathbf{I}} C \cup \{d\} \vdash p'$$

as in the premise of Condition **I**4b. This must be because

$$C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \overset{\mathsf{out}(c,d)}{\rightsquigarrow} p'$$

and $d \in \mathsf{supp}([C \hookrightarrow C \cup \{d\}]p)$; since C supports $[C \hookrightarrow C \cup \{d\}]p$ we have $d \in C$ and Condition **I**4b is satisfied.

We will show that Condition **I**6 is satisfied for input transitions; other modes of communication are handled in a similar way. Suppose that

$$C \cup D \vdash [C \hookrightarrow C \cup D]p \overset{c?d}{\rightsquigarrow_{\mathbf{I}}} C' \cup D \vdash p'$$

is induced, and $d \notin D$. It follows that $C' \cup D = C \cup \{d\} \cup D$ and that $C \cup \{d\} \cup D \vdash [C \hookrightarrow C \cup \{d\} \cup D]p \overset{\mathsf{in}(c,d)}{\rightsquigarrow} p'$. By Condition i1 we have $c \in C$, and so $\mathsf{in}(c,d) \in L(C \cup \{d\})$. By Condition i2, then, we have $p'' \in P(C \cup \{d\})$ with $[C \cup \{d\} \hookrightarrow C \cup \{d\} \cup D]p'' = p'$ and, as $\rightsquigarrow$ is a subsheaf, it follows that $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \overset{\mathsf{in}(c,d)}{\rightsquigarrow} p''$. So $C \vdash p \overset{c?d}{\rightsquigarrow_{\mathbf{I}}} C \cup \{d\} \vdash p''$ by definition. Thus Condition **I**6 is satisfied.

C.2 The i*-LTS induced by an* **I***-LTS*

We now show that the induced i-LTS $\longrightarrow_{\mathsf{i}}$ is indeed a subsheaf satisfying Conditions i1–i3.

To see that $\longrightarrow_{\mathsf{i}}$ is a subfunctor of $P \times L \times P$, suppose that $C \vdash p \xrightarrow{\ell}_{\mathsf{i}} p'$. This transition must be induced by the **I**-LTS, and by Condition **I**1 we have $C = C' \cup \mathsf{dat}(\ell)$ with $q \in P(C')$ such that $p = [C' \hookrightarrow C]q$ and that

$$C' \vdash q \xrightarrow{\ell} C \vdash p' \qquad .$$

Consider $\imath : C \to D$ in **I**, and observe that $\imath = (C \xrightarrow{\imath|_C} \imath(C) \hookrightarrow D)$. By Condition **I**3,

$$\imath(C') \vdash [\imath|_{C'}]q \xrightarrow{[\imath|_C]\ell} \imath(C') \cup \imath(\mathsf{dat}(\ell)) \vdash [\imath|_C]p' \qquad .$$

Let $D' = D \setminus \imath(C)$; so $\imath(\mathsf{dat}(\ell)) \cap D' = \emptyset$. Note that $\mathsf{dat}([\imath]\ell) = \imath(\mathsf{dat}(\ell))$, and that $[\imath|_C]\ell = [\imath]\ell$. By Condition **I**5,

$$\begin{aligned} &\imath(C') \cup D' \vdash [\imath(C') \hookrightarrow \imath(C') \cup D'][\imath|_{C'}]q \xrightarrow{[\imath]\ell} \\ &\quad \imath(C') \cup D' \cup \imath(\mathsf{dat}(\ell)) \vdash [\imath(C') \cup \imath(\mathsf{dat}(\ell)) \hookrightarrow \imath(C') \cup D' \cup \imath(\mathsf{dat}(\ell))][\imath|_C]p' \end{aligned}$$

and so

$$\begin{aligned} &\imath(C') \cup D' \cup \imath(\mathsf{dat}(\ell)) \vdash [\imath(C') \hookrightarrow \imath(C') \cup D' \cup \imath(\mathsf{dat}(\ell))][\imath|_{C'}]q \\ &\qquad \xrightarrow{[\imath]\ell}_{\mathsf{i}} [\imath(C') \cup \imath(\mathsf{dat}(\ell)) \hookrightarrow \imath(C') \cup D' \cup \imath(\mathsf{dat}(\ell))][\imath|_C]p' \end{aligned}$$

is induced. But $\imath(C') \cup D' \cup \imath(\mathsf{dat}(\ell)) = D$, so

$$D \vdash [\imath][C' \hookrightarrow C]q \xrightarrow{[\imath]\ell}_{\mathsf{i}} [\imath]p' \qquad .$$

