CS 174: Combinatorics and Discrete Probability Fall 2012

Homework 7 Solutions

Problem 1. *(Exercise 6.10 from MU – 6 points)* A family of subsets $\mathcal{F}$ of $\{1, 2, \ldots, n\}$ is called an *antichain* if there is no pair of sets A and B in $\mathcal{F}$ satisfying $A \subset B$.

(a) Given an example of $\mathcal{F}$ where $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$.

Solution: Choose every subset of size $\lfloor n/2 \rfloor$.

(b) Let f_k be the number of sets in $\mathcal{F}$ with size k. Show that

$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1.$$

(*Hint*: Choose a random permutation of the numbers from 1 to n, and let $X_k = 1$ if the first k numbers in your permutation yeild a set in $\mathcal{F}$. If $X = \sum_{k=0}^{n} X_k$, what can you say about X?)

Solution: Following the hint, choose a random permutation of $(1, \ldots, n)$. Let $X_k = 1$ if the first k numbers yield a set in $\mathcal{F}$, and let $X = \sum_{k=0}^{n} X_k$.

Note that $\Pr(X_k = 1) = \frac{f_k}{\binom{n}{k}}$. Furthermore, for only one value of k can $X_k = 1$, which means that $\mathbb{E}[X] \leq 1$. Therefore,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{n} \mathbb{E}[X_k] \\
&= \frac{f_k}{\binom{n}{k}} \\
&\leq 1
\end{aligned}$$

(c) Argue that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for any antichain $\mathcal{F}$.

Solution: For a fixed n, the binomial coefficient is maximized at $\binom{n}{\lfloor n/2 \rfloor}$. Therefore,

$$\frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} f_k \leq \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1$$

which implies that

$$\sum_{k=0}^{n} f_k \leq \binom{n}{\lfloor n/2 \rfloor}$$

The result follows since $|\mathcal{F}| = \sum_{k=0}^{n} f_k$.

Problem 2. *(Exercise 6.14 from MU – 6 points)* Consider a graph in $G_{n,p}$, with $p = 1/n$. Let X be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

$$Pr(X \geq 1) \leq 1/6$$

and that

$$\lim_{n\to\infty} \Pr(X \geq 1) \geq 1/7$$

(*Hint*: Use the conditional expectation inequality.)

Solution: Let $C_1, \ldots, C_{\binom{n}{3}}$ be an enumeration of all subsets of 3 vertices in the graph. Let $X = \sum_{i=1}^{\binom{n}{3}} X_i$, where $X - i$ is 1 if C_i is a triangle and 0 otherwise. We have $\mathbb{E}[X_i] = \Pr[X_i = 1] = p^3$, so $\mathbb{E}[X] = \binom{n}{3}p^3$. So by Markov's inequality,

$$\Pr[X \geq 1] \leq \binom{n}{3}p^3 = \binom{n}{3}(1/n)^3 \leq 1/6$$

To use the conditional expectation inequality, we mimic the argument from the previous part to get

$$\mathbb{E}[X|X_i = 1] = 1 + \binom{n-3}{3}p^3 + \binom{n-3}{2}p^3 + \binom{n-3}{1}p^2$$

and then show

$$\begin{aligned}
\Pr[X \geq 1] &\geq \sum_{i=1}^{\binom{n}{3}} \frac{\Pr[X_i = 1]}{\mathbb{E}[X|X_i = 1]} \\
&= \frac{\binom{n}{3}p^3}{1 + \binom{n-3}{3}p^3 + \binom{n-3}{2}p^3 + \binom{n-3}{1}p^2} \\
&= \frac{\binom{n}{3}(1/n)^3}{1 + \binom{n-3}{3}(1/n)^3 + \binom{n-3}{2}(1/n)^3 + \binom{n-3}{1}(1/n)^2} \\
&\to \frac{1/6}{1 + 1/6 + 0 + 0} \quad \text{as } n \to \infty \\
&= 1/7
\end{aligned}$$

Problem 3 *(Exercise 6.18 from MU – 6 points)* Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colours, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbours u of v such that c lies in $S(u)$. Prove that there is a proper colouring of G assigning to each vertex v a colour from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colours assigned to u and v are different. You may want to let $A_{u,v,c}$ be the event that u and v are both coloured with colour c and then consider the family of such events.

