Contextuality, Cohomology, and Paradox

(arXiv:1502.03097)

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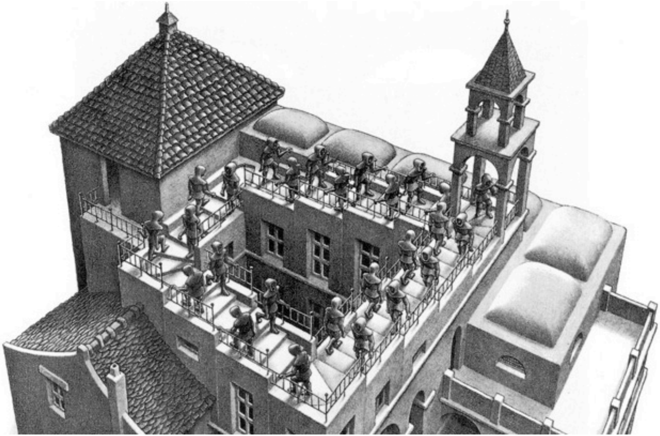
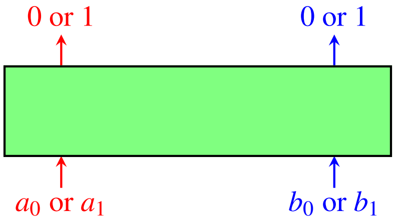
Outline

1. Topological model for contextuality.
2. Cohomology: Contextuality is like “impossible figures”.
3. Relation to QM no-go theorems.



Bell Non-Locality

Bell-type setup. Input-output box for $(2, 2, 2)$ scenario:



Distribution $p(o_A, o_B \mid a_i, b_j)$ for each **context** $\{a_i, b_j\}$.

So a probability table:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$3/8$	$1/8$	$1/8$	$3/8$
(a_1, b_0)	$3/8$	$1/8$	$1/8$	$3/8$
(a_1, b_1)	$1/8$	$3/8$	$3/8$	$1/8$

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Possiblility table: non-zero $\mapsto$ 1 ("possible")
0 $\mapsto$ 0 ("impossible").

Support of a probability table is a possibility table.

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E.g. model by Hardy 1993:

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No local probability table has this support.
(Logical non-locality / contextuality implies probabilistic one.)

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- i.e. a distribution over deterministic

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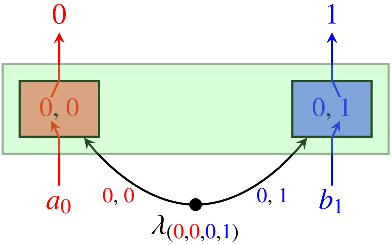

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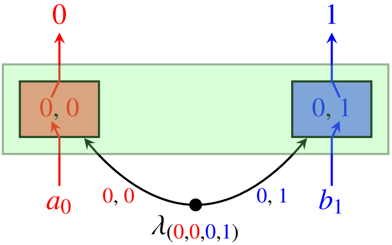

- i.e. the table is a convex combination of the deterministic tables for such λ's.

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Topology on the set of measurements.

Topological Model for Contextuality

Topological spaces of variables and of their values.

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- measurements and outcomes
- sentences and truth values
- questions and answers

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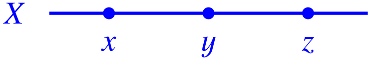
For each variable x,



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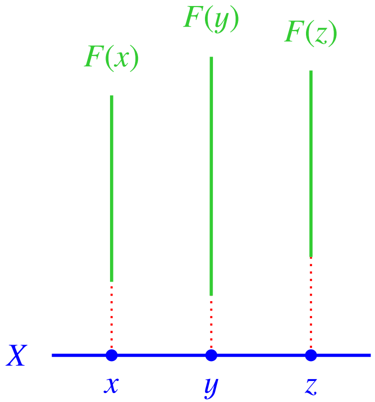
For each variable x,
a dependent type
$F(x)$ of values.



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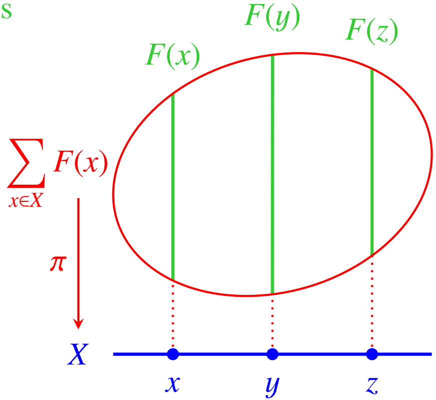
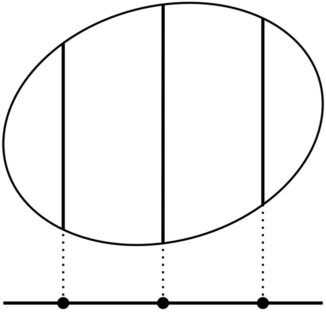
"Bundle" $\sum_{x \in X} F(x)$



When we ask several questions,
answers may obey constraints:



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- laws of physics,
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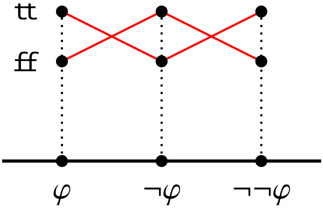
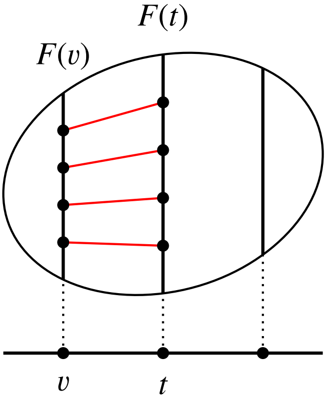
- laws of logic





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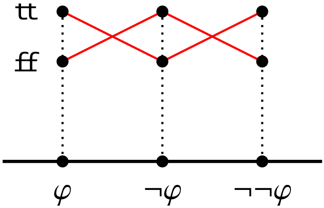

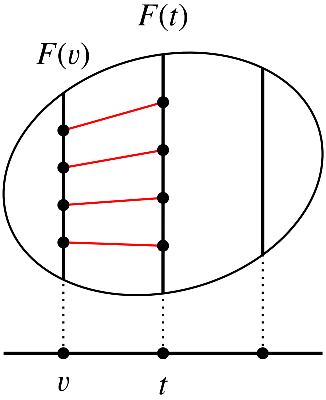

Distinguish good and bad ways of connecting dots in bundles
. . . just like “continuous sections”!

