From the Kochen-Specker theorem to noncontextuality inequalities without assuming determinism

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(based on arXiv:1506.04150 [quant-ph] (2015),
joint work with Rob Spekkens.)

Rob says “Hi” to everyone!

Motivation

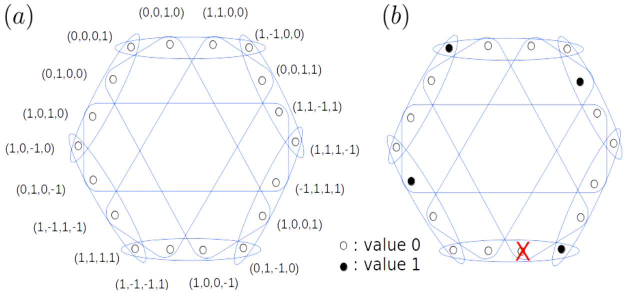
To turn the Kochen-Specker theorem, a no-go result precluding deterministic noncontextual models of quantum theory, into an experimentally testable noncontextuality inequality whose violation rules out a noncontextual model of nature (rather than the theory we currently believe best describes nature).[1] Crucially, determinism is not assumed.[2]

[1] Just as Bell's theorem allows a test of local causality independent of the validity of quantum theory, we seek a test of noncontextuality that is theory-independent.

[2] R.W. Spekkens, Contextuality for preparations, transformations, and unsharp measurements, Phys. Rev. A 71, 052108 (2005).

The Kochen-Specker theorem

Original proof of the KS theorem: 117 rays in 3d Hilbert space.[3] To illustrate our approach, we use the proof due to Cabello et al.[4], requiring 18 rays in 4d:



[3] S. Kochen and E. P. Specker, The Problem of Hidden Variables in Quantum Mechanics, J. Math. Mech. 17, 59 (1967).

[4] A. Cabello, J. Estebaranz, and G. Garcia-Alcaine, Bell-Kochen-Specker theorem: A proof with 18 vectors, Physics Letters A 212, 183 (1996).

Obstacles to a robust noncontextuality inequality

- KS theorem - *about* quantum theory, not general operational theories. Operationalization needed.
- Even after operationalization, need to relax the perfect predictability ideal.

Towards a noncontextuality inequality

1. Operationalize the KS theorem,
2. Define noncontextuality without outcome determinism,
3. Justify outcome determinism for perfectly predictable measurements:

$$\text{universal noncontextuality} \bigwedge \text{operational equivalences} \bigwedge \text{perfect correlation} \Rightarrow \text{contradiction} \tag{1}$$

4. Contend with the lack of perfect predictability in real experiments:

$$\text{universal noncontextuality} \bigwedge \text{operational equivalences} \Rightarrow \text{failure of perfect correlation} \tag{2}$$

Operational theory

$(\mathcal{P}, \mathcal{M}, p)$, where $p : (\mathcal{M}, \mathcal{P}) \to [0,1]$ is the probability $p(k|M, P)$ that $k \in \mathcal{K}_M$ occurs when $M \in \mathcal{M}$ is implemented following $P \in \mathcal{P}$. For each M:

$$\sum_{k \in \mathcal{K}_M} p(k|M, P) = 1 \quad \forall P \in \mathcal{P}. \tag{3}$$

$[k|M]$ denotes the event: outcome k occurs for measurement M.

Ontological model of an Operational theory

(Λ, μ, ξ), where each preparation $P \in \mathcal{P}$ is associated with a distribution $\mu(\lambda|P) \in [0,1]$ such that $\sum_{\lambda \in \Lambda} \mu(\lambda|P) = 1$ for all $P \in \mathcal{P}$, each $[k|M]$ with the probability $\xi(k|M,\lambda) \in [0,1]$ that $[k|M]$ occurs when the ontic state of the system is λ, and for each $M \in \mathcal{M}$:

$$\sum_{k \in \mathcal{K}_M} \xi(k|M,\lambda) = 1 \quad \forall \lambda \in \Lambda. \tag{4}$$

Assumption of outcome determinism: for any $[k|M]$, $\xi(k|M,\lambda) \in \{0,1\} \forall \lambda \in \Lambda$.