Solution: The solution to this problem is an application of the Lovaxa local lemma. Let us check the three requirements necessary to apply the lemma:

(a) $\Pr(A_{u,v,c}) \leq \Pr(u \text{ is color } c \mid v \text{ is color } c)\Pr(v \text{ is color } c) \leq \frac{1}{8r} \cdot \frac{1}{8r} = \frac{1}{64r^2}$

(b) $A_{u,v,c}$ may only depend on events $A_{u',v',c'}$ where either $u = u'$ or $v = v'$. Note that u has at most $8r$ colours, and hence at most $8r^2$ neighbours with which it can share a colour (this includes v). A symmetric argument holds for v. However, this means that the total number of events that $A_{u,v,c}$ can be dependent on is at most $16r^2$ (note that this is despite the double counting $A_{u,v,c'}$).

(c) $4dp \leq 4 \cdot 16r^2 \cdot \frac{1}{64r^2} = 1$.

So applying the Lovasz local lemma, we have:

$$\Pr\left(\bigcap_{u,v,c} \bar{A}_{u,v,c}\right) > 0$$

In other words, there exists a coloring such that no neighboring vertices have the same color.

Problem 4 *(12 points)* In this problem we will see that the value $p = \ln(n)/n$ is a *threshold* property that a random graph in the $G_{n,p}$ model has an isolated vertex, *i.e.* a vertex with no adjacent edges. That is, we will prove that

$$\lim_{n\to\infty} \Pr[G \text{ has an isolated vertex}] = \begin{cases} 0 & \text{if } p = \omega(\frac{\ln(n)}{n}) \\ 1 & \text{if } p = o(\frac{\ln(n)}{n}) \end{cases}.$$

(a) Let X be the random variable denoting the number of isolated vertices in G. Write down the expectation of X as a function of n and p.

Solution: Let X_i be the r.v. indicating whether vertex i is isolated. Then

$$\mathbb{E}[X_i] = (1-p)^{n-1}$$

and by linearity of expectation, $\mathbb{E}[X] = n(1-p)^{n-1}$.

(b) Show that $\mathbb{E}[X] \to 0$ for $p = \omega(\frac{\ln(n)}{n})$, and that $\mathbb{E}[X] \to \infty$ for $p = o(\frac{\ln(n)}{n})$.

Solution: Write $p = a \cdot \frac{\ln n}{n}$. Note that

$$\begin{aligned}
\mathbb{E}[X] &= n(1-p)^{n-1} \\
&= n\left(1 - a \cdot \frac{\ln n}{n}\right)^{n-1} \\
&\to ne^{-a\ln n} \\
&= n^{1-a}
\end{aligned}$$

The case $p = o\left(\frac{\ln n}{n}\right)$ is equivalent to $a = o(1)$, and thus

$$\mathbb{E}[X] \sim n^{1-o(1)} \to \infty$$

The second case, $p = \omega\left(\frac{\ln n}{n}\right)$ is equivalent to $a = \omega(1)$, and thus

$$\mathbb{E}[X] \sim n^{-(\omega(1)-1)} \to 0$$

(c) Deduce from part (b) that $\Pr[G$ has an isolated vertex$] \to 0$ for $p = \omega(\ln(n)/n)$.

Solution: By Markov's inequality we have $\Pr[X \geq 1] \leq E[X]$, which by part (b) goes to zero in the case $p = \omega\left(\frac{\ln n}{n}\right)$. Hence $\Pr[X > 0] \to 0$ as required.

(d) Compute $\mathbf{Var}(X)$ as a function of n and p.

Solution: For any $i \neq j$, $\mathbb{E}[X_i X_j] = (1-p)^{2n-3}$ (there are $2n-3$ possible edges adjacent to either i or j). Hence

$$\mathbb{E}[X^2] = \sum_{i,j} \mathbb{E}[X_i X_j] = n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3}$$

Therefore

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[x]^2 = n(1-p)^{n-1} + n(1-p)^{2n-3}(np-1)$$

(e) Deduce from parts (b) and (d) that $\Pr[G$ has an isolated vertex$] \to 1$ for $p = o(\ln(n)/n)$.

Solution: By Chebyshev's inequality,

$$\begin{aligned}
\Pr[X = 0] &\leq \Pr[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]] \\
&\leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \\
&= \frac{1}{\mathbb{E}[X]} + \frac{np-1}{n(1-p)}
\end{aligned}$$

Now for $p = o\left(\frac{\ln n}{n}\right)$ we know from part (b) that $\mathbb{E}[X] \to \infty$, so the first term here goes to zero. And the second term is $\frac{np-1}{n(1-p)} \leq \frac{p}{1-p}$, which certainly goes to zero for $p = o\left(\frac{\ln n}{n}\right)$. Hence we have $\Pr[X = 0] \to 0$, i.e. $\Pr[X > 0] \to 1$ as required.