Hardy model:

	00	01	10	11
a_0b_0	1	1	1	1
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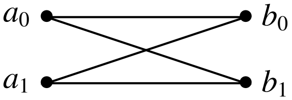

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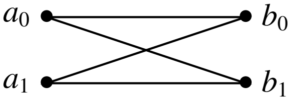

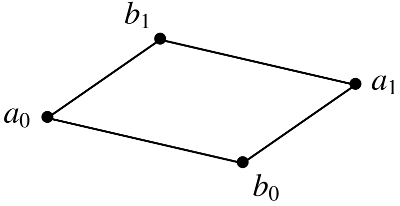

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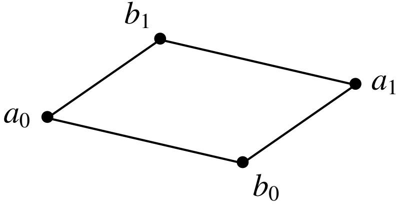

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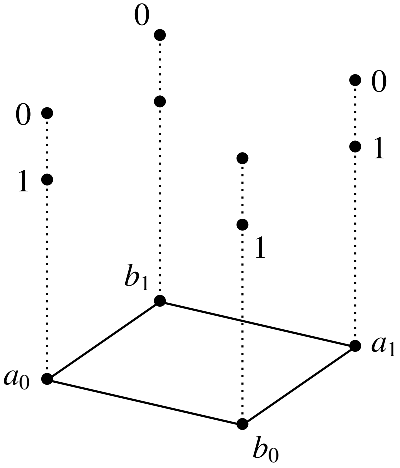

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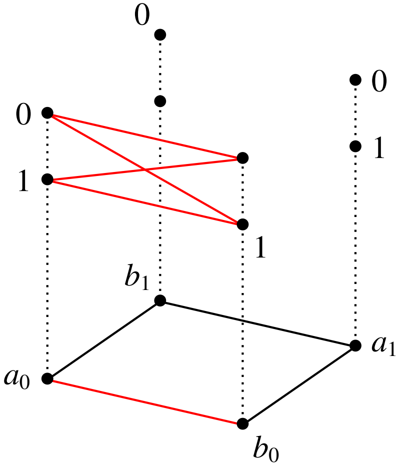

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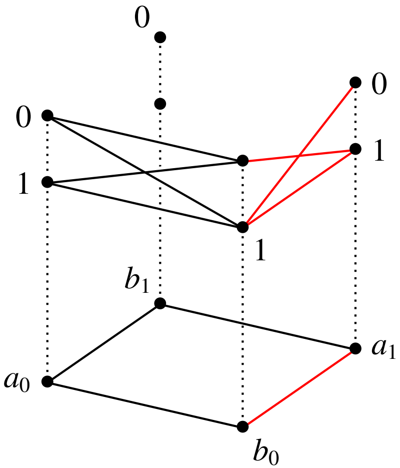

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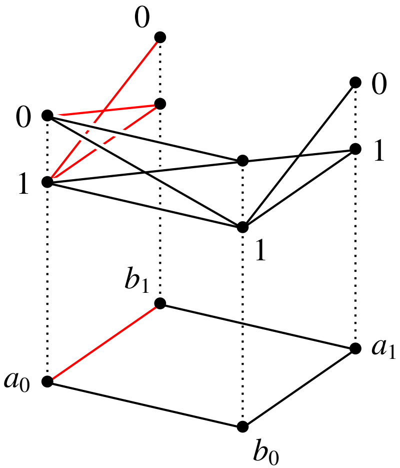

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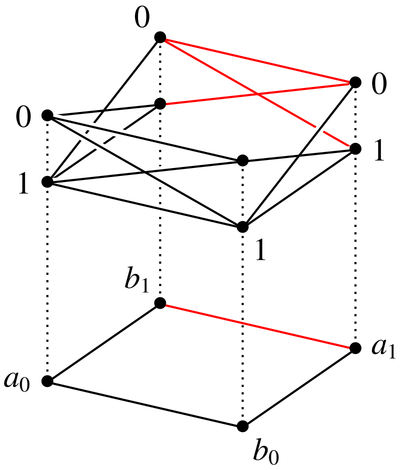

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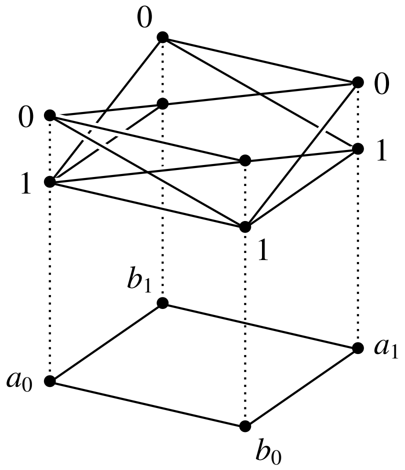

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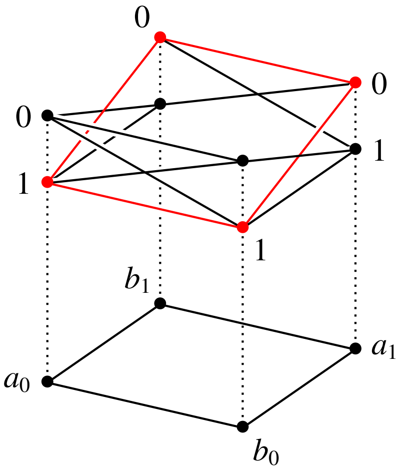

Global section: $\lambda_{(a_0,a_1,b_0,b_1)\mapsto(1,0,1,0)}$.

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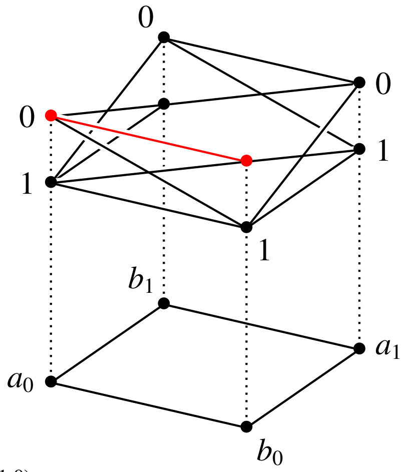

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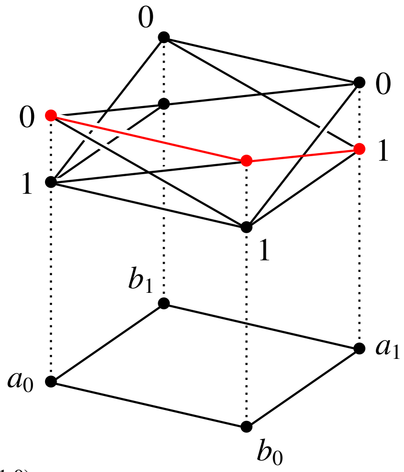

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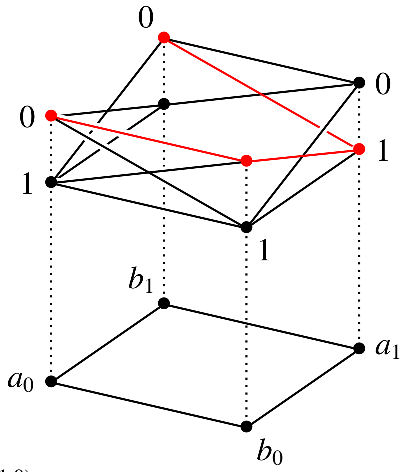