Since $[C' \hookrightarrow C]q = p$, we have $D \vdash [\imath]p \xrightarrow{[\imath]\ell}_{\mathsf{i}} [\imath]p'$, and so $\longrightarrow_{\mathsf{i}}$ is a subfunctor of $P \times L \times P$.

Now we show that $\longrightarrow_{\mathsf{i}}$ is in fact a sub*sheaf* of $P \times L \times P$. Suppose that $D \subseteq C$ supports $C \vdash p \xrightarrow{\ell}_{\mathsf{i}} p'$. So we have $q, q' \in P(D)$ with $p = [D \hookrightarrow C]q$, $p' = [D \hookrightarrow C]q'$, and we know that $\mathsf{ch}(\ell) \cup \mathsf{dat}(\ell) \subseteq D$. We must show that $D \vdash q \xrightarrow{\ell}_{\mathsf{i}} q'$. By Condition **I**1, we must have C' with $C = C' \cup \mathsf{dat}(\ell)$ such that this transition is induced by $C' \vdash p'' \xrightarrow{\ell} C \vdash p'$ for some $p'' \in P(C')$ with $[C' \hookrightarrow C]p'' = p$. Since both C' and D support p, we have $r \in P(C' \cap D)$ with $[C' \cap D \hookrightarrow C]r = p$, and $[C' \cap D \hookrightarrow D]r = q$ and $[C' \cap D \hookrightarrow C']r = p''$. So we have

$$C' \vdash [C' \cap D \hookrightarrow C']r \xrightarrow{\ell} C \vdash p' \qquad .$$

Since $\mathsf{dat}(\ell) \subseteq D \subseteq C = C' \cup \mathsf{dat}(\ell)$, we have $D = (C' \cap D) \cup \mathsf{dat}(\ell)$. Since $(C' \setminus (C' \cap D)) \cap \mathsf{dat}(\ell) = \emptyset$, Condition **I**6 gives $r' \in P(D)$ with $[D \hookrightarrow C]r' = p'$ and

$$C' \cap D \vdash r \xrightarrow{\ell} D \vdash r' \qquad .$$

Now r', q' present two gluings of p' at D; the sheaf condition requires a unique such and so $r' = q'$. So $D \vdash q \xrightarrow{\ell}_{\mathsf{i}} q'$ is induced, and it follows that $\longrightarrow_{\mathsf{i}}$ is a subsheaf.

We now show that Conditions i1–i3 are satisfied.

Suppose that $C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{\ell}_{\mathsf{i}} p'$ is induced, as in the premise of Condition i1, and suppose that $\ell = \mathsf{in}(c, d)$. By definition, and by Condition **I**1, we have C'' and $q \in P(C'')$ with $C \cup D = C'' \cup \{d\}$, and

$$[C \hookrightarrow C \cup D]p = [C'' \hookrightarrow C \cup D]q \quad \text{and} \quad C'' \vdash q \xrightarrow{c?d} C'' \cup \{d\} \vdash p' \qquad .$$

By Condition **I**4a,

$$C'' \cup \{d\} \vdash [C'' \hookrightarrow C'' \cup \{d\}]q \xrightarrow{c?d} C'' \cup \{d\} \vdash p' \qquad .$$

Now, both C and C'' support $[C \hookrightarrow C \cup D]p$; since P is a sheaf we have $p'' \in P(C \cap C'')$ with $[C \cap C'' \hookrightarrow C'' \cup \{d\}]p'' = [C'' \hookrightarrow C'' \cup \{d\}]q$. Condition **I**6 gives us $q'' \in P((C \cap C'') \cup \{d\})$ for which

$$(C \cap C'') \cup \{d\} \vdash [C \cap C'' \hookrightarrow (C \cap C'') \cup \{d\}]p'' \xrightarrow{c?d} (C \cap C'') \cup \{d\} \vdash q'' \qquad .$$

Applying Condition **I**4a again gives $C \cap C'' \vdash p'' \xrightarrow{c?d} (C \cap C'') \cup \{d\} \vdash q''$, and by Condition **I**1, $c \in C \cap C''$, so $c \in C$ as required.

