QUANTUM MEASUREMENTS FROM A LOGICAL POINT OF VIEW

Olivier Brunet — olivier.brunet@normalesup.org
Grenoble, France

July 17, 2015

QPL
Oxford, U.K.

PRESENTATION OUTLINE

A Logic about Measurement

Models

Two Theorems on Models

Sequences of Outcomes

A LOGIC ABOUT MEASUREMENT

Logical Constructor

$$\mathrm{Mes}(s, \mathcal{O}, p, t)$$

s	–	Measured system
$\mathcal{O}$	–	Measured **sharp** observable
$p \in \mathcal{O}$	–	Outcome
t	–	Resulting system

$$\neg\big(\exists\, s, \mathcal{O}, t \colon \operatorname{Mes}(s, \mathcal{O}, \bot, t)\big)$$

$$\forall\, p \neq \bot, \exists\, s, \mathcal{O}, t \colon \operatorname{Mes}(s, \mathcal{O}, p, t)$$

$$\forall\, s, \mathcal{O}, \exists\, p, t \colon \operatorname{Mes}(s, \mathcal{O}, p, t)$$

Definition

$$s \blacktriangleright p \overset{\Delta}{\iff} \neg\left(\exists\, \mathcal{O}, t\colon \mathrm{Mes}(s, \mathcal{O}, p^{\perp}, t)\right)$$

Definition

$$s \blacktriangleright p \overset{\Delta}{\iff} \neg\big(\exists\, \mathcal{O}, t:\ \mathrm{Mes}(s, \mathcal{O}, p^{\perp}, t)\big)$$

Meaning in a Hilbert Space

If a system s is in a state $|\varphi\rangle$,
then s $\blacktriangleright$ p corresponds to $|\varphi\rangle \in p$, or $\Pi_p|\varphi\rangle = |\varphi\rangle$.

$$s \blacktriangleright \top$$

$$\neg(s \blacktriangleright \bot)$$

$$\forall p \neq \bot, \exists s : s \blacktriangleright p$$

Claim

The **certainty/impossibility** of an outcome is **independent** of the measured observable.

Claim

The **certainty/impossibility** of an outcome is **independent** of the measured observable.

Motivation: the Born Rule

The probability of obtaining outcome P in state $|\varphi\rangle$ is $\langle\varphi|\Pi_P|\varphi\rangle$.

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Justification

$$\{p, p^{\perp} \wedge q, q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Justification

$$\{p, p^{\perp} \wedge q, q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Justification

$$\{p, p^{\perp} \wedge q, q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Justification

$$\{p, p^{\perp} \wedge q, q^{\perp}\}$$

$$\{q, q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Justification

$$\{p, p^\perp \wedge q, q^\perp\}$$

$$\{q, q^\perp\}$$

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

Justification

$$\{p, p^\perp \wedge q, q^\perp\}$$

$$\{q, q^\perp\}$$

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Justification

$$\{p, p^\perp \wedge q, p^\perp \wedge q^\perp\}$$
$$\{q, p \wedge q^\perp, p^\perp \wedge q^\perp\}$$

$$\{p \wedge q, p \wedge q^\perp, p^\perp \wedge q, p^\perp \wedge q^\perp\}$$

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Justification

$$\{p, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

$$\{q, p \wedge q^{\perp}, p^{\perp} \wedge q^{\perp}\}$$

$$\{p \wedge q, p \wedge q^{\perp}, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Justification

$$\{p, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

$$\{q, p \wedge q^{\perp}, p^{\perp} \wedge q^{\perp}\}$$

$$\{p \wedge q, p \wedge q^{\perp}, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Justification

$$\{p, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

$$\{q, p \wedge q^{\perp}, p^{\perp} \wedge q^{\perp}\}$$

$$\{p \wedge q, p \wedge q^{\perp}, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Justification

$$\{p, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

$$\{q, p \wedge q^{\perp}, p^{\perp} \wedge q^{\perp}\}$$

$$\{p \wedge q, p \wedge q^{\perp}, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

Axiom

If $s \blacktriangleright p$, $s \blacktriangleright q$ and p is compatible with q, then $s \blacktriangleright p \wedge q$.

