Basic Mapping Theorem

THEOREM. $\Lambda_{\varepsilon|\beta|}(\alpha I + \beta A) = \alpha + \beta\Lambda_\varepsilon(A)$ for all $\alpha, \beta \in \mathbb{C}$.

Notation. Set addition is defined componentwise: $S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$.
This result holds for any norm.

Proof. Suppose $z \in \Lambda_{\varepsilon|\beta|}(\alpha I + \beta A)$. The result is trivial if $\beta = 0$ or $\varepsilon = 0$. Otherwise, the definition of pseudospectra implies that

$$\begin{aligned} \frac{1}{\varepsilon|\beta|} &\le \|(zI - (\alpha I + \beta A))^{-1}\| \\ &= \frac{1}{|\beta|}\left\|\left(\frac{z-\alpha}{\beta} I - A\right)^{-1}\right\|. \end{aligned}$$

Thus, $z \in \Lambda_{\varepsilon|\beta|}(\alpha I + \beta A)$ is equivalent to $(z-\alpha)/\beta \in \Lambda_\varepsilon(A)$. ∎

History. These basic identities appeared in [Tre99b]. This result *does not* simply generalize to higher degree polynomials.

Bibliography.

[Tre99b] L. N. Trefethen. *Spectra and pseudospectra: The behavior of non-normal matrices and operators.* In *The Graduate Student's Guide to Numerical Analysis*, M. Ainsworth, J. Levesley, and M. Marletta, eds., Springer-Verlag, Berlin, 1999.