CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 7

Due: Thursday, October 25, 2012 by **9:30am**

Instructions: *You should upload your homework solutions on bspace. You are strongly encouraged to type out your solutions using LaTeX. You may also want to consider using mathematical mode typing in some office suite if you are not familiar with LaTeX. If you must handwrite your homeworks, please write clearly and legibly. We will not grade homeworks that are unreadable. You are encouraged to work in groups of 2-4, but you* **must** *write solutions on your own. Please review the homework policy carefully on the class homepage.*

Note: You *must* justify all your answers. In particular, you will get no credit if you simply write the final answer without any explanation.

Problem 1. *(Exercise 6.10 from MU – 6 points)* A family of subsets $\mathcal{F}$ of $\{1, 2, \ldots, n\}$ is called an *antichain* if there is no pair of sets A and B in $\mathcal{F}$ satisfying $A \subset B$.

(a) Given an example of $\mathcal{F}$ where $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$.

(b) Let f_k be the number of sets in $\mathcal{F}$ with size k. Show that

$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1.$$

(*Hint*: Choose a random permutation of the numbers from 1 to n, and let $X_k = 1$ if the first k numbers in your permutation yeild a set in $\mathcal{F}$. If $X = \sum_{k=0}^{n} X_k$, what can you say about X?)

(c) Argue that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for any antichain $\mathcal{F}$.

Problem 2. *(Exercise 6.14 from MU – 6 points)* Consider a graph in $G_{n,p}$, with $p = 1/n$. Let X be the number of triangles in the graph, where a trianle is a clique with three edges. Show that

$$Pr(X \geq 1) \leq 1/6$$

and that

$$\lim_{n \to \infty} \Pr(X \geq 1) \geq 1/7$$

(*Hint*: Use the conditional expectation inequality.)

Problem 3 *(Exercise 6.18 from MU – 6 points)* Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colours, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbours u of v such that c lies in $S(u)$. Prove that there is a proper colouring of G assigning to each vertex v a colour from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colours assigned to u and v are different. You may want to let $A_{u,v,c}$ be the event that u and v are both coloured with colour c and then consider the family of such events.

Problem 4 *(12 points)* In this problem we will see that the value $p = \ln(n)/n$ is a *threshold* property that a random graph in the $G_{n,p}$ model has an isolated vertex, *i.e.* a vertex with no adjacent edges. That is, we will prove that

$$\lim_{n\to\infty} \Pr[G \text{ has an isolated vertex}] = \begin{cases} 0 & \text{if } p = \omega(\frac{\ln(n)}{n}) \\ 1 & \text{if } p = o(\frac{\ln(n)}{n}) \end{cases}.$$

(a) Let X be the random variable denoting the number of isolated vertices in G. Write down the expectation of X as a function of n and p.

(b) Show that $\mathbb{E}[X] \to 0$ for $p = \omega(\frac{\ln(n)}{n})$, and that $\mathbb{E}[X] \to \infty$ for $p = o(\frac{\ln(n)}{n})$.

(c) Deduce from part (b) that $\Pr[G \text{ has an isolated vertex}] \to 0$ for $p = \omega(\ln(n)/n)$.

(d) Compute $\mathbf{Var}(X)$ as a function of n and p.

(e) Deduce from parts (b) and (d) that $\Pr[G \text{ has an isolated vertex}] \to 1$ for $p = o(\ln(n)/n)$.