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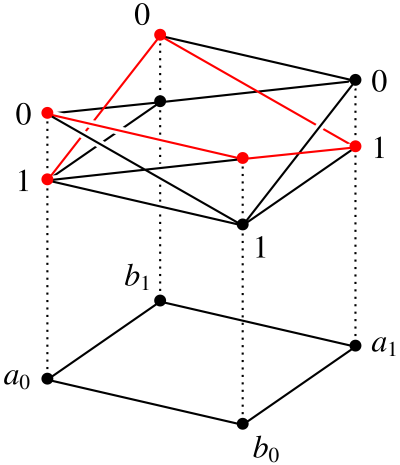

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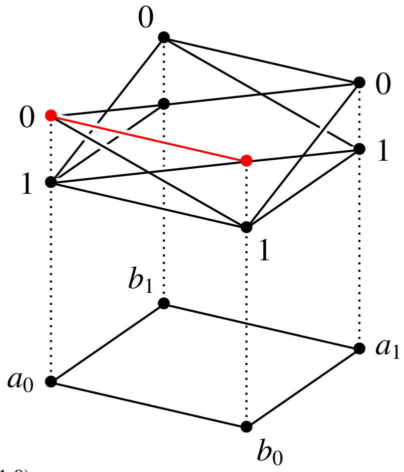

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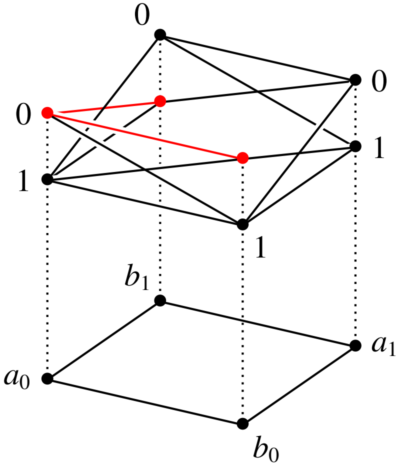

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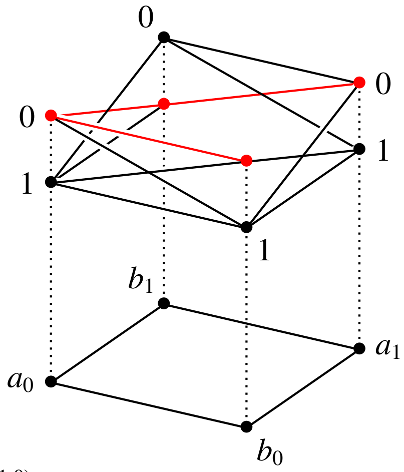

Global section: $\lambda_{(a_0,a_1,b_0,b_1)\mapsto(1,0,1,0)}$.

Hardy model:

	00	01	10	11
$a_0 b_0$	1	1	1	1
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$a_1 b_0$	0	1	1	1
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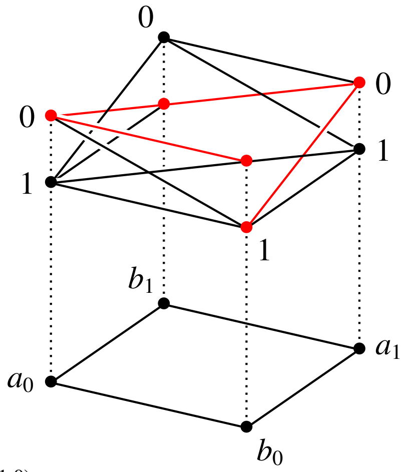

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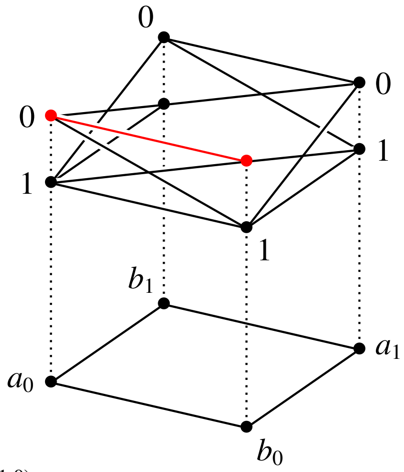

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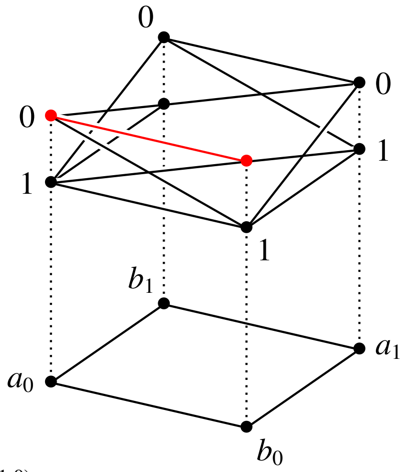

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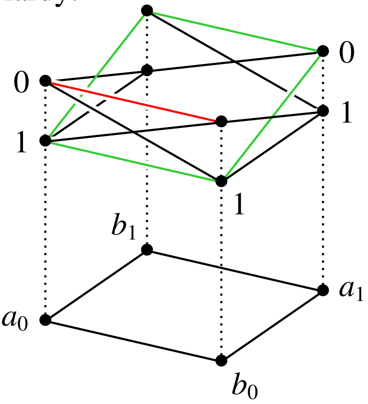

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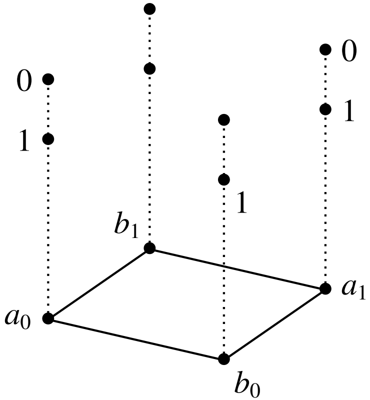

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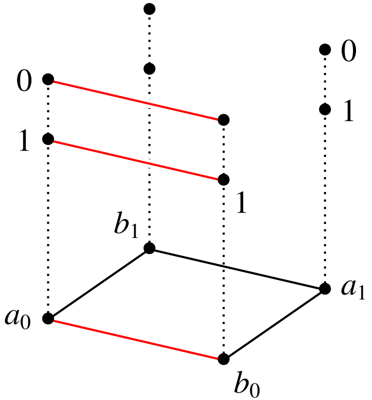

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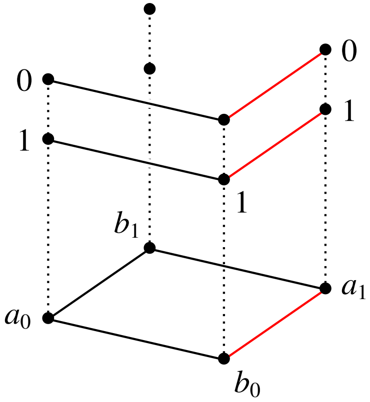