Condition $\mathsf{i}1$ is proved for the other modes of communication in a similar way; for output, Condition $\mathbf{I}4b$ is required.

Turning now to Condition $\mathsf{i}2$; suppose that $C \cup D \vdash [C \hookrightarrow C \cup D]p \xrightarrow{[C \hookrightarrow C \cup D]\ell}_{\mathsf{i}} p'$. For clarity, we suppose that $\ell = \mathsf{out}(c, d)$, but the other cases are handled in a very similar manner. By assumption we must have $d \in C$, and by definition (using Condition $\mathbf{I}1$) there must exist C'' and $q \in P(C'')$ with $C \cup D = C'' \cup \{d\}$, $[C \hookrightarrow C \cup D]p = [C'' \hookrightarrow C \cup D]q$, and $C'' \vdash q \xrightarrow{c!d} C'' \cup \{d\} \vdash p'$. Now, both C and C'' support $[C \hookrightarrow C \cup D]p$; since P is a sheaf, we have $p'' \in P(C \cap C'')$ with $[C \cap C'' \hookrightarrow C]p'' = p$ and $[C \cap C'' \hookrightarrow C'']p'' = q$. Since $d \in C$, we know that $d \notin C'' \setminus (C \cap C'')$; thus Condition $\mathbf{I}6$ gives us $q' \in P((C \cap C'') \cup \{d\})$ such that $[(C \cap C'') \cup \{d\} \hookrightarrow C \cup D]q' = p'$. For $q'' = [(C \cap C'') \cup \{d\} \hookrightarrow C]q' \in P(C)$ we have $[C \hookrightarrow C \cup D]q'' = p'$, and so Condition $\mathsf{i}2$ holds.

Condition $\mathsf{i}3$ results from Condition $\mathbf{I}2$, as follows. Suppose that

$$C \vdash p \xrightarrow{\mathsf{in}(c,d)}_{\mathsf{i}} p'$$

is induced. By definition, and by Condition $\mathbf{I}1$, we have C' and $q \in P(C')$ with $C = C' \cup \{d\}$, $p = [C' \hookrightarrow C]q$, and $C' \vdash q \xrightarrow{c?d} C \vdash p'$. By Condition $\mathbf{I}2$, for every $e \in \mathcal{N}$ we have $p'' \in P(C' \cup \{e\})$ such that $C' \vdash q \xrightarrow{c?e} C' \cup \{e\} \vdash p''$. If $d \in C'$, or otherwise by Condition $\mathbf{I}5$,

$$C \vdash p \xrightarrow{c?e} C \cup \{e\} \vdash [C' \cup \{e\} \hookrightarrow C \cup \{e\}]p''$$

and hence

$$C \cup \{e\} \vdash [C \hookrightarrow C \cup \{e\}]p \xrightarrow{\mathsf{in}(c,e)}_{\mathsf{i}} [C' \cup \{e\} \hookrightarrow C \cup \{e\}]p''$$

is induced, as required.

C.3 The bijective correspondence between i*-LTSs and* $\mathbf{I}$*-LTSs*

The equality $\rightsquigarrow = (\rightsquigarrow_{\mathbf{I}})_{\mathsf{i}}$ follows straightforwardly from the definitions and Condition $\mathbf{I}1$. To prove $\longrightarrow = (\longrightarrow_{\mathsf{i}})_{\mathbf{I}}$, Conditions $\mathbf{I}4a$ and $\mathbf{I}4b$ are also needed. For instance, suppose that $C \vdash p \xrightarrow{c!d} C \cup \{d\} \vdash p'$ with $d \notin \mathsf{supp}(p)$. In this case, then, $C \setminus \{d\}$ supports p, so, since P is a sheaf, we have $p'' \in P(C \setminus \{d\})$ with $[C \setminus \{d\} \hookrightarrow C]p'' = p$. If $d \in C$, Condition $\mathbf{I}4b$ gives $d \in C \setminus \{d\}$, which is absurd; so $d \notin C$. Now, by definition, we have $C \cup \{d\} \vdash [C \hookrightarrow C \cup \{d\}]p \xrightarrow{\mathsf{out}(c,d)}_{\mathsf{i}} p'$. So, by definition, $C \vdash p \xrightarrow{c!d}_{\mathsf{i}\mathbf{I}} C \cup \{d\} \vdash p'$, as required.