An ontological model of an operational theory must be empirically adequate, that is:

$$p(k|M,P) = \sum_{\lambda\in\Lambda} \xi(k|M,\lambda)\mu(\lambda|P) \tag{5}$$

for all $P \in \mathcal{P}, M \in \mathcal{M}$. This is how an operational theory and its ontological model fit together.

Operational equivalence of experimental procedures

- $[k|M]$ and $[k'|M']$ operationally equivalent ($[k|M] \simeq [k'|M']$) if no preparation procedure yields differing outcome probabilities for them, i.e.,

$$\forall P \in \mathcal{P} : p(k|M, P) = p(k'|M', P). \tag{6}$$

- P and P' operationally equivalent ($P \simeq P'$) if no measurement event $[k|M]$ yields differing outcome probabilities for them, i.e.,

$$p(k|M, P) = p(k|M, P') \quad \forall k \in \mathcal{K}_M, (M, \mathcal{K}_M) \in \mathcal{M}. \tag{7}$$

What is a ‘context’?

- Any distinction between two operationally equivalent experimental procedures.[5]
- Measurement contexts: (a) whether M_1 is jointly measured with M_2 (M_{12}) or with M_3 (M_{13}), where $M_1^{(2)} \simeq M_1^{(3)} \simeq M_1$, (b) different operationally equivalent ways of implementing a fair coin flip measurement.[6]
- Preparation contexts: (a) different convex decompositions: $\frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$, (b) different purifications: $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|_{AB} = \mathrm{Tr}_C|\phi\rangle\langle\phi|_{AC}$.

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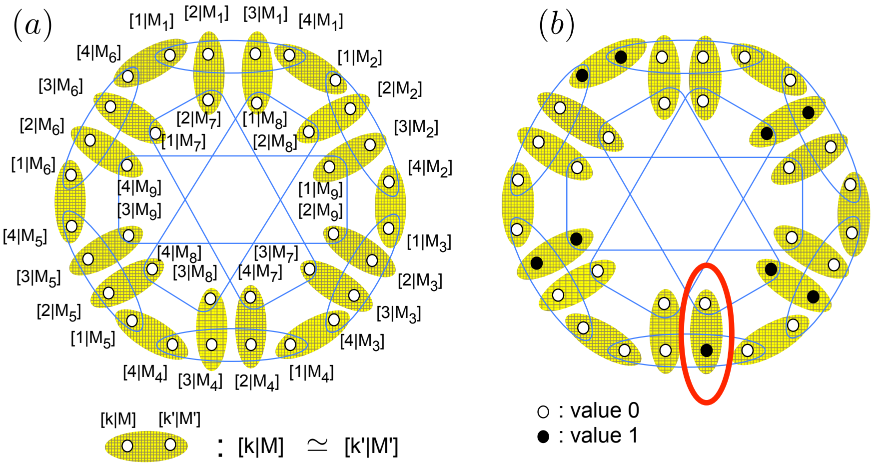
[5] A distinction that doesn't make a difference, operationally. ‘Contextuality’: this distinction sometimes *necessarily* makes a difference in any ontological model underlying the operational statistics.

[6] M. D. Mazurek, M. F. Pusey, R. Kunjwal, K. J. Resch, and R. W. Spekkens, An experimental test of noncontextuality without unwarranted idealizations, arXiv:1505.06244 [quant-ph] (2015).

Operationalizing KS-noncontextuality



An ontological model (Λ, μ, ξ) of an operational theory $(\mathcal{P}, \mathcal{M}, p)$ is KS-noncontextual if

1. operational equivalence of events implies equivalent representations in the model, i.e., $[k|M] \simeq [k'|M'] \Rightarrow \xi(k|M, \lambda) = \xi(k'|M', \lambda)$ for all $\lambda \in \Lambda$ (measurement noncontextuality), and
2. the model is outcome-deterministic, $\xi(k|M, \lambda) : \Lambda \to \{0, 1\}$.