Justification

$$\{p, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

$$\{q, p \wedge q^{\perp}, p^{\perp} \wedge q^{\perp}\}$$

$$\{p \wedge q, p \wedge q^{\perp}, p^{\perp} \wedge q, p^{\perp} \wedge q^{\perp}\}$$

Axiom

If p and q are compatible, $s \blacktriangleright p$ and $\mathrm{Mes}(s, \mathcal{O}, q, t)$, then $t \blacktriangleright p$.

Axiom

If p and q are compatible, $s \blacktriangleright p$ and $\mathrm{Mes}(s, \mathcal{O}, q, t)$, then $t \blacktriangleright p$.

Justification

p and q are compatible iff Π_p and Π_q commute

If $\Pi_p|\varphi\rangle = |\varphi\rangle$, then $\Pi_p\big(\Pi_q|\varphi\rangle\big) = \Pi_q|\varphi\rangle$.

Definition

$$\forall\, a, b \in L, \qquad a \,\&\, b \triangleq b \wedge (a \vee b^{\perp})$$

Definition

$$\forall\, a, b \in L, \qquad a \,\&\, b \triangleq b \wedge (a \vee b^{\perp})$$

Intuition

It's the lattice theoretic equivalent of the orthogonal projection.

SASAKI PROJECTION

Definition

$$\forall\, a, b \in L, \qquad a \,\&\, b \triangleq b \wedge (a \vee b^{\perp})$$

Intuition

It's the lattice theoretic equivalent of the orthogonal projection.

"Simplifications"

Weak Noncontextuality 2	$s \blacktriangleright p$ and $s \blacktriangleright q \implies s \blacktriangleright p \,\&\, q$
Compatible Preservation	$s \blacktriangleright p$ and $\mathrm{Mes}(s, \mathcal{O}, q, t) \implies t \blacktriangleright p \,\&\, q$

Axioms of $\mathcal{T}_L$

$$s \blacktriangleright \top$$

$$\neg(s \blacktriangleright \bot)$$

$$p \neq \top \implies \exists s : s \blacktriangleright p$$

$$s \blacktriangleright p \text{ and } p \leq q \implies s \blacktriangleright q$$

$$s \blacktriangleright p \text{ and } s \blacktriangleright q \implies s \blacktriangleright p \,\&\, q$$

$$s \blacktriangleright p \text{ and } \mathrm{Mes}(s, \mathcal{O}, q, t) \implies t \blacktriangleright p \,\&\, q$$

MODELS

Reminder

$$\forall\, s, p, \quad s \blacktriangleright p \overset{\Delta}{\iff} \forall\, t, \neg\big(\exists\, \mathcal{O} : \mathrm{Mes}(s, \mathcal{O}, p^{\perp}, t)\big)$$

$$\forall\, s, p, q, t, \quad s \blacktriangleright p \text{ and } \big(\exists\, \mathcal{O} : \mathrm{Mes}(s, \mathcal{O}, q, t)\big) \implies t \blacktriangleright p \,\&\, q$$

Definition

A **model** of $\mathcal{T}_L$ is a pair $\mathfrak{G} = (A, M)$ consisting of

– a set A

– a relation M on $A \times L \times A$

Definition

A **model** of $\mathcal{T}_L$ is a pair $\mathfrak{G} = (A, M)$ consisting of

- a set A
- a relation M on $A \times L \times A$

$$\exists\, \mathcal{O} : \ \mathrm{Mes}(s, \mathcal{O}, p, t) \quad \text{translates as} \quad M(a, p, b)$$

$$a \blacktriangleright_{\mathfrak{G}} p \overset{\Delta}{\iff} \neg\big(\exists\, b \in A : M(a, p^{\perp}, b)\big)$$

