A Relationship Between Pseudospectra and the Numerical Range

THEOREM. $\Lambda_\varepsilon(A) \subseteq W(A) + \Delta_\varepsilon$.

Notation. $W(A) \equiv \{x^*Ax : \|x\|_2 = 1\}$ is the numerical range of A.
$\Delta_\delta \equiv \{z \in \mathbb{C} : |z| \le \delta\}$ is the closed disk of radius δ.
$\Lambda_\varepsilon(A)$ is the 2-norm ε-pseudospectrum of A.
Set addition is defined componentwise: $S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$.

Proof. Suppose $z \in \Lambda_\varepsilon(A)$. By definition, there exists some E with $\|E\|_2 \le \varepsilon$ such that $z \in \Lambda(A+E)$. It follows that there exists some non-zero vector v such that $(A+E)v = zv$. Then

$$z = \frac{v^*(A+E)v}{v^*v} = \frac{v^*Av}{v^*v} + \frac{v^*Ev}{v^*v},$$

where the first term is in $W(A)$ and the magnitude of the second term never larger than ε. $\blacksquare$

History. This is a version of a standard result from operator theory. Marshall Stone published it in Theorem 4.20 of his 1932 book [Sto32]. It also appeared in Theorem V.3.2 in Kato's treatise [Kat80]. Apparently Reddy, Schmid, and Henningson were the first to publish this result using the language of pseudospectra [RSH93]. This pseudospectral version is also in Theorem 4.6.2 of the monograph by Gustafson and Rao [GR97], who give a proof similar to the one presented here.

Bibliography.

[GR97] K. E. Gustafson and D. K. M. Rao. *Numerical Range: The Field of Values of Linear Operators and Matrices.* Springer-Verlag, New York, 1980.

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[RSH93] S. C. Reddy, P. J. Schmid, and D. S. Henningson. *Pseudospectra of the Orr-Sommerfeld operator.* SIAM J. Appl. Math. **53** (1993), 15–47.