CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 3

Due: Thursday, September 20, 2012 by **9:30am**

Instructions: *You should upload your homework solutions on bspace. You are strongly encouraged to type out your solutions using LaTeX. You may also want to consider using mathematical mode typing in some office suite if you are not familiar with LaTeX. If you must handwrite your homeworks, please write clearly and legibly. We will not grade homeworks that are unreadable. You are encouraged to work in groups of 2-4, but you* **must** *write solutions on your own. Please review the homework policy carefully on the class homepage.*

Note: You *must* justify all your answers. In particular, you will get no credit if you simply write the final answer without any explanation.

Problem 1. *(Exercise 3.5 from MU)* Given any two random variables X and Y, by the linearity of expectation we have $\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y]$. Prove that, when X and Y are independent, $\mathbf{Var}[X - Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$.

Problem 2. *(Exercise 3.15 from MU)* Let the random variable X be representable as a sum of random variables $X = \sum_{i=1}^{n} X_i$. Show that, if $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$ for every pair of i and j with $1 \leq i < j \leq n$, then $\mathbf{Var}[X] = \sum_{i=1}^{n} \mathbf{Var}[X_i]$.

Problem 3. (Exercise 3.19) Let Y be a non-negative integer-valued random variable with positive expectation. Prove

$$\frac{\mathbb{E}[Y]^2}{\mathbb{E}[Y^2]} \leq \Pr[Y \neq 0] \leq \mathbb{E}[Y]$$

Problem 4. *(Exercise 3.20 from MU)*

(a) Chebyshev's inequality uses the variance of a random variable to bound its deviation from its expectation. We can also use higher moments. Suppose that we have a random variable X and an even integer k for which $\mathbb{E}[(X - \mathbb{E}[X])^k]$ is finite. Show that

$$\Pr\left(|X - \mathbb{E}[X]| \geq t \sqrt[k]{\mathbb{E}[(X - \mathbb{E}[X])^k]}\right) \leq \frac{1}{t^k}$$

(b) Why is it difficult to derive a similar inequality when k is odd?

Problem 5. *(Exercise 3.21 from MU)* A fixed point of a permutation $\pi : [1, n] \rightarrow [1, n]$ is a value for which $\pi(x) = x$. Find the variance in the number of fixed points of a permutation chosen uniformly at random from all permutations. (*Hint*: Let X_i be 1 if $\pi(i) = i$, so that $\sum_{i=1}^{n} X_i$ is the number of fixed points. You cannot use linearity to find $\mathbf{Var}[\sum_{i=1}^{n} X_i]$, but you can calculate it directly.)

Problem 6. *(Exercise 3.25 from MU)* The weak law of large numbers states that, if $X_1, X_2, X_3, \ldots$ are independent and identically distributed random variables with mean μ and standard deviation σ, then for any constant $\epsilon > 0$ we have

$$\lim_{n \to \infty} \Pr\left(\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) = 0.$$

Use Chebychev's inequality to prove the weak law of large numbers.