VERIFICATION OF AXIOMS

$$\forall a \in A, \quad a \blacktriangleright_{\mathfrak{G}} \top$$

$$\forall a \in A, \quad \neg(a \blacktriangleright_{\mathfrak{G}} \bot)$$

$$\forall p \in L, \quad p \neq \top \implies \exists a \in A : a \blacktriangleright_{\mathfrak{G}} p$$

$$\forall a \in A, \forall p, q \in L, \quad a \blacktriangleright_{\mathfrak{G}} p \text{ and } p \leq q \implies a \blacktriangleright_{\mathfrak{G}} q$$

$$\forall a \in A, \forall p, q \in L, \quad a \blacktriangleright_{\mathfrak{G}} p \text{ and } a \blacktriangleright_{\mathfrak{G}} q \implies a \blacktriangleright_{\mathfrak{G}} p \,\&\, q$$

$$\forall a, b \in A, \forall p, q \in L, \quad a \blacktriangleright_{\mathfrak{G}} p \text{ and } M(a, q, b) \implies b \blacktriangleright_{\mathfrak{G}} p \,\&\, q$$

Definition

Given a Hilbert space $\mathcal{H}$, we define the model $\mathfrak{H}_{\mathcal{H}} = (A_{\mathfrak{H}}, M_{\mathfrak{H}})$ by

$$A_{\mathfrak{H}} \overset{\Delta}{=} \{|\varphi\rangle \mid \langle\varphi|\varphi\rangle = 1\}$$

$$M_{\mathfrak{H}}(|\varphi\rangle, p, |\psi\rangle) \overset{\Delta}{\iff} \Pi_p|\psi\rangle \neq |0\rangle \text{ and } |\varphi\rangle = \frac{\Pi_p|\psi\rangle}{\|\Pi_p|\psi\rangle\|}$$

Definition

Given a Hilbert space $\mathcal{H}$, we define the model $\mathfrak{H}_{\mathcal{H}} = \left(A_{\mathfrak{H}}, M_{\mathfrak{H}}\right)$ by

$$A_{\mathfrak{H}} \overset{\Delta}{=} \left\{|\varphi\rangle \;\middle|\; \langle\varphi|\varphi\rangle = 1\right\}$$

$$M_{\mathfrak{H}}\left(|\varphi\rangle, p, |\psi\rangle\right) \overset{\Delta}{\iff} \Pi_p|\psi\rangle \neq |0\rangle \text{ and } |\varphi\rangle = \frac{\Pi_p|\psi\rangle}{\left\|\Pi_p|\psi\rangle\right\|}$$

Verification Relation

$$|\varphi\rangle \blacktriangleright_{\mathfrak{H}} p \iff |\varphi\rangle \in p$$

Definition

Given an *orthomodular lattice* L, we define the model $\mathfrak{L}_L = (A_{\mathfrak{L}}, M_{\mathfrak{L}})$ by

$$A_{\mathfrak{L}} \triangleq L^\star \qquad \text{where} \qquad L^\star \triangleq L \setminus \{\bot\}$$

$$M_{\mathfrak{L}}(a, p, b) \stackrel{\Delta}{\iff} b \leq a \,\&\, p$$

Definition

Given an *orthomodular lattice* L, we define the model $\mathfrak{L}_L = (A_{\mathfrak{L}}, M_{\mathfrak{L}})$ by

$$A_{\mathfrak{L}} \triangleq L^{\star} \qquad \text{where} \qquad L^{\star} \triangleq L \setminus \{\bot\}$$

$$M_{\mathfrak{L}}(a, p, b) \overset{\Delta}{\iff} b \leq a \,\&\, p$$

Verification Relation

$$a \blacktriangleright_{\mathfrak{L}} p \iff p \leq a$$

TWO THEOREMS ON MODELS

Definition

Given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, for all $a \in A$,

$$[\![a]\!]_{\mathfrak{G}} \triangleq \left\{p \in L \,\middle|\, a \blacktriangleright_{\mathfrak{G}} p\right\}$$

Theorem[1]

If $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$ with $\dim \mathcal{H} \geq 3$, then for all $a \in A$, $[\![a]\!]_{\mathfrak{G}}$ contains at most one vector ray.

