Contextuality, Cohomology and Paradox

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This note summarises the results in:

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Background. Contextuality is one of the key characteristic features of quantum mechanics. Recent advances suggest that it provides the "magic" ingredient enabling quantum computation [5]. However, the study of quantum contextuality has largely been carried out in a concrete, example-driven fashion, which makes it appear highly specific to quantum mechanics.

Recent work by the present authors [1, 2] and others [3] has exposed the general mathematical structure of contextuality, enabling more general and systematic results. The key idea from [1] is to understand contextuality as arising where we have a family of data which is *locally but not globally consistent*. This can be understood and very effectively visualised (see fig. 1a) in topological terms. We have a base space of *contexts* (typically sets of variables which can be jointly measured or observed); and local (i.e. contextual) data or observations — typically valuations of the variables in the context — are fibred over the base space, forming the total space. In this setting, contextuality arises as the absence of *global sections*, or valuations on all the variables that reconcile the local data. In topological language, we can say that the bundle space of observations is "twisted", resulting in a topological *obstruction* to forming a global section. This perspective provides a unifying description of a number of phenomena which at first sight seem very different, including quantum contextuality as well as phenomena in *classical* computation such as the failure of the universal relation assumption in database theory, and the non-existence of solutions for constraint satisfaction problems.

In this work, we develop our unified viewpoint in two ways:

1. We develop an algebraic notion of contextuality, in terms of global inconsistency of a (locally consistent) system of linear equations satisfied by the possible valuations, which we call an *All-vs-Nothing argument*. We show that such arguments are always manifested topologically, as witnessed by cohomological obstructions. This constitutes an extensive generalisation of the examples in [1], including a large class of Kochen–Specker models. Our main result establishes a chain of implications between algebraic, topological, and logical forms of contextuality.
2. We relate contextuality to *logical paradoxes*: we find a direct connection between the structure of quantum contextuality and classic semantic paradoxes such as "Liar cycles" [4].

All-vs-nothing arguments and the cohomology of contextuality. Our first contribution begins with the identification of a powerful type of contextuality proof, which we call an *All-vs-Nothing (AvN) argument*[1]. These proofs of contextuality are strong in the sense that they do not require inequalities (e.g. Bell inequalities).

[1] The term "All-vs-Nothing" originated Mermin's description of his non-locality argument [6] based on the GHZ experiment, which is a particular instance of the kind of argument to which our definition gives precise meaning.

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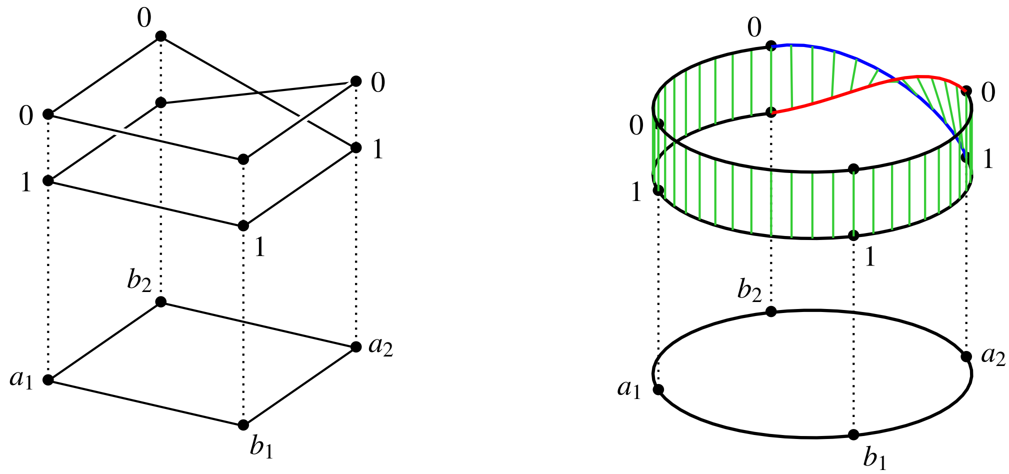

(a) Bundle diagram for the PR box

(b) The PR box embedded into the Möbius strip

Figure 1: The PR box and the Möbius strip

For example, consider the usual Bell scenario: this involves two agents, Alice and Bob, who have access to measurements $\{a_1, a_2\}$ and $\{b_1, b_2\}$ respectively, and each measurement gives one of two possible outcomes. This gives rise to an *empirical model*, i.e. sets of possible outcome pairs, one for each joint measurement, corresponding to the empirical data of the experiment. Such models obey local constraints, imposed by the no-signalling principle (in this case, meaning that the possibility of individual measurement outcomes are not dependent on which context that measurement is performed in).

An example of an AvN argument in this scenario is given by the *Popescu–Rohrlich (PR) box*[2]. The local empirical data satisfies the constraints expressed by the following equations over $\mathbb{Z}_2$:

$$a_1 \oplus b_1 = 0, \qquad a_1 \oplus b_2 = 0, \qquad a_2 \oplus b_1 = 0, \qquad a_2 \oplus b_2 = 1.$$

However, this system of equations is inconsistent: regardless of the values assigned to $a_1, \ldots, b_2$, the left-hand sides sum to 0 (since each variable occurs twice) whereas the right-hand sides sum to 1. In this way, the model is seen to be *strongly contextual*: there can be no global assignment of outcomes to measurements consistent with the local constraints.

