CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 1

Due: Thursday, September 6, 2012 by **9:30am**

Instructions: *You should upload your homework solutions on bspace. You are strongly encouraged to type out your solutions using LaTeX. You may also want to consider using mathematical mode typing in some office suite if you are not familiar with LaTeX. If you must handwrite your homeworks, please write clearly and legibly. We will not grade homeworks that are unreadable. You are encouraged to work in groups of 2-4, but you* **must** *write solutions on your own. Please review the homework policy carefully on the class homepage.*

Note: You *must* justify all your answers. In particular, you will get no credit if you simply write the final answer without any explanation.

Problem 1. *(Exercise 1.1 from MU)* We flip a fair coin ten times. Find the probability of the following events.

(a) The number of heads and the number of tails are equal.

(b) There are more heads than tails.

(c) The i^{th} flip and the $(11-i)^{th}$ flip are the same for $i = 1, \ldots, 5$.

(d) We flip at least four consecutive heads.

Problem 2. *(Exercise 1.6 from MU)* Consider the following balls-and-bin game. We start with one black ball and one white ball in a bin. We repeatedly do the following: choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same colour. We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n-1$.

Problem 3. *(Exercise 1.8 from MU)* Suppose you choose an integer uniformly at random from the range $[1, 1,000,000]$. Using the inclusion-exclusion principle, determine the probability that the number chosen is divisible by one or more of 4, 6, and 9.

Problem 4. Suppose 3 coins are tossed. Each coin has an equal probability of head or tail, but are *not* independent.

(a) What are the minimum and maximum values of the probability of three heads?

(b) Now assume that all *pairs* of coins are mutually independent. What are the minimum and maximum values of the probability of three heads?

Problem 5. *(Exercise 1.13 from MU)* A medical company touts its new test for a certain genetic disorder. The false negative rate is small: if you have the disorder, the probability that the test returns a positive result is 0.999. The false positive rate is also small: if you do not have the disorder, the probability that the test returns a positive result is only 0.005. Assume that 2% of the population has the disorder. If a person chosen uniformly at random from the population is tested and the result comes back positive, what is the probability that the person has the disorder?

Problem 6. *(Exercise 1.18 from MU)* We have a function $F : \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$. We know that, for $0 \leq x, y \leq n-1$, $F((x+y) \bmod n) = (F(x) + F(y)) \bmod m$. The only way we have for evaluating F is to use a lookup table that stores the values of F. Unfortunately, an Evil Adversary has changed the value of $1/5$ of the table entries when were were not looking.

Describe a simple randomized algorithm that, given an input z, outputs a value that equals $F(z)$ with probability at least $1/2$. Your algorithm should work for every value of z, regardless of what values the Adversary changed. Your algorithm should use as few lookups and as little computation as possible.

Suppose you are allowed to repeat your initial algorithm three times. What should you do in this case, and what is the probability that your enhanced algorithm returns the correct answer?