[1] arXiv:quant-ph/0610066

Kochen-Specker Theorem

Pick **exactly** one element in each maximal orthogonal family of vectors

Kochen-Specker	117 vectors in dimension 3
Peres	33 vectors in dimension 3
Cabello	17 vectors in dimension 4

Kochen-Specker Theorem

Pick **exactly** one element in each maximal orthogonal family of vectors

Kochen-Specker	117 vectors in dimension 3
Peres	33 vectors in dimension 3
Cabello	17 vectors in dimension 4

Here

Pick **at most** one element in each maximal orthogonal family of vectors

2 vectors in dimension 3

Theorem

If $3 \leq \dim \mathcal{H} < \infty$ and $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$, then

$$\forall\, a \in A, \exists\, e(a) \in L(\mathcal{H}) : \llbracket a \rrbracket_{\mathfrak{G}} = e(a)^{\uparrow}$$

Theorem

If $3 \leq \dim \mathcal{H} < \infty$ and $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$, then

$$\forall\, a \in A, \exists\, e(a) \in L(\mathcal{H}) :\ [\![a]\!]_{\mathfrak{G}} = e(a)^{\uparrow}$$

Consequence

If $3 \leq \dim \mathcal{H} < \infty$, in $\mathcal{T}_{L(\mathcal{H})}$,

$$s \blacktriangleright p \ \text{ and } \ s \blacktriangleright q \implies s \blacktriangleright p \wedge q$$

$$\left\{ [\![a]\!]_{\mathfrak{G}} \;\middle|\; a \in A \right\} \simeq L(\mathcal{H})$$

$$\left\{ [\![a]\!]_{\mathfrak{G}} \;\middle|\; a \in A \right\} \simeq L(\mathcal{H})$$

[The Metaphysical Disaster is] the error of ***equating properties*** *of a physical system on the one hand* ***with experimentally testable propositions*** *about the system on the other hand.*

Unfortunately, this is precisely what is done in conventional Hilbert-space based quantum mechanics where both properties and experimentally testable propositions are represented by projection operators.

(David Foulis, private communication)

SEQUENCES OF OUTCOMES

A **word** on L is a **finite sequence of elements** of L

Definition

Given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, a word $\mathbf{p} = p_1 p_2 \cdots p_n$ is in $\ell(\mathfrak{G})$ iff

$$\exists\, a_0, a_1, \ldots, a_n \in A :$$

$$M(a_0, p_1, a_1) \;\text{ and }\; \cdots \;\text{ and }\; M(a_{n-1}, p_n, a_n)$$

$$a_0 \xrightarrow{p_1} a_1 \xrightarrow{p_2} a_2 \xrightarrow{p_3} \cdots \xrightarrow{p_{n-1}} a_{n-1} \xrightarrow{p_n} a_n$$

Theorem

If $3 \leq \dim \mathcal{H} < \infty$,

$$\ell(\mathfrak{L}_{L(\mathcal{H})}) = \ell(\mathfrak{H}_{\mathcal{H}})$$

Theorem

If $3 \leq \dim \mathcal{H} < \infty$ and $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$, then

$$\ell(\mathfrak{G}) = \ell(\mathfrak{L}_{L(\mathcal{H})}) = \ell(\mathfrak{H}_{\mathcal{H}})$$

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathfrak{L}_{L(\mathcal{H})}) = \ell(\mathfrak{H}_H)$$

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathfrak{L}_{L(\mathcal{H})}) = \ell(\mathfrak{H}_H) = \ell(\mathcal{H})$$

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathfrak{L}_{L(\mathcal{H})}) = \ell(\mathfrak{H}_H) = \ell(\mathcal{H})$$

Given a word $p_1 p_2 \cdots p_n$ on $L(\mathcal{H})$,

$$\begin{aligned} p_1 p_2 \cdots p_n \in \ell(\mathcal{H}) &\iff p_1 \mathbin{\&} p_2 \mathbin{\&} \cdots \mathbin{\&} p_n \neq \bot \\ &\iff \Pi_{p_n} \Pi_{p_{n-1}} \cdots \Pi_{p_1} \neq 0 \end{aligned}$$

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathcal{H})$$

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathcal{H})$$

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathcal{H})$$

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

Possibilistically yes.