D Proof of Theorem 16

We start by showing that $\Sigma_! : \mathbf{fsNSet} \to \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ is essentially surjective. Recalling that every $P \in \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ decomposes up to isomorphism as $\langle P \rangle_!$ with $\langle P \rangle \in \mathbf{Set}^{\mathbf{B}}$, we need only show that for every $Q \in \mathbf{Set}^{\mathbf{B}}$ there exists $\oint Q \in \mathbf{fsNSet}$ such that $\Sigma \oint Q \cong Q$.

Define $| \oint Q |$ to be the quotient set $(\int Q)_{/\approx}$ where $\approx$ is the equivalence relation with $(C, p) \approx (D, q)$ if and only if there exists $\beta : C \xrightarrow{\sim} D$ in $\mathbf{B}$ such that $[\beta]p = q$.

To each $(D, q) \in \int Q$ associate the subgroup $\mathfrak{S}_{(D,q)}$ of $\mathsf{Sym}(\mathcal{N})$ consisting of all the permutations that restrict to automorphisms on D and fix $q \in QD$. That is,

$$\mathfrak{S}_{(D,q)} = \left\{ \sigma \in \mathsf{Sym}(\mathcal{N}) \;\middle|\; \begin{array}{c} \sigma|_D \text{ is an automorphism} \\ \text{such that } [\sigma|_D]q = q \end{array} \right\} .$$

For a section $s : | \oint Q | \to \int Q$ of the quotient map $\int Q \twoheadrightarrow | \oint Q |$, define a named-set with symmetries $\oint_s Q$ by

$$\oint_s Q \;=\; \Big(\, | \oint Q | \, , \; \mathfrak{S}_s = \{ \mathfrak{S}_{sx} \}_{x \in | \oint Q |} \Big) .$$

This named-set is finitely supported. Indeed,

> for $s[C, p] = (D, q)$, we have that D is least among the finite supports of $[C, p]$ in $\oint_s Q$

as we now show:

(1) D supports $[C, p]$ because every $\sigma \in \mathsf{Sym}(\mathcal{N})$ such that $\sigma|_D = \mathrm{id}_D$ is clearly in $\mathfrak{S}_{(D,q)}$.
(2) Assume that $E \subseteq \mathcal{N}$ is a finite set supporting $[C, p]$ in $\oint_s Q$, and suppose that there is a $d \in \mathcal{N}$ which is in D but not in E. Then, there is $\sigma \in \mathsf{Sym}(\mathcal{N})$ such that (i) $\sigma|_E = \mathrm{id}_E$ and (ii) $\sigma(d) \notin D$. By assumption and (i), it follows that $\sigma \in \mathfrak{S}_{(D,q)}$. Hence, $\sigma|_D$ is an automorphism, contradicting (ii).

We now exhibit an isomorphism between Q and $\Sigma \oint_s Q$ in $\mathbf{Set}^{\mathbf{B}}$.

Define a mapping $QC \to \Sigma(\oint_s Q)C$ as follows

$$p \mapsto \Big(\, [C, p] \, , \; \sigma \mathfrak{S}_{(D,q)} \, \Big) \tag{D.1}$$

where $(D,q) = s[C,p]$ and $\sigma \in \mathsf{Sym}(\mathcal{N})$ is such that $\sigma(D) = C$ and $[\sigma|_D]q = p$. (Note that, as $(D,q) \approx (C,p)$, such a σ always exists. Note also that, for $\tau \in \mathsf{Sym}(\mathcal{N})$ such that $\tau(D) = C$ and $[\tau|_D]q = p$, we have that $(\sigma^{-1}\tau)|_D = \sigma^{-1}|_C\,\tau|_D$ is an automorphism on D that fixes q, so that $\sigma\mathfrak{S}_{(D,q)} = \tau\mathfrak{S}_{(D,q)}$. Further recall from above that $\sigma^{-1}(C) = D$ is least among the finite supports of $[C,p]$ in $\oint_s Q$.)