Defining noncontextuality without outcome determinism

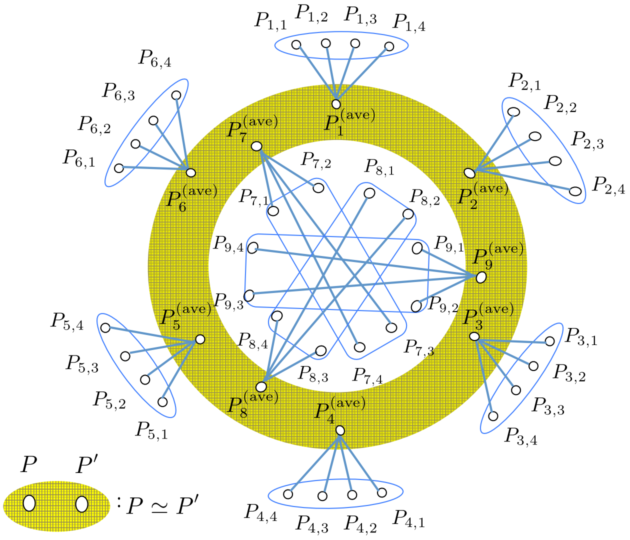
- Identity of indiscernables: context-independence at the operational level should imply context-independence at the ontological level.
- A Kochen-Specker contradiction cannot be derived from measurement noncontextuality alone.
- How do we fix this? Enter preparation noncontextuality.

Justifying outcome determinism for perfectly predictable measurements

- Preparation noncontextuality: $P \simeq P' \Rightarrow \mu(\lambda|P) = \mu(\lambda|P')$ for all $\lambda \in \Lambda$.
- Assumption of universal noncontextuality: preparations and measurements.
- PNC $\bigwedge$ QT $\Rightarrow$ ODSM.[7] We outline how this argument plays out for an operational theory in the present scenario.

[7] RWS, Phys. Rev. A 71, 052108 (2005). Abandoning outcome determinism in this case amounts to allowing preparation contextuality.

Preparation procedures and their operational equivalences



- Suppose $\forall i, \forall k : p(k|M_i, P_{i,k}) = 1$. (perfect correlation or perfect predictability).
- Since $P_i^{(\mathrm{ave})} \simeq P_{i'}^{(\mathrm{ave})}$ for all $i, i' \in \{1, 2, \ldots, 9\}$, and $\mu(\lambda|P_i^{(\mathrm{ave})}) = \frac{1}{4}\sum_{k=1}^{4} \mu(\lambda|P_{i,k})$, we have by preparation noncontextuality:

$$\frac{1}{4}\sum_{k=1}^{4} \mu(\lambda|P_{i,k}) \equiv \nu(\lambda) \quad \forall i \in \{1, 2, \ldots, 9\}. \tag{8}$$

- Empirical adequacy requires that

$$\forall i, \forall k : \sum_{\lambda} \xi(k|M_i, \lambda)\mu(\lambda|P_{i,k}) = 1. \tag{9}$$

This immediately implies that $\xi(k|M_i, \lambda) = 1$ for all λ in the support of $P_{i,k}$, i.e., $\lambda \in \{\Lambda|\mu(\lambda|P_{i,k}) > 0\}$. Since this is true for all i, k, and since every λ in the support of $\nu(\lambda)$ appears in the support of some $P_{i,k}$ (for each i), we must have $\forall i, \forall k : \xi(k|M_i, \lambda) \in \{0, 1\}$ for all $\lambda \in \{\Lambda|\nu(\lambda) > 0\}$.

In justifying outcome determinism, we have revised the operational content of the Kochen-Specker theorem:

$$\text{universal noncontextuality} \bigwedge \text{operational equivalences} \bigwedge \text{perfect correlation} \Rightarrow \text{contradiction} \tag{10}$$

This leads to a natural formulation of a noncontextuality inequality when perfect correlation fails.