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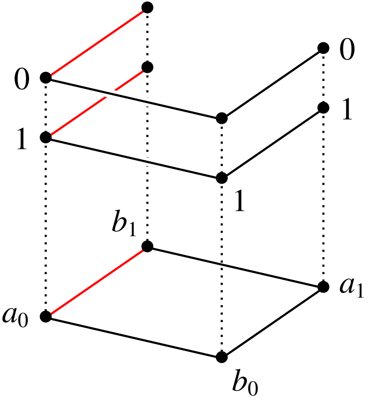

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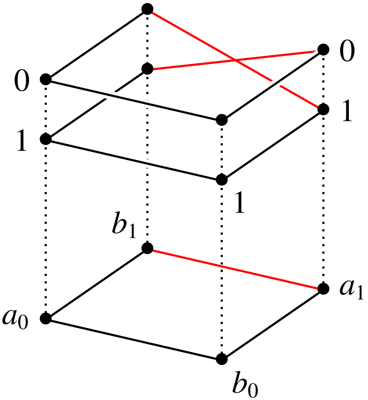

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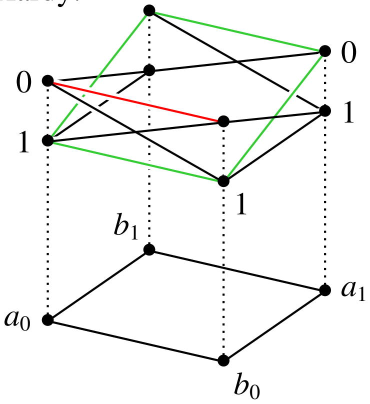

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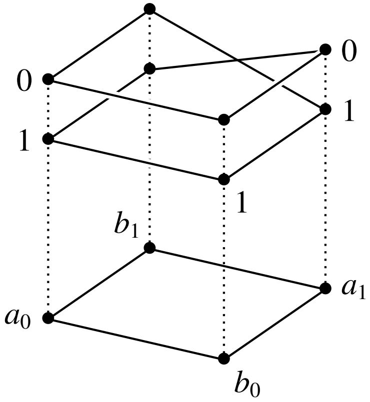

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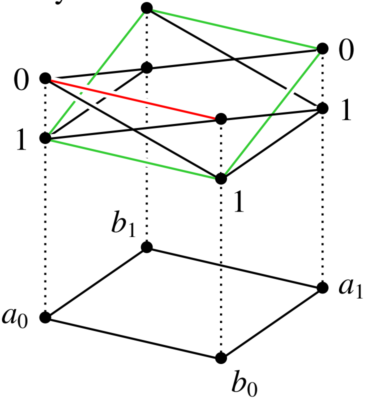

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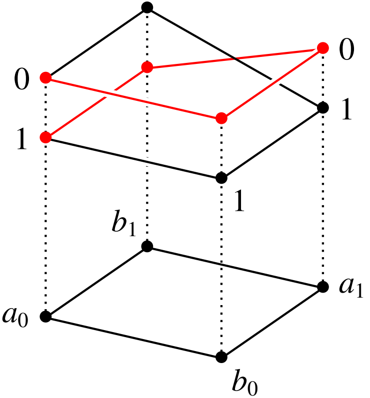

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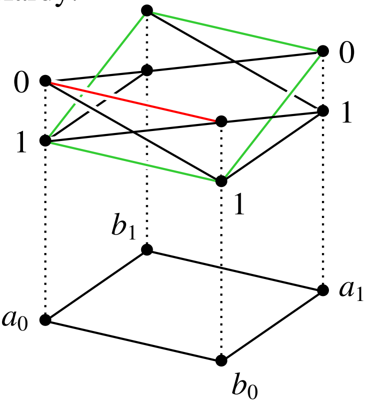

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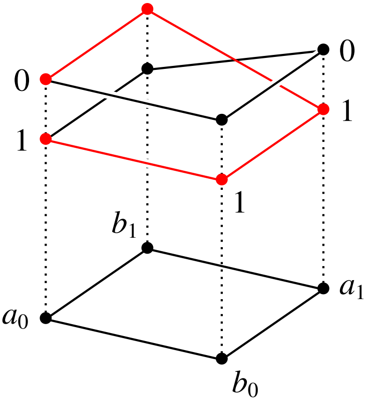

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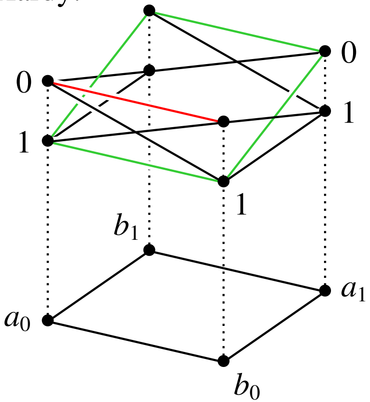

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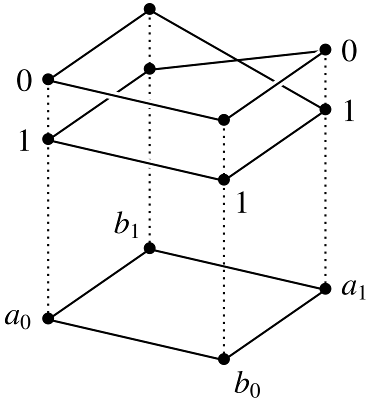

Logical contextuality: Not all sections extend to global.

Strong contextuality: No global section at all.

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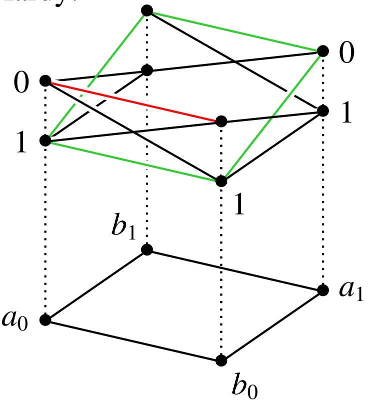

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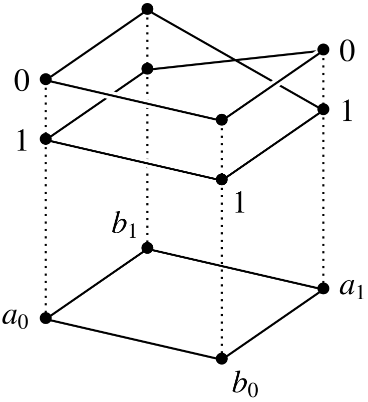

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Hieararchy of contextuality:

Probabilistic $\supsetneq$ **Logical** $\supsetneq$ **Strong contextuality**

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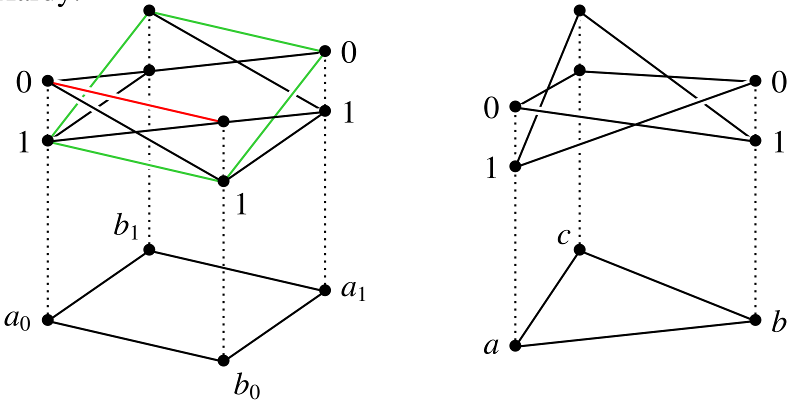

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Contextuality in Logical Paradoxes

Read bundles $\pi : \sum_{x \in X} F(x) \to X$ in logic terms:

$$x \in X \quad \text{are sentences,}$$
$$\mathtt{tt}, \mathtt{ff} \in F(x) \quad \text{are truth values.}$$

Contextuality in Logical Paradoxes

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$x \in X$ are sentences,

$\mathtt{tt}, \mathtt{ff} \in F(x)$ are truth values.