We show that (D.1) is injective. To this end, let $p, p' \in QC$ be such that

$$\Big(\ [C,p]\ ,\ \sigma\mathfrak{S}_{(D,q)}\ \Big) = \Big(\ [C,p']\ ,\ \sigma'\mathfrak{S}_{(D,q)}\ \Big)$$

where $(D,q) = s[C,p] = s[C,p']$, and $\sigma(D) = \sigma'(D) = C$ and $[\sigma|_D]q = p$, $[\sigma'|_D]q = p'$. Then, $\sigma^{-1}\sigma' \in \mathfrak{S}_{(D,q)}$ and so $\sigma^{-1}|_C\,\sigma'|_D = (\sigma^{-1}\sigma')|_D$ is an automorphism such that $[\sigma^{-1}|_C][\sigma'|_D]q = q$. Hence,

$$p\ =\ [\sigma|_D]q\ =\ [\sigma'|_D]q\ =\ p'$$

as required.

We show that (D.1) is surjective. Let $\Big(\ [D,q]\ ,\ \sigma\mathfrak{S}_{s[D,q]}\ \Big) \in \Sigma(\oint_s Q)C$, so that $\sigma^{-1}(C)$ is least among the finite supports of $[D,q]$ in $\oint_s Q$. If $(D',q') = s[D,q]$ then, as D' is the least support of $[D,q]$ in $\oint_s Q$, it follows that $D' = \sigma^{-1}(C)$. Thus, we have $\sigma|_{D'} : D' \xrightarrow{\sim} C$ in $\mathbf{B}$, and $[\sigma|_{D'}]q' \in QC$ maps to $\Big(\ \big[C, [\sigma|_{D'}]q'\big]\ ,\ \sigma\mathfrak{S}_{(D',q')}\ \Big) = \Big(\ [D',q']\ ,\ \sigma\mathfrak{S}_{(D',q')}\ \Big) = \Big(\ [D,q]\ ,\ \sigma\mathfrak{S}_{s[D,q]}\ \Big)$.

We show that (D.1) is natural. For $(C,p) \in \int Q$, $(D,q) = s[C,p]$, and $\beta : C \xrightarrow{\sim} C'$ in $\mathbf{B}$ we need to show that

$$\Big(\ \big[C', [\beta]p\big]\ ,\ \sigma'\mathfrak{S}_{(D,q)}\ \Big) = \Big(\ [C,p]\ ,\ \tau\sigma\mathfrak{S}_{(D,q)}\ \Big)$$

where $\sigma'(D) = C'$ and $[\sigma'|_D]q = [\beta]p$, $\sigma(D) = C$ and $[\sigma|_D]q = p$, and $\tau|_C = \beta$. Clearly, $\big[C', [\beta]p\big] = [C,p]$. As for the identity $\sigma'\mathfrak{S}_{(D,q)} = \tau\sigma\mathfrak{S}_{(D,q)}$, note that the automorphism

$$(\sigma^{-1}\tau^{-1}\sigma')|_D = \Big(D \xrightarrow[\sim]{\sigma'|_D} C' \xrightarrow[\sim]{\tau^{-1}|_{C'} = \beta^{-1}} C \xrightarrow[\sim]{\sigma^{-1}|_C} D\Big)$$

is such that

$$\begin{aligned}
[(\sigma^{-1}\tau^{-1}\sigma')|_D]q &= [\sigma^{-1}|_C][\tau^{-1}|_{C'}][\sigma'|_D]q \\
&= [\sigma^{-1}|_C][\beta^{-1}][\beta]p \\
&= [\sigma^{-1}|_C]p \\
&= q
\end{aligned}$$

Thus, $\sigma^{-1}\tau^{-1}\sigma' \in \mathfrak{S}_{(D,q)}$ and we are done.