Contending with the lack of perfect predictability in real experiments

$$\text{universal noncontextuality} \bigwedge \text{operational equivalences}$$
$$\Rightarrow \text{failure of perfect correlation} \tag{11}$$

$\therefore$ our noncontextuality inequality bounds

$$A \equiv \frac{1}{36} \sum_{i=1}^{9} \sum_{k=1}^{4} p(k|M_i, P_{i,k}). \tag{12}$$

Bounding the average predictability

$$
\begin{aligned}
A &= \frac{1}{36}\sum_{i=1}^{9}\sum_{k=1}^{4}\sum_{\lambda}\xi(k|M_i,\lambda)\mu(\lambda|P_{i,k}) \\
&\leq \frac{1}{9}\sum_{i=1}^{9}\sum_{\lambda}\zeta(M_i,\lambda)\frac{1}{4}\sum_{k=1}^{4}\mu(\lambda|P_{i,k}) \\
&\quad (\text{where } \zeta(M_i,\lambda) \equiv \max_{k'\in\mathcal{K}_M}\xi(k'|M,\lambda)) \\
&= \sum_{\lambda}\frac{1}{9}\left(\sum_{i=1}^{9}\zeta(M_i,\lambda)\right)\nu(\lambda) \\
&\leq \max_{\lambda}\frac{1}{9}\left(\sum_{i=1}^{9}\zeta(M_i,\lambda)\right) \\
&= \frac{5}{6} \qquad (13)
\end{aligned}
$$

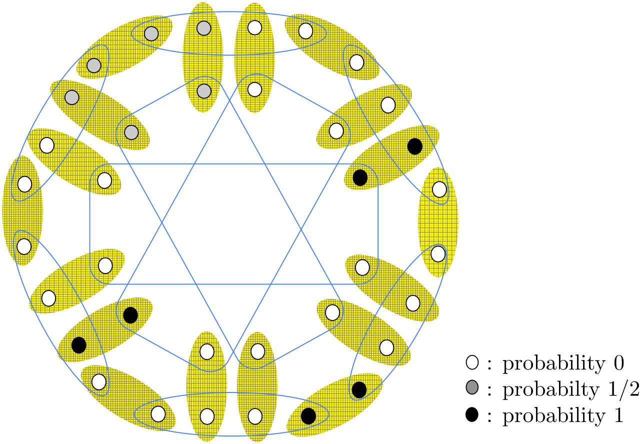


Figure : An extremal vertex of 146-vertex, 9-dimensional polytope. It makes at most 6 measurements have deterministic outcomes but the remaining 3 have a max-probability of $\frac{1}{2}$ each: $\frac{1}{9}(6.1 + 3.\frac{1}{2}) = \frac{5}{6}$. One can think of the 146 vertices as the space of ontic states Λ, since their convex hull characterizes all possible probabilistic models on the hypergraph.

Noise robustness: why trivial POVMs are not a problem

- Assuming the experiment is well-modelled by quantum theory: $p(k|M_i, P_{i,k}) = \mathrm{Tr}(E_{k|M_i}\rho_{i,k})$, where $E_{k|M_i} \geq 0, \sum_k E_{k|M_i} = I, \rho_{i,k} \geq 0$, and $\mathrm{Tr}\rho_{i,k} = 1$.
- In the ideal limit of (noiseless) projective measurements, we have $E_{k|M_i} = \Pi_{i,k}$ and $\rho_{i,k} = \Pi_{i,k}$, where $\Pi_{i,k}$ is a rank 1 projector, so that $\forall i, \forall k : p(k|M_i, P_{i,k}) = 1$ (perfect correlation is satisfied) and $A = 1$: operational equivalences $\bigwedge$ perfect correlation $\Rightarrow$ contextuality.
- Consider a simple depolarizing channel acting on the preparation: $\mathcal{D}_p(\cdot) = pI(\cdot)I + (1-p)\frac{I}{4}\mathrm{Tr}(\cdot)$. Equivalently, the adjoint of this channel acts on the measurement.

The deviation from the noiseless ideal is given by

$$\rho_{i,k} = \mathcal{D}_{p_1}(\Pi_{i,k}) = p_1 \Pi_{i,k} + (1 - p_1)\frac{I}{4},$$

$$E_{k|M_i} = \mathcal{D}^{\dagger}_{p_2}(\Pi_{i,k}) = p_2 \Pi_{i,k} + (1 - p_2)\frac{I}{4}.$$

The channel between the preparation and measurement introduces noise characterized by p_1 and p_2. It then follows that $p(k|M_i, P_{i,k}) = p_1 p_2 + (1 - p_1 p_2)\frac{1}{4}$ and therefore

$$A = \frac{1}{4} + \frac{3}{4} p_1 p_2.$$

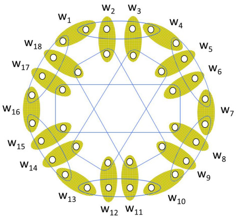
Clearly $A > \frac{5}{6}$ (contextuality!) if and only if $p_1 p_2 > \frac{7}{9}$. In the completely noisy case (trivial POVMs!) $p_1 p_2 = 0$ and $A = \frac{1}{4}$ and the noncontextuality inequality is satisfied. On the other hand, in the noiseless ideal limit (rank 1 projectors!) $p_1 p_2 = 1$ and $A = 1$, maximally violating the noncontextuality inequality.