This leads to the general definition of an *AvN argument*: the joint outcomes of a set of measurements satisfy an inconsistent system of R-linear equations, over a ring R (in the case of the PR box, $R = \mathbb{Z}_2$). Our main result here is that AvN arguments, which are algebraic in flavour, can be captured using topological methods. This can be seen intuitively in the bundle diagram of fig. 1a. Suppose that we attempt to build a global assignment by tracing a path through the fibres: if we start at $a_1 \mapsto 1$, then we are forced to take the route $b_1 \mapsto 1$, then $a_2 \mapsto 1$, then $b_2 \mapsto 0$, then $a_1 \mapsto 0$, and hence we obtain a contradiction, and so on. More formally, the bundle picture of fig. 1a can be formalised as a presheaf varying over the contexts. Using Čech cohomology, we have shown that if an empirical model admits an AvN argument, then cohomological obstructions witness its strong contextuality. More precisely, for each local section s, we define an element $\gamma(s)$ in the first cohomology group. We show that if $\gamma(s) \neq 0$, then s cannot be extended to a global section, and we have a witness for contextuality. Hence AvN arguments arise when

[2]We consider the PR box here for ease of visualisation. Though is not quantum mechanically realisable, the argument we set out is formally similar to the Mermin-GHZ argument, and serves as a representative illustration.

the bundle for the empirical model contains a "twist" (which appears, in some sense, to be analogous to the non-orientability of a Möbius strip: cf. fig. 1b).

Theorem (6.1 in our paper[3]). *Let $\mathscr{S}$ be an empirical model. Then:*

$$\mathsf{AvN}_R(\mathscr{S}) \implies \mathsf{CSC}_R(\mathscr{S}) \implies \mathsf{CSC}_{\mathbb{Z}}(\mathscr{S}) \implies \mathsf{SC}(\mathscr{S}) .$$

This chain of implications can be read as follows: if a model admits an AvN argument (an algebraic obstruction), then it has a cohomological witness of contextuality (a topological obstruction), and this in turn implies strong contextuality (a logical obstruction).

Logical paradoxes as strong contextuality. Our second contribution concerns the relation between contextuality and logic, which we demonstrate with the classic semantic paradoxes of Liar cycles. Here we consider the Liar cycle with $n = 4$ sentences, though a similar treatment applies in the general case.[4]

Imagine a cloister with four corners named b_2, a_1, b_1, a_2. At each corner, a notice carrying a sentence is posted:

- The sentence at b_2 reads: "The sentence at a_1 is true."
- The sentence at a_1 reads: "The sentence at b_1 is true."
- The sentence at b_1 reads: "The sentence at a_2 is true."
- The sentence at a_2 reads: "The sentence at b_2 is false."

Assuming, say, b_2 to be true leads to a contradiction, but assuming b_2 to be false also leads to a contradiction, by the following derivations (where 1 and 0 are Boolean truth values):

$$b_2 = 1 \overset{\text{by def of } b_2}{\Longrightarrow} a_1 = 1 \overset{\text{by def of } a_1}{\Longrightarrow} b_1 = 1 \overset{\text{by def of } b_1}{\Longrightarrow} a_2 = 1 \overset{\text{by def of } a_2}{\Longrightarrow} b_2 = 0 \neq 1$$
$$b_2 = 0 \overset{\text{by def of } b_2}{\Longrightarrow} a_1 = 0 \overset{\text{by def of } a_1}{\Longrightarrow} b_1 = 0 \overset{\text{by def of } b_1}{\Longrightarrow} a_2 = 0 \overset{\text{by def of } a_2}{\Longrightarrow} b_2 = 1 \neq 0$$

In short, this possible cloister presents us with a paradoxical combination of sentences that we cannot consistently interpret. Importantly, this is more serious than saying that the four sentences are jointly inconsistent—which merely rules out the "truth value" assignment $(1,1,1,1)$. The paradox is that every assignment to the four sentences is ruled out.

From the definition of the sentences in the cloister (e.g. b_2 is defined so that it is true if and only if a_1 is), we can read off the following Boolean equations:

$$b_2 = a_1, \qquad a_1 = b_1, \qquad b_1 = a_2, \qquad a_2 = \neg b_2.$$

This logical paradox exhibits *exactly the same phenomenon* as the strong contextuality of the PR box.

Outlook. Our work provides several avenues for further research. Firstly, we have found examples of strong contextuality which are not All-vs-Nothing arguments. However it remains open whether quantum theory realises only All-vs-Nothing arguments. Secondly, our work provides topological techniques for the study of logical paradoxes, which may be useful in the classification of such results.

[3] Cf. complete paper for appropriate definitions.

[4] $n = 1$ gives the Liar paradox: "This sentence is false."

References

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