“West is true”

“North is false”

“South is true”

“East is true”

Contextuality in Logical Paradoxes

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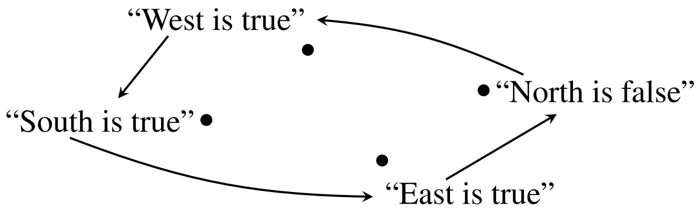

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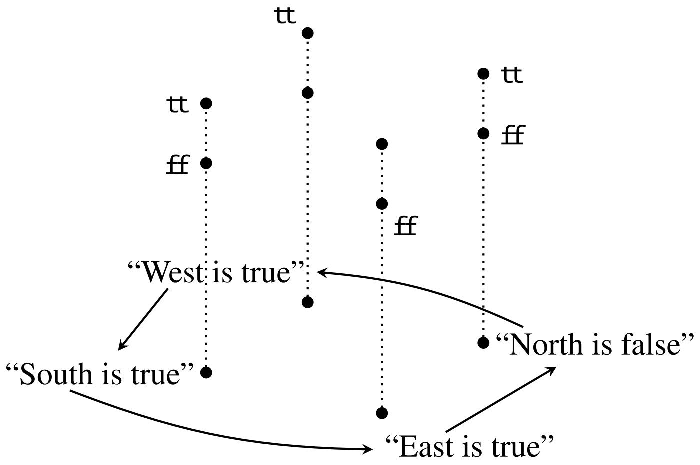

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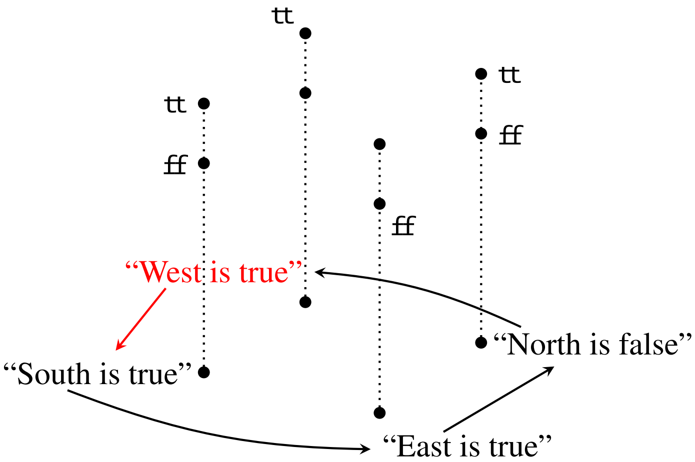

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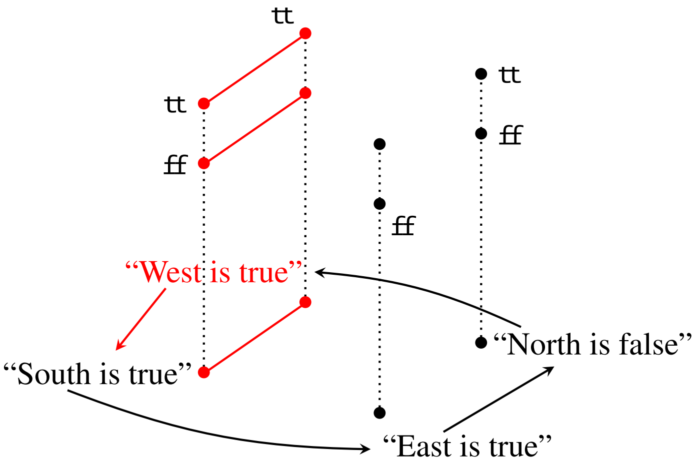

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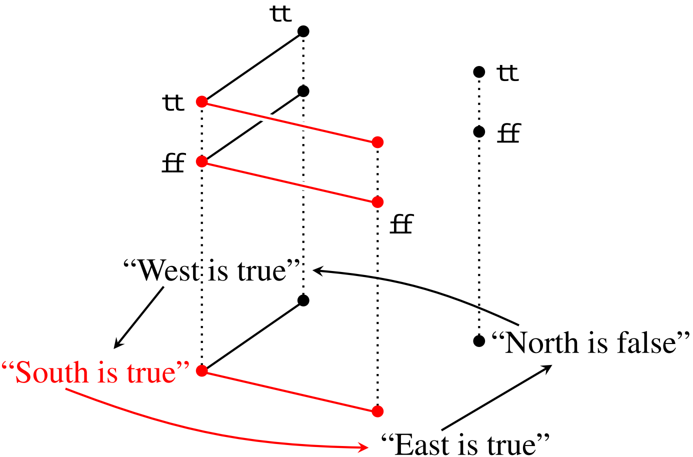

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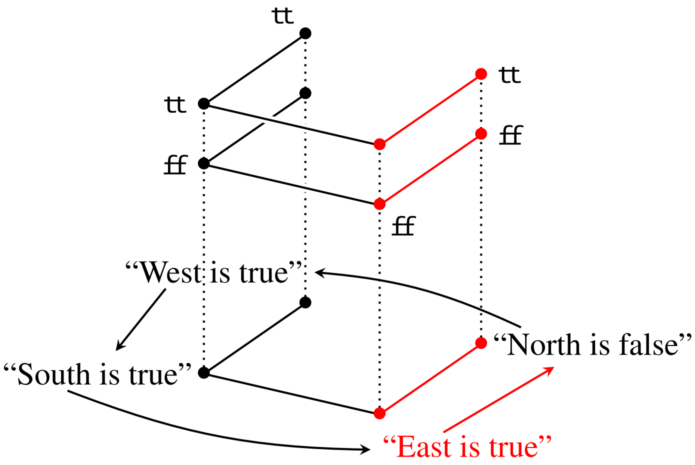

