CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 10

Due: Thursday, November 29, 2012 by **9:30am**

Instructions: *You should upload your homework solutions on bspace. You are strongly encouraged to type out your solutions using LaTeX. You may also want to consider using mathematical mode typing in some office suite if you are not familiar with LaTeX. If you must handwrite your homeworks, please write clearly and legibly. We will not grade homeworks that are unreadable. You are encouraged to work in groups of 2-4, but you* **must** *write solutions on your own. Please review the homework policy carefully on the class homepage.*

Note: You *must* justify all your answers. In particular, you will get no credit if you simply write the final answer without any explanation.

Problem 1. *(Exercise 10.6 from MU – 8 points)* The problem of counting the number of solutions to a knapsack instance can be defined as follows: Given items with sizes $a_1, \dots, a_n > 0$ and an integer $b > 0$, find the number of vectors $(x_1, x_2, \dots, x_n) \in \{0,1\}^n$, such that $\sum_{i=1}^n a_i x_i \le b$. The number b can be thought of as the size of a knapsack, and the x_i denote whether or not each item is put into the knapsack. Counting solutions corresponds to counting the number of different sets of items that can be placed in the knapsack without exceeding its capacity.

(a) A naïve way of counting the number of solutions to this problem is to repeatedly choose $(x_1, \dots, x_n) \in \{0,1\}^n$ uniformly at random. If f is the fraction of valid solutions, then return $f \cdot 2^n$. Argue why this is not a good strategy in general; in particular, argue that it will work poorly when each a_i is 1 and $b = \sqrt{n}$.

(b) Consider a Markov chain, $X_0, X_1, \dots,$ on vectors $(x_1, \dots, x_n) \in \{0,1\}^n$. Suppose that X_j is $(x_1, \dots, x_n)$. At each time step, the Markov chain chooses $i \in \{1, \dots, n\}$ uniformly at random. If $x_i = 1$, then X_{j+1} is obtained from X_j by setting x_i to 0. If $x_i = 0$, then X_{j+1} is obtained from X_j by setting x_i to 1 if doing so maintains the restriction $\sum_{i=1}^n a_i x_i \le b$. Otherwise, $X_{j+1} = X_j$.

Argue that this Markov chain has a uniform stationary distribution whenever $\sum_{i=1}^n a_i > b$. Be sure to argue that the chain is irreducible and aperiodic.

(c) Argue that, if we have an FPAUS for the knapsack problem, then we can derive an FPRAS for the problem. To set up the problem properly, assume without loss of generality that $a_1 \le a_2 \le \dots \le a_n$. Let $b_0 = 0$ and $b_i = \sum_{j=1}^i a_i$. Let $\Omega(b_i)$ be the set of vectors $(x_1, \dots, x_n) \in \{0,1\}^n$ that satisfy $\sum_{i=1}^n a_i x_i \le b_i$. Let k be the smallest integer such that $b_k \ge b$. Consider the equation

$$|\Omega(b)| = \frac{|\Omega(b)|}{|\Omega(b_{k-1})|} \times \frac{|\Omega(b_{k-1}|}{|\Omega(b_{k-2})|} \times \cdots \times \frac{|\Omega(b_1)|}{|\Omega(b_0)|} \times |\Omega(b_0)|$$

You will need to argue that $|\Omega(b_{i-1})|/|\Omega(b_i)|$ is not too small. Specifically, argue that $|\Omega(b_i)| \le (n+1)|\Omega(b_{i-1})|$.

Problem 2. *(Exercise 10.7 from MU – 6 points)* An alternative definition of an ϵ-uniform sample of Ω is as follows: A sampling algorithm generates an ϵ-uniform sample w if, for all $x \in \Omega$,

$$\frac{|\Pr(w = x) - 1/|\Omega||}{1/|\Omega|} \leq \epsilon.$$

Show that an ϵ-uniform sample under this definition yields an ϵ-uniform sample as given in Definition 10.3.

Problem 3. *(Exercise 10.12 from MU – 6 points)* The following generalization of the Metropolis algorithm is due to Hastings. Suppose that we have a Markov chain on a state space Ω given by the transition matrix $\mathbf{Q}$ and that we want to construct a Markov chain on this state space with a stationary distribution $\pi_x = b(x)/B$, where for all $x \in \Omega$, $b(x) > 0$, and $B = \sum_{x \in \Omega} b(x)$ is finite. Define a new Markov chain as follows: When $X_n = x$, generate a random variable Y with $\Pr(Y = y) = Q_{x,y}$. Notice that Y can be generated by simulating one step of the original Markov chain. Set X_{n+1} to Y with probability

$$\min\left(\frac{\pi_y Q_{y,x}}{\pi_x Q_{x,y}}, 1\right),$$

and otherwise set X_{n+1} to X_n. Argue that, if this chain is aperiodic and irreducible, then it is also time reversible and has a stationary distribution given by the π_x.

Problem 4. *(10 points)* In this problem we will use a different fingerprinting technique to solve the pattern matching problem. The idea is to map any bit string s into a 2×2 matrix $\mathbf{M}(s)$ as follows:

- For the empty string ϵ, $\mathbf{M}(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- $\mathbf{M}(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
- $\mathbf{M}(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- For non-empty strings x and y, $\mathbf{M}(xy) = \mathbf{M}(x) \times \mathbf{M}(y)$.

Show that this fingerprint function has the following properties.

1. $\mathbf{M}(x)$ is well-defined for all $x \in \{0,1\}^*$.
2. $\mathbf{M}(x) = \mathbf{M}(y) \Rightarrow x = y$.
3. For $x \in \{0,1\}^n$, the entries in $\mathbf{M}(x)$ are bounded by Fibonacci number, F_n. (Where the Fibonacci numbers are defined by the recurrence, $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. You may have to use a slightly clever induction to prove this.)

By considering the matrices $\mathbf{M}(x)$ modulo a suitable prime p, show how you would perform efficient randomized pattern matching.