We have thus established that the functor $\Sigma_! : \mathbf{fsNSet} \to \mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$ is essentially surjective. It remains to be shown that it is also full and faithful.

To see that $\Sigma_!$ is full, consider some $\alpha : \Sigma_!(X, H) \to \Sigma_!(X', H')$ in $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$, and let $(x', J_x) = \alpha_{\mathsf{supp}_H(x)}(x, H_x)$ for each $x \in X$; define $m(x) = x'$ and let $K_x = J_x$. We must show that the pair (m, K) is a valid morphism of named-sets. Suppose $K_x = \sigma_x H'_{m\,x}$; we will show that $H_x \subseteq \sigma_x H'_{m\,x}\sigma_x^{-1}$. Consider some $\sigma \in H_x$. Then $\alpha_{\sigma(\mathsf{supp}_H(x))}(x, \sigma H_x) = \alpha_{\mathsf{supp}_H(x)}(x, H_x)$. The action of $\Sigma_!(X, H)$ and naturality of α gives $\alpha_{\mathsf{supp}_H(x)}(x, H_x) = [\sigma|_{\mathsf{supp}_H(x)}](\alpha_{\mathsf{supp}_H(x)}(x, H_x))$. That is to say, $[\sigma|_{\mathsf{supp}_H(x)}](m\,x, K_x) = (m\,x, K_x)$. We know that σ extends $\sigma|_{\mathsf{supp}_H(x)}$, and so $(m\,x, \sigma_x H'_{m\,x}) = (m\,x, \sigma\sigma_x H'_{m\,x})$. So $\sigma_x H'_{m\,x} = \sigma\sigma_x H'_{m\,x}$. Thus $\sigma_x^{-1}\sigma\sigma_x \in H'_{m\,x}$, and we have shown that $H_x \subseteq \sigma_x H'_{m\,x}\sigma_x^{-1}$. Thus the pair (m, K) is a valid morphism of named-sets.

We still have to show that $\Sigma_!(m, K) = \alpha$. Consider $(x, \sigma H_x) \in \Sigma_!(X, H)C$. We have $\Sigma_!(m, K)_C(x, \sigma H_x) = (m\,x, \sigma K_x)$. Since σ extends $\sigma|_{\sigma^{-1}(C)}$, we have $\Sigma_!(m, K)_C(x, \sigma H_x) = [\sigma|_{\sigma^{-1}(C)}](m\,x, K_x)$. Since $\Sigma_!(X, H)$ maps inclusions to inclusions, and $\mathsf{supp}_H x \subseteq C$, and α is natural with respect to the inclusion maps, we have $\Sigma_!(m, K)_C(x, \sigma H_x) = [\sigma|_{\sigma^{-1}(C)}](\alpha_{\sigma^{-1}(C)}(x, H_x))$. Since α is natural with respect to the bijection $\sigma|_{\sigma^{-1}(C)}$, we have $\Sigma_!(m, K)_C(x, \sigma H_x) = \alpha_C([\sigma|_{\sigma^{-1}(C)}](x, H_x))$. The action of $\Sigma_!(X, H)$ gives us $\Sigma_!(m, K)_C(x, \sigma H_x) = \alpha_C(x, \sigma H_x)$. So $\Sigma_!(m, K) = \alpha$, and $\Sigma_!$ is full.

To see that $\Sigma_!$ is faithful, consider two maps of named-sets $(m, K),\ (m', K') : (X, H) \to (X', H')$ and suppose that $\Sigma_!(m, K) = \Sigma_!(m', K')$. That is, for all $C \in \mathbf{I}$ and $(x, \sigma H_x) \in \Sigma_!(X, H)C$, we have $(m\,x, \sigma K_x) = (m'\,x, \sigma K'_x)$. In particular, by taking $\sigma = \mathrm{id}_{\mathcal{N}}$, we have $(m\,x, K_x) = (m'\,x, K'_x)$ for all $x \in X$. So $m = m'$ and $K = K'$, and $\Sigma_!$ is faithful.

Thus, the category **fsNSet** of finitely supported named-sets with symmetries is equivalent to the Schanuel topos $\mathrm{Sh}(\mathbf{I}^{\mathrm{op}})$, and Theorem 16 is proved.