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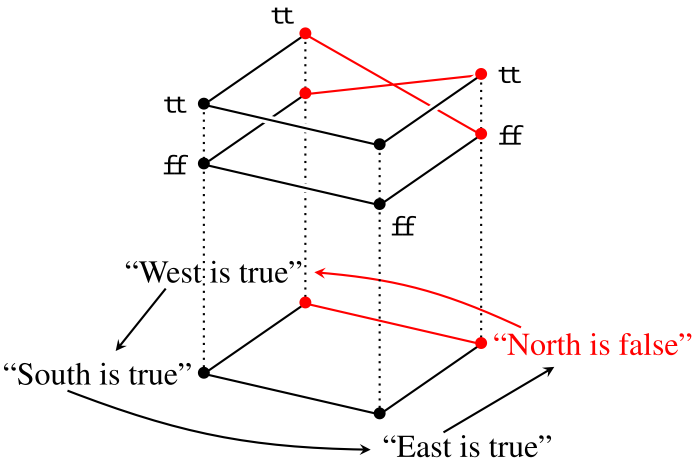

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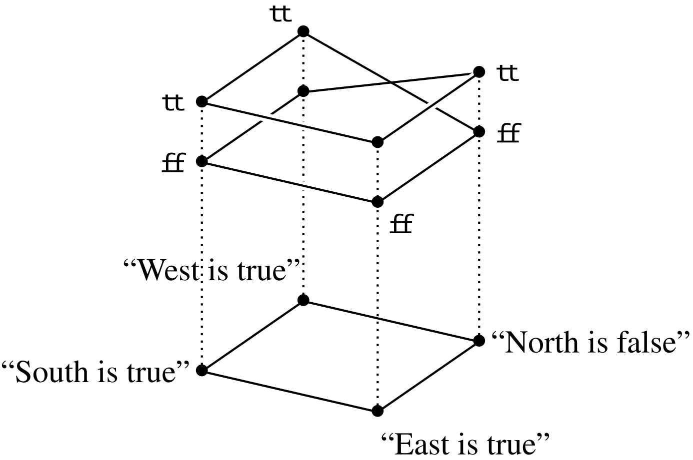

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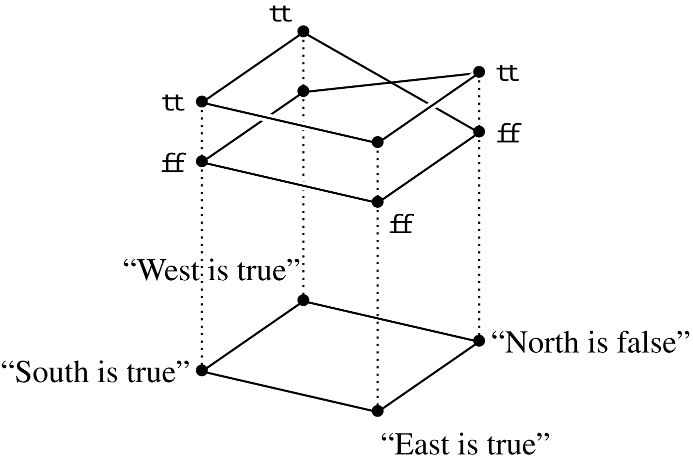
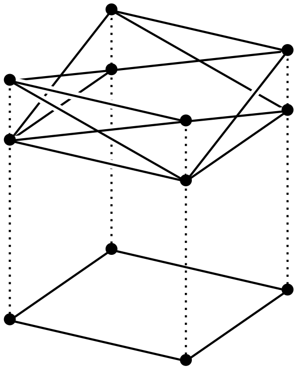


This type of logical paradoxes (incl. the Liar Paradox) have the same topology as "paradoxes" of (strong) contextuality.

How to Formally Define . . .

Bundles that correspond to no-signalling possibility tables.



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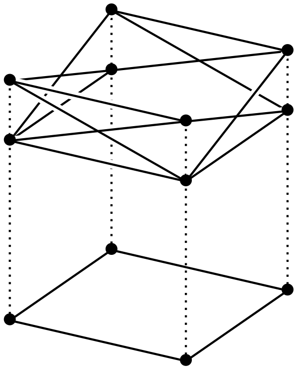

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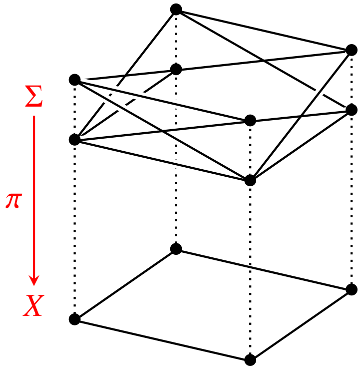

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$$\pi : \sum_{x \in X} F(x) \to X.$$

(With some axioms, e.g. no-signalling.)

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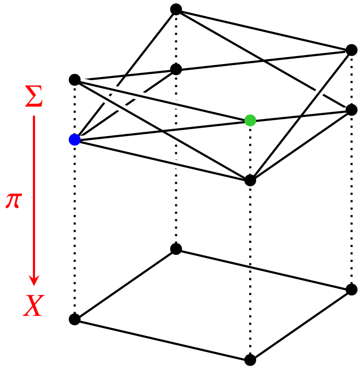

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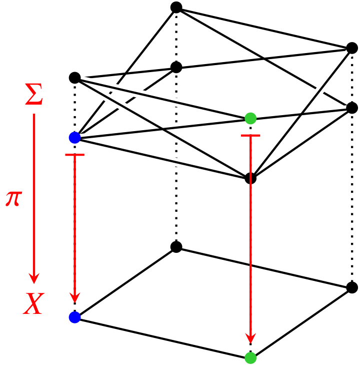

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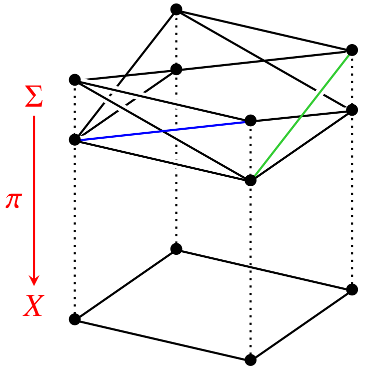

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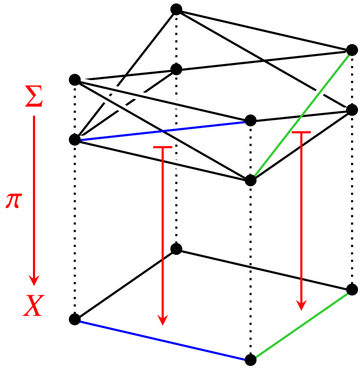

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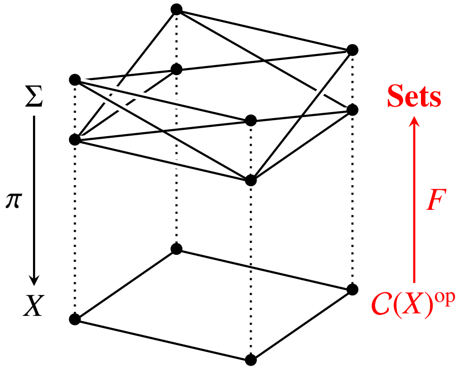

