

## Homework 1 Solutions

**Problem 1.** (*Exercise 1.1 from MU*) We flip a fair coin ten times. Find the probability of the following events. [*Total: 5 points*]

- (a) The number of heads and the number of tails are equal. [*1 point*]

**Solution:** There must be 5 tails and 5 heads. There are  $\binom{10}{5}$  ways to pick 5 heads out of 10 coins flips, and a total of  $2^{10}$  head/tail sequences (of length 10). Therefore the probability of 5 heads is  $\binom{10}{5}/2^{10} = 63/256$

- (b) There are more heads than tails. [*1 point*]

**Solution:** Let  $H$  be the number of heads and  $T$  be the number of tails. We know by symmetry that

$$P[H > T] = P[H < T]$$

and by the axiom of trichotomy,

$$P[H < T] + P[H = T] + P[H > T] = 1$$

So

$$\begin{aligned} 2P[H > T] &= 1 - P[H = T] \\ &= 193/256 \\ \Rightarrow P[H > T] &= 193/512 \end{aligned}$$

- (c) The  $i^{th}$  flip and the  $(11 - i)^{th}$  flip are the same for  $i = 1, \dots, 5$ . [*1 point*]

**Solution:** In this case flips 6–10 are completely determined by flips 1–5, so there are only  $2^5$  different combinations demonstrating this symmetry, giving the probability  $2^5/2^{10} = 1/32$

- (d) We flip at least four consecutive heads. [*2 points*].

**Solution:** Let  $S(h, k)$  be the number of head/tail sequences of length  $k$ , not containing 4 consecutive heads, which end on exactly  $h$  heads. Then we have the dynamic programming recurrence:

$$\begin{aligned} S(0, 0) &= 1 \\ S(h, 0) &= 0 \quad \forall h \geq 1 \\ S(0, k) &= \sum_{h=0}^3 S(h, k-1) \quad \forall k \geq 1 \\ S(h, k) &= S(h-1, k-1) \quad \forall h, k \geq 1 \end{aligned}$$

Then the answer is

$$1 - \left( \sum_{h=0}^3 S(h, 10)/2^{10} \right) = 1 - 773/1024 = 251/1024$$

**Problem 2.** (*Exercise 1.6 from MU*) Consider the following balls-and-bin game. We start with one black ball and one white ball in a bin. We repeatedly do the following: choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same colour. We repeat until there are  $n$  balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and  $n - 1$ . [5 points]

**Solution:** Let  $W_n$  be the number of white balls after the  $n$ th ball has been added. The proposition is trivial for  $n = 2$ . Suppose it holds for  $n = k$ , i.e.

$$P[W_k = i] = \frac{1}{k-1}, \quad i = 1, \dots, k-1$$

Then adding the  $k + 1$ th ball,

$$\begin{aligned} P[W_{k+1} = i] &= P[W_k = i-1]P[\text{select white}|W_k = i-1] + P[W_k = i]P[\text{select black}|W_k = i] \\ &= \frac{1}{k-1} \frac{i-1}{k} + \frac{1}{k-1} \frac{k-i}{k} \\ &= \frac{1}{k} \quad \forall i = 1, \dots, k \end{aligned}$$

**Problem 3.** (*Exercise 1.8 from MU*) Suppose you choose an integer uniformly at random from the range  $[1, 1000000]$ . Using the inclusion-exclusion principle, determine the probability that the number chosen is divisible by one or more of 4, 6, and 9.

**Solution:** Let  $X$  be a uniform number in the given range and let  $E_k$  be the event that  $X$  is divisible by  $k$ . Then for a set of divisors  $D$ ,

$$P \left[ \bigcap_{k \in D} E_k \right] = \frac{\lfloor 10^6 / \text{LCM}(D) \rfloor}{10^6}$$

where LCM is the lowest positive integer divisible by all members of  $D$ . Then

$$\begin{aligned} P[E_4 \cup E_6 \cup E_9] &= P[E_4] + P[E_6] + P[E_9] - P[E_4 \cap E_6] - P[E_4 \cap E_9] - P[E_6 \cap E_9] + P[E_4 \cap E_6 \cap E_9] \\ &= \frac{1}{10^6} (250000 + 166666 + 111111 - 83333 - 27777 - 55555 + 27777) \\ &= 0.388889 \end{aligned}$$

**Problem 4.** Suppose 3 coins are tossed. Each coin has an equal probability of head or tail, but are *not* independent. [Total 5 points]

- (a) What are the minimum and maximum values of the probability of three heads? [2 points]

**Solution:** The minimum probability is 0. This is achieved as follows: The total possible space  $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ . Let  $\Pr[HTH] = 1/2$  and  $\Pr[THT] = 1/2$ . Then  $\Pr[HHH] = 0$ , but for each coin, the probability that the coin is  $H$  is exactly  $1/2$ .

