Computational Learning Theory
Michaelmas Term 2022

Problem Sheet 2

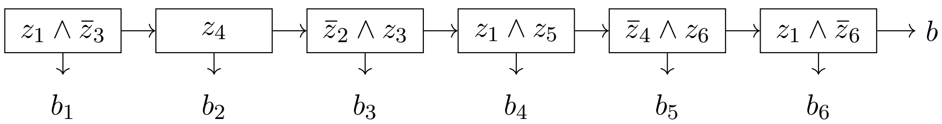
Instructions: The problem sheets are designed to increase your understanding of the material taught in the lectures, as well as to prepare you for the final exam. You should attempt to solve the problems on your own after reading the lecture notes and other posted material, where applicable. Problems marked with an asterisk are optional. Once you have given sufficient thought to a problem, if you are stuck, you are encouraged to discuss with others in the course and with the lecturer during office hours. You are *not permitted* to search for solutions online.

1 Learning Decision Lists

A k-decision list over n boolean variables $z_1, \ldots, z_n$, is defined by an ordered list

$$L = (t_1, b_1), (t_2, b_2), \ldots, (t_l, b_l),$$

and a bit b, where each t_i is a term (conjunction) of at most k literals (positive or negative) and each $b_i \in \{0, 1\}$. For $\mathbf{x} \in \{0, 1\}^n$ the value $L(\mathbf{x})$ is defined to be b_j, where j is the smallest index satisfying $t_j(\mathbf{x}) = 1$ and $L(\mathbf{x}) = b$ if no such index exists. Pictorially, a decision list can be depicted as shown below. As we move from left to right, the first time a term is satisfied, the corresponding b_j is output, if none of the terms is satisfied the default bit b is output.



Give an *efficient* PAC learning algorithm for the class of decision lists. As a first step, argue that it is enough to just consider the case where all the terms have length 1, *i.e.,* in fact they are just literals.

2 Learning Convex Sets in $[0, 1]^2$

In this question we will consider the learnability of convex sets. Let us consider the domain to be $X = [0, 1]^2$, the unit square in the plane. For $S \subset X$ a convex set, let $c_S : X \to \{0, 1\}$, where $c_S(x) = 1$ if $x \in S$ and 0 otherwise. Let $C = \{c_S \mid S \text{ convex subset of } X\}$ be the concept class defined by convex sets of $[0, 1]^2$.

1. Show that the VC dimension of C is ∞. This shows that the concept class of convex sets of $[0, 1]^2$ cannot be learnt by an algorithm (efficiently or otherwise) that uses a sample whose size is bounded by a polynomial in $1/\epsilon$ and $1/\delta$ alone.

2. We will consider a restriction of PAC-learning where the learning algorithm is only required to work for a specific distribution D over X. Show that if D is the uniform distribution

over $[0, 1]^2$, then the concept class of convex sets is *efficiently* PAC-learnable in this restricted sense, where efficiency means running time (and sample complexity) bounded by a polynomial in $\frac{1}{\epsilon}$ and $\frac{1}{\delta}$.

3 Growth Function

Prove that for any $d \in \mathbb{N}$, there is a concept class C such that $\mathsf{VCD}(C) = d$, and that for any $m \in \mathbb{N}$, $\Pi_C(m) = \Phi(m, d)$.