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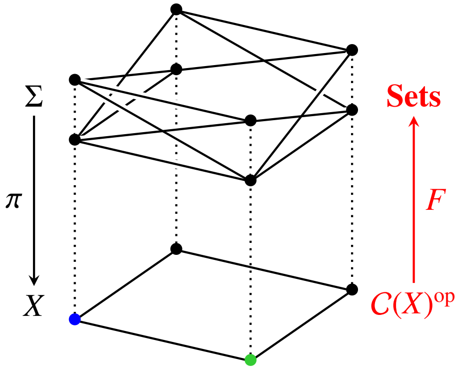

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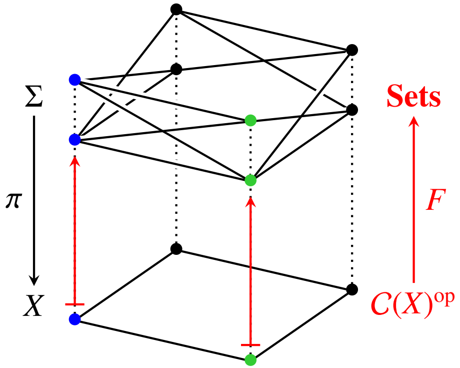

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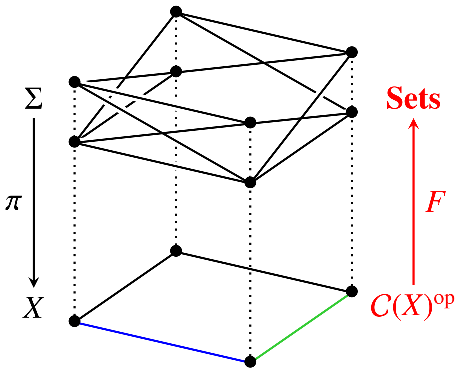

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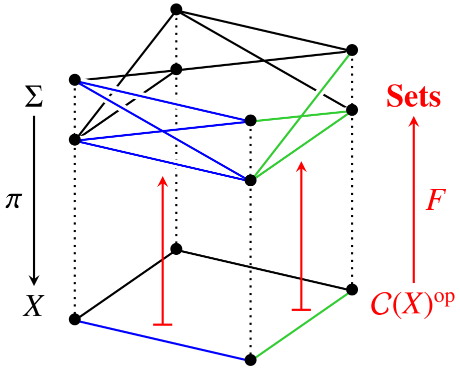

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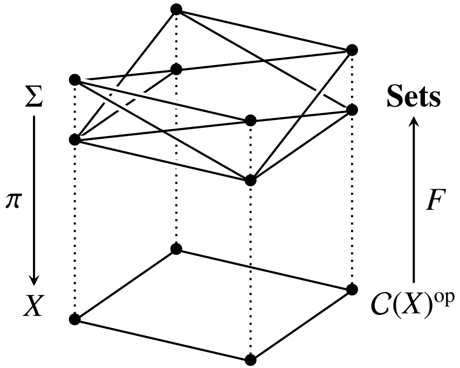

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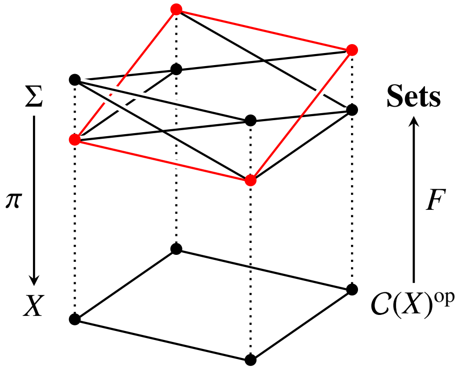

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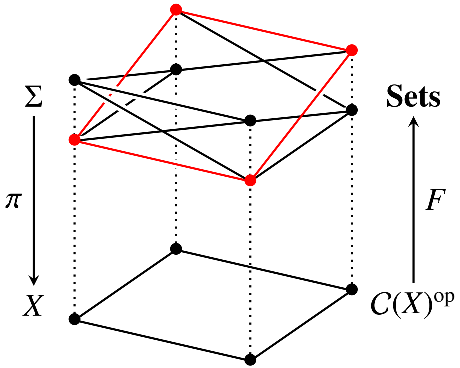

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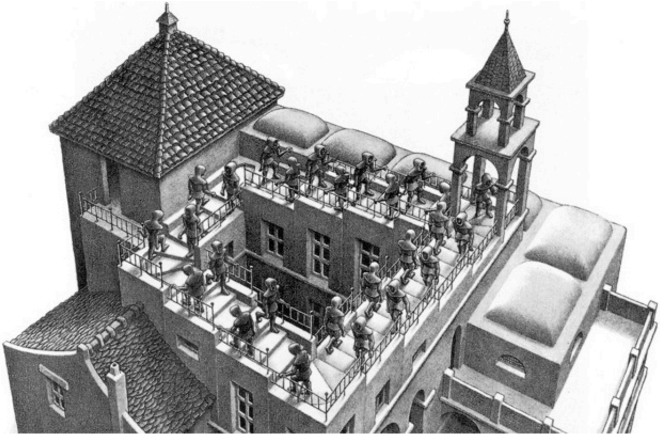
2 makes it possible to apply cohomology.

Cohomology of Contextuality

Local consistency, global inconsistency…



Penrose 1991, "On the Cohomology of Impossible Figures".

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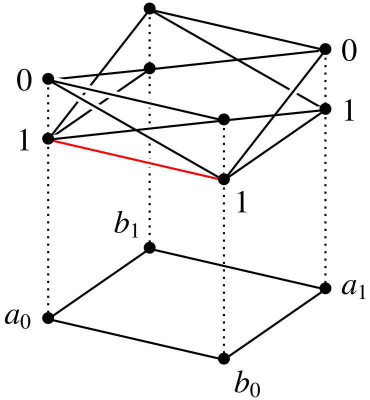
"Čech cohomology" gives a group homomorphism γ that assigns to each section s an "obstruction" γ_s s.th.