The maximum probability is  $1/2$  since the first coin will land tails with probability at least  $1/2$ . This probability is achievable if all three coins are equal, i.e.  $\Pr[HHH] = \Pr[TTT] = 1/2$ .

- (b) Now assume that all *pairs* of coins are mutually independent. What are the minimum and maximum values of the probability of three heads? [3 points]

**Solution:** The minimum probability is still 0 (suppose the third coin is tails if the first two coins are equal, heads if they are different). Thus we have  $\Pr[HHT] = \Pr[TTT] = \Pr[HTH] = \Pr[THH] = 1/4$ . Then,  $\Pr[HHH] = 0$  and it is easily checked that any of the three pairs of coins are independent.

The maximum probability is  $1/4$  since the first two coins must be independent and will both land heads with probability at most  $1/4$ . A probability of  $1/4$  for three heads is achievable if the third coin is heads when the first two are equal and tails when they aren't. Thus:  $\Pr[HHH] = \Pr[TTH] = \Pr[HTT] = \Pr[THT] = 1/4$ . Again, it can easily be checked that any pair of coin tosses are independent.

**Problem 5.** (*Exercise 1.13 from MU*) A medical company touts its new test for a certain genetic disorder. The false negative rate is small: if you have the disorder, the probability that the test returns a positive result is 0.999. The false positive rate is also small: if you do not have the disorder, the probability that the test returns a positive result is only 0.005. Assume that 2% of the population has the disorder. If a person chosen uniformly at random from the population is tested and the result comes back positive, what is the probability that the person has the disorder? [5 points]

**Solution:** Let  $D$  be the event the person has the disorder and  $T$  be the event they test positive. Then

$$\begin{aligned} P[D|T] &= P[D \cap T]/P[T] \\ &= \frac{P[T|D]P[D]}{P[T|D]P[D] + P[T|\neg D]P[\neg D]} \\ &= \frac{0.999 \cdot 0.02}{0.999 \cdot 0.02 + 0.005 \cdot 0.98} \\ &= 0.8030547 \end{aligned}$$

**Problem 6.** (*Exercise 1.18 from MU*) We have a function  $F : \{0, \dots, n-1\} \rightarrow \{0, \dots, m-1\}$ . We know that, for  $0 \leq x, y \leq n-1$ ,  $F((x+y) \bmod n) = (F(x) + F(y)) \bmod m$ . The only way we have for evaluating  $F$  is to use a lookup table that stores the values of  $F$ . Unfortunately, an Evil Adversary has changed the value of  $1/5$  of the table entries when we were not looking.

Describe a simple randomized algorithm that, given an input  $z$ , outputs a value that equals  $F(z)$  with probability at least  $1/2$ . Your algorithm should work for every value of  $z$ , regardless

of what values the Adversary changed. Your algorithm should use as few lookups and as little computation as possible.

Suppose you are allowed to repeat your initial algorithm three times. What should you do in this case, and what is the probability that your enhanced algorithm returns the correct answer?

**Solution:**

1. [3 points] Choose  $x$  uniformly from  $\{0, \dots, n-1\}$  and let  $y = z - x \bmod n$ . Then output the value  $F(z)$  as

$$F(z) = F(x) + F(y) \bmod m$$

Note that  $x$  and  $y = z - x$  are both a uniformly random number in the range  $\{0, \dots, n-1\}$ , but they are *not independent*. So, we have  $\Pr["F(x) \text{ is corrupted}"] = \Pr["F(y) \text{ is corrupted}"] = 1/5$ . Then by *union bound*

$$P[\text{error}] = P["F(x) \text{ is corrupted"} \cup "F(y) \text{ is corrupted"}] \leq 2/5$$

2. [2 points] If we can repeat the algorithm three times, take the majority vote (or the first if no majority exists). Then the probability of error requires at least two of the runs to go wrong.

Suppose the probability that the algorithm runs once, on input  $z$  is wrong is exactly  $p_z$ . We know that  $p_z \leq 2/5$ . Then the probability that when the algorithm is run three times that at least two runs are wrong is exactly  $\binom{3}{2}p_z^2(1-p_z) + \binom{3}{3}p_z^3$ . *If we increase the probability of being wrong, the probability of being wrong twice can only increase. So since  $p_z \leq 2/5$ , we have*

$$\binom{3}{2}p_z^2(1-p_z) + \binom{3}{3}p_z^3 \leq \binom{3}{2}(2/5)^2(3/5) + \binom{3}{3}(2/5)^3 = 0.352$$

So the answer will be correct with probability  $\geq 0.648$ .

**Note:** The first term is actually only  $2(2/5)^2(3/5)$ , since if  $F(z)$  is correct for the first run, then the answer returned is actually correct. No points will be deducted for not making this observation.