Equivalent Definitions of Pseudospectra

THEOREM. Let $\|\cdot\|$ be a matrix norm induced by a vector norm.
The following definitions of pseudospectra are equivalent.

(I) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \|(zI-A)^{-1}\| \geq \varepsilon^{-1}\}$.

(II) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : z \in \Lambda(A+E) \text{ for some } E \text{ with } \|E\| \leq \varepsilon\}$.

(III) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \text{there exists } v \in \mathbb{C}^n \text{ with } \|v\| = 1 \text{ such that } \|(A-zI)v\| \leq \varepsilon\}$.

If $\|\cdot\|$ is the 2-norm, then the following definition is also equivalent:

(IV) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \sigma_{\min}(zI-A) \leq \varepsilon\}$.

Notation. If $zI-A$ is singular, we define $\|(zI-A)^{-1}\| = \infty$.
$\sigma_{\min}(\cdot)$ denotes the minimum singular value.

Proof. First note that if $z \in \Lambda(A)$, then z satisfies all of the above criteria for belonging to $\Lambda_\varepsilon(A)$. Thus, for the rest of the proof assume that $z \notin \Lambda(A)$.

(I)$\Longrightarrow$(III): Suppose $\|(zI-A)^{-1}\| \geq \varepsilon^{-1}$. Then there exists $u \in \mathbb{C}^n$ such that $\|(zI-A)^{-1}u\|/\|u\| = \|(zI-A)^{-1}\|$. Define $\widehat{v} = (zI-A)^{-1}u$, so that

$$\varepsilon^{-1} \leq \frac{\|(zI-A)^{-1}u\|}{\|u\|} = \frac{\|\widehat{v}\|}{\|(zI-A)\widehat{v}\|},$$

Then $v = \widehat{v}/\|\widehat{v}\|$ is a unit vector satisfying $\|(zI-A)v\| \leq \varepsilon$.

(III)$\Longrightarrow$(II): Suppose there exists $v \in \mathbb{C}^n$ with $\|v\| = 1$ such that $\|(A-zI)v\| \leq \varepsilon$. Then let $u \in \mathbb{C}^n$ be a unit vector satisfying $(A-zI)v = \widehat{\varepsilon}u$ for $\widehat{\varepsilon} \leq \varepsilon$. We can complete this part of the proof if we have a vector w satisfying $w^*v = 1$ with $\|w\| = 1$. For general norms,the existence of such a w follows from the theory of dual norms (a corollary to the Hahn–Banach theorem) [DKS93, p.221–223], [Wil86]. With such a w we can write $zv = Av - \widehat{\varepsilon}uw^*v = (A - \widehat{\varepsilon}uw^*)v$, which means that $z \in \Lambda(A+E)$ for $E = \widehat{\varepsilon}uw^*$ satisfying $\|E\| \leq \varepsilon$.

(II)$\Longrightarrow$(I): Suppose $z \in \Lambda(A+E)$ for some E with $\|E\| \leq \varepsilon$. Then there exists a unit vector $v \in \mathbb{C}^n$ such that $(A+E)v = zv$. Rearranging and inverting, $v = (zI-A)^{-1}Ev$ and

$$1 = \|v\| = \|(zI-A)^{-1}Ev\| \leq \|(zI-A)^{-1}\|\|E\| \leq \|(zI-A)^{-1}\|\varepsilon,$$

implying that $\|(zI-A)^{-1}\| \geq \varepsilon^{-1}$.

(I)$\Longleftrightarrow$(IV): If $\|\cdot\|$ is the 2-norm, then this follows from the characterization of the 2-norm of the inverse of a matrix as its smallest singular value. ■

History. These ideas were described for the 2-norm by Varah [Var79, Def. 3.1] and Trefethen [Tre90]. The key to the proof in its full generality is the use of dual norms, as discussed by Wilkinson [Wil86] and by van Dorsselaer, Kraaijevanger, and Spijker [DKS93]. This last citation remains the most complete source for a detailed description of these equivalences.

Bibliography.

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