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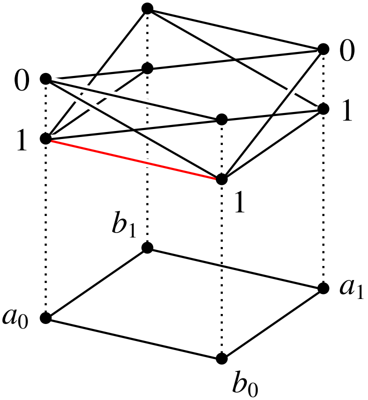

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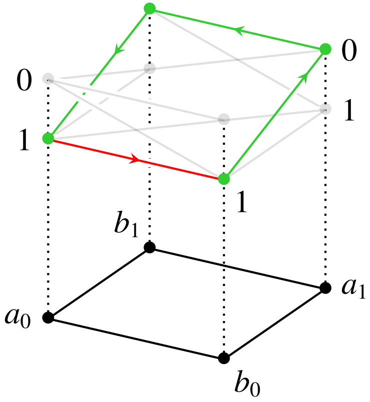

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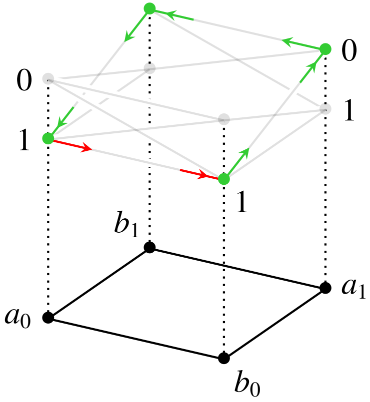

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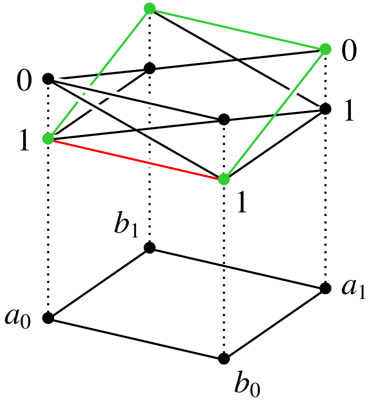

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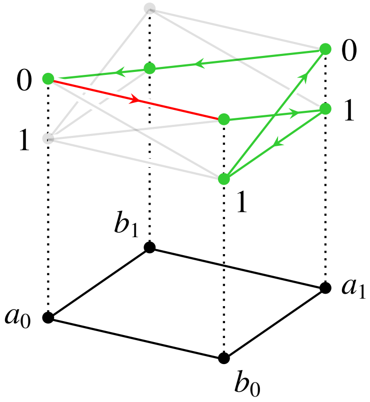

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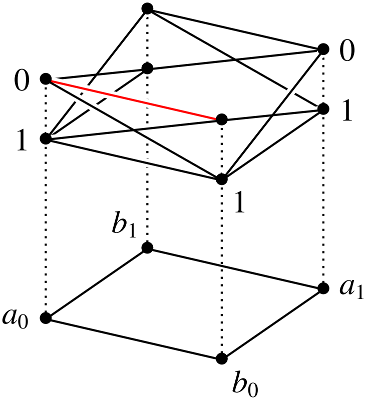

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- False positives,
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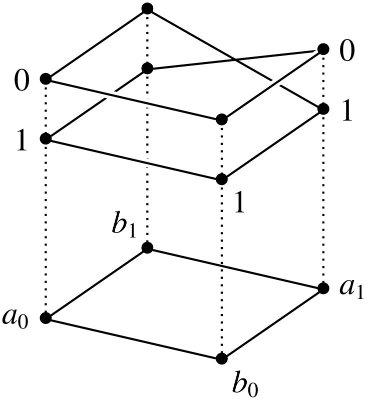
- Works for many cases;
  e.g. PR box:



“All vs Nothing” Argument

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Joint outcomes may / may not satisfy parity equations:

$(0,0) \quad \rightsquigarrow \quad x \oplus y = 0$

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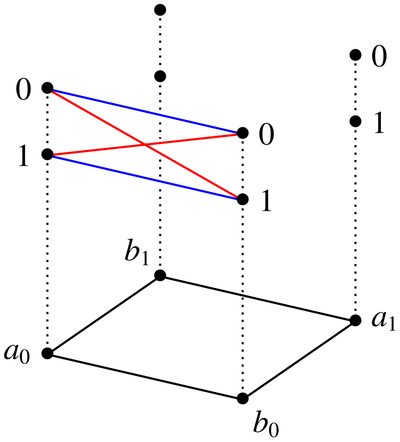

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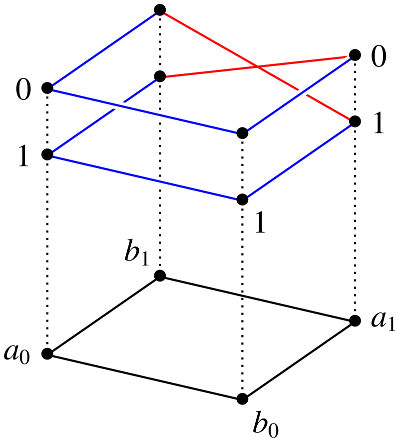

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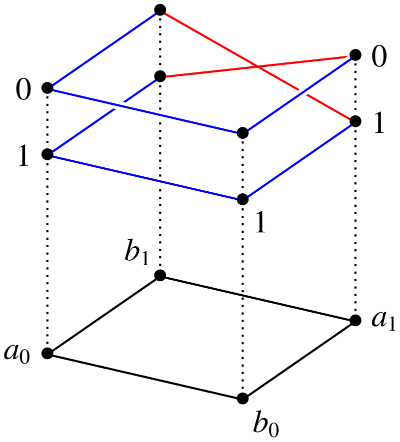

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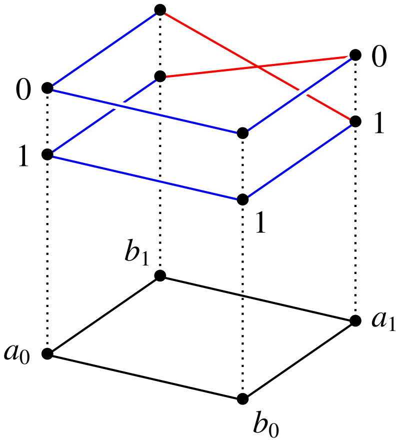

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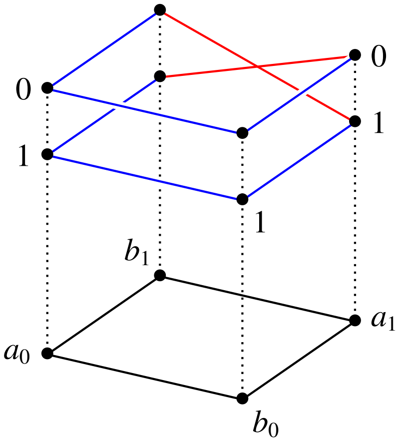

The equations are inconsistent,

i.e. no global assignment to a_0, a_1, b_0, b_1,

i.e. strongly contextual!

"All vs nothing" arguments in QM can be formulated the same way.

- GHZ state:

$$
\begin{aligned}
a_0 \oplus b_0 \oplus c_0 &= 0 \\
a_0 \oplus b_1 \oplus c_1 &= 1 \\
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$$\bigoplus \text{LHS's} = 0 \neq 1 = \bigoplus \text{RHS's}$$

- Kochen-Specker-type:

  18 variables, each occurs twice, so $\bigoplus$ LHS's = 0;
  9 equations, all of parity 1, so $\bigoplus$ RHS's = 1.

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- Equations are inconsistent if a subset of them is s.th.
  - coefficients k of each variable x add up to 0,
  - parities p do not.

"Strongly contextual by AvN argument" is explained by "strongly contextual by cohomology":

Theorem.

Let $\mathcal{M}$ be a no-signalling bundle model. Then

- $\mathcal{M}$ admits a generalized AvN argument in a ring R

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References

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