Takeaway

- Operational KS: universal noncontextuality $\bigwedge$ operational equivalences $\bigwedge$ perfect predictability $\Rightarrow$ contradiction.
- This allows us to graduate to a *theory-independent* noncontextuality inequality from an uncolourability proof of the KS theorem.
- This noncontextuality inequality is NOT a traditional Kochen-Specker inequality. A Kochen-Specker inequality for a KS-uncolourable hypergraph is anyway an oxymoron.
- It tolerates noisy preparations and measurements but if an experiment doesn't suppress noise sufficiently, the inequality cannot be violated. This simple criterion of operational meaningfulness in the presence of noise is not satisfied by KS inequalities (in cases where they are well-defined).
- No artificial restriction to "sharp" measurements, however defined, is needed.

That's a wrap!

Bonus slides: Critiquing a previous proposal[8]



$$
\begin{aligned}
\alpha' \equiv\ & \langle w_1 \oplus w_2 \oplus w_3 \oplus w_4 \rangle + \langle w_4 \oplus w_5 \oplus w_6 \oplus w_7 \rangle \\
+\ & \langle w_7 \oplus w_8 \oplus w_9 \oplus w_{10} \rangle + \langle w_{10} \oplus w_{11} \oplus w_{12} \oplus w_{13} \rangle \\
+\ & \langle w_{13} \oplus w_{14} \oplus w_{15} \oplus w_{16} \rangle + \langle w_{16} \oplus w_{17} \oplus w_{18} \oplus w_1 \rangle \\
+\ & \langle w_{18} \oplus w_2 \oplus w_9 \oplus w_{11} \rangle + \langle w_3 \oplus w_5 \oplus w_{12} \oplus w_{14} \rangle \\
+\ & \langle w_6 \oplus w_8 \oplus w_{15} \oplus w_{17} \rangle \\
\leq\ & 8. \qquad (14)
\end{aligned}
$$

[8] A. Cabello, Experimentally testable state independent quantum contextuality, Phys. Rev. Lett. 101, 210401 (2008)

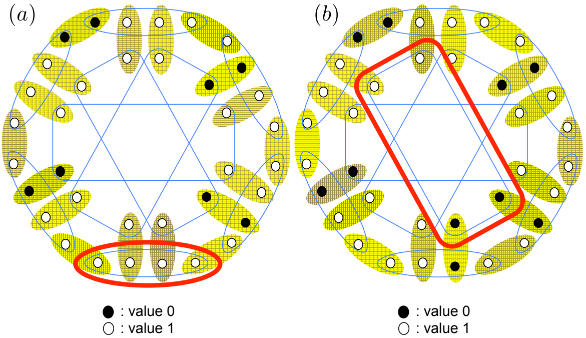


Figure : Unphysical assignments

- Obtained by considering the 2^{18} assignments to the vector $(w_1, w_2, \ldots, w_{18}) \in \{0,1\}^{18}$ and noting that none of these assignments beats the upper bound.
- BUT: $w_i \in \{0,1\}$, and the physical assignments to w_i in an edge are 1000, 0100, 0010, and 0001. KS theorem already precludes such assignments.

- Clearly, the 2^{18} deterministic assignments considered in deriving this inequality are not valid probabilistic assignments (hence, unphysical).
- A violation, $\alpha' > 8$, is therefore *necessary* for any valid probablistic assignment. It says nothing about contextuality, quantum or otherwise: $\alpha' \leq 7$ is ruled out by logic alone, no experiment is needed.
- For possible responses to this criticism, and their inadequacy, read the paper!

That's a bubble wrap!