Computational Learning Theory
Hilary Term 2018
Week 5

Problem Sheet 3

Instructions: The problem sheets are designed to increase your understanding of the material taught in the lectures, as well as to prepare you for the final exam. You should attempt to solve the problems on your own after reading the lecture notes and other posted material, where applicable. Once you have given sufficient thought to a problem, if you are stuck, you are encouraged to discuss with others in the course and with the lecturer during office hours. Please avoid posting on Piazza until the Wednesday before the submission deadline. You are *not permitted* to search for solutions online. Questions marked with an asterisk (*) are optional.

1 Learning Convex Sets in $[0,1]^2$

In this question we will consider the learnability of convex sets. Let us consider the domain to be $X = [0,1]^2$, the unit square in the plane. For $S \subset X$ a convex set, let $c_S : X \to \{0,1\}$, where $c_S(x) = 1$ if $x \in S$ and 0 otherwise. Let $C = \{c_S \mid S \text{ convex subset of } X\}$ be the concept class defined by convex sets of $[0,1]^2$.

1. Show that the VC dimension of C is ∞. This shows that the concept class of convex sets of $[0,1]^2$ is not PAC-learnable (efficiently or otherwise).

2. We will consider a restriction of PAC-learning where the learning algorithm is only required to work for a specific distribution D over X. Show that if D is the uniform distribution over $[0,1]^2$, then the concept class of convex sets is PAC-learnable. (*Hint*: Consider the algorithm that simply outputs the convex hull of positive points as the output hypothesis.)

2 Growth Function

Prove that for any $d \in \mathbb{N}$, there is a concept class C such that $\mathsf{VCD}(C) = d$, and that for any $m \in \mathbb{N}$, $\Pi_C(m) = \Phi_d(m)$.

3 VC Dimension of Linear Halfspaces in $\mathbb{R}^n$

We will show that the concept class of linear halfspaces in $\mathbb{R}^n$ has VC-dimension $n+1$.

1. Give a set of $n+1$ points in $\mathbb{R}^n$ that is shattered by the class of linear halfspaces.

2. We want to show that no set of $m = n+2$ points in $\mathbb{R}^n$ can be shattered by the class of linear halfspaces. For this you can use what is called as Radon's theorem, described below.

3.* Prove Radon's theorem.

Given a set $S = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n$, the convex hull of S is the set,

$$\{z \in \mathbb{R}^n \mid \exists \lambda_1, \ldots, \lambda_m \in [0, 1], \sum_{i=1}^{m} \lambda_i = 1, z = \sum_{i=1}^{m} \lambda_i x_i\}$$

Radon's Theorem: Let $m \geq n + 2$, then S must have two disjoint subsets S_1 and S_2 whose convex hulls intersect.