Some Nearly Quantum Theories

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Outline

- I Euclidean Jordan algebras
- II Composites of EJAs
- III Categories of embedded EJAs

I. Euclidean Jordan Algebras

Euclidean Jordan Algebras

A euclidean Jordan algebra (EJA) is a finite-dimensional real inner product space A with commutative bilinear product $a, b \mapsto a \cdot b$, having identity element u, and satisfying, $\forall a, b, c \in A$,

- $a \bullet (a^2 \bullet b) = a^2 \bullet (a \bullet b)$;
- $\langle a \bullet b, c \rangle = \langle a, b \bullet c \rangle$

Examples (and notation):

$R_n := M_n(\mathbb{R})_{\text{sa}}, \;\; C_n := M_n(\mathbb{C})_{\text{sa}}, \;\; Q_n := M_n(\mathbb{H})_{\text{sa}}, \;\; E_3 := M_3(\mathbb{O})_{\text{sa}}$

all with $a \bullet b = \frac{1}{2}(ab + ba)$ and $\langle a, b \rangle = \text{Tr}(ab)$.

Euclidean Jordan Algebras

Theorem [Jordan, von Neumann, Wigner, 1934] All EJAs are direct sums of the following *simple* types:

- R_n, C_n, Q_n, E_3 as above;
- *Spin Factors:* V_n = the euclidean space $\mathbb{R} \times \mathbb{R}^n$, with Jordan product
$$(t, \mathbf{x}) \bullet (s, \mathbf{y}) = (ts + \langle \mathbf{x}, \mathbf{y} \rangle, t\mathbf{y} + s\mathbf{x}).$$

An EJA A is special iff it's a Jordan subalgebra of $M_n(\mathbb{C})_{sa}$ for some n. A is special iff it has no E_3 summand.

EJAs as Probabilistic Models

An EJA A is also an order-unit space with $A_+ = \{a^2 | a \in A\}$ and order-unit u. Thus, A can be viewed as a probabilistic model:

- Effects are elements $a \in A_+$ with $a \leq u$
- States are positive linear functionals $\alpha : A \to \mathbb{R}$ with $\alpha(u) = 1$; $\alpha(a) =$ probability of effect a in state α.
- Observables are sets $\{a_1, ..., a_n\}$ of effects summing to u.

Can also model continuous dynamics via one-parameter subgroups of $G(A) =$ identity conmponent of group of order-automorphisms.

$R_n, C_n, Q_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$ QM systems; $V_n =$ generalized bit with n-dimensional "Bloch sphere". In particular:

$$V_2 = R_2 \text{ (rebit)}, \ V_3 = C_2 \text{ (qubit)}, \ V_5 = Q_2 \text{ (quabit)}.$$

EJAs as Probabilistic Models

The order-unit space structure determines the Jordan structure:

Theorem [Koecher 1958; Vinberg 1961] *An order-unit space* (A, u) *arises from an EJA as above iff the cone* A_+ *is*

- homogeneous *(order-isomorphisms act transitively on interior of* A_+*);*
- self-dual *(*$\exists$ *inner product such that* $a \in A_+$ *iff* $\langle a, b \rangle \geq 0$ *for all* $b \in A_+$*).*

Homogeneity and self-duality can be motivated in various ways (e.g., AW 2012, Barnum-Mueller-Ududec 14). So EJAs form a *reasonably natural* class of probabilistic models!

II. Composites of EJAs

Composites of EJAs

In general, there's no reasonable way to make $A \otimes B$ into an EJA.

A *composite* of EJAs A and B: an EJA AB, plus bilinear mapping $A \times B \to AB$, $(a, b) \mapsto a \odot b$, such that

(a) $u_A \odot u_B = u_{AB}$;

(b) $\langle a \odot b | x \odot y \rangle = \langle a | x \rangle \langle b | y \rangle$

(c) $\phi \in G(A), \psi \in G(B) \Rightarrow \exists\ \phi \odot \psi \in G(AB)$ with

$$(\psi \odot \phi)(a \odot b) = \psi(a) \odot \phi(b).$$

(d) $\{a \odot b | (a, b) \in A \times B\}$ generates AB as a Jordan algebra.

Theorem 1: For A, B nontrivial, AB exists $\Rightarrow$ A, B, AB all special!

Embedded JC algebras

An *embedded JC algebra (EJC)* is a pair $(A, \mathbf{M}_A)$:

- $\mathbf{M}_A$ a (unital) $*$-subalgebra of $M_n(\mathbb{C})$ for some n;
- A a (unital) Jordan sub-algebra of $(\mathbf{M}_A)_{\text{sa}}$.

EJCs have a canonical product:

$$(A, \mathbf{M}_A) \odot (B, \mathbf{M}_B) := (A \odot B, \mathbf{M}_A \otimes \mathbf{M}_B),$$

$$A \odot B = J(A \otimes B) \leq (\mathbf{M}_A \otimes \mathbf{M}_B)_{\text{sa}}$$

where $J(X)$ = Jordan subalgebra of $\mathbf{M}_{\text{sa}}$ generated by $X \subseteq \mathbf{M}_{\text{sa}}$.

Notation: From now on, overload A for $(A, \mathbf{M}_A)$.

Theorem 2: *$A \odot B$ is a composite in our sense.*

Theorem 3: *$A, B \mapsto A \odot B$ is naturally associative.*

Standard and Universal Embeddings

Special EJAs have both standard and universal embeddedings.

Standard embeddings for simple (special) EJAs:

- $R_n, C_n \leq M_n(\mathbb{C})_{\mathrm{sa}}$;
- $\mathbb{H} \leq M_2(\mathbb{C})$, so $Q_n \leq M_{2n}(\mathbb{C})_{\mathrm{sa}}$;
- $V_n \leq M_{2^k}(\mathbb{C})_{\mathrm{sa}}$ for $n = 2k, 2k+1$
  (Details: MG's quantum lunch talk on the 24th).

Standard and Universal Embeddings

Universal embedding $A \mapsto C^*(A)_{sa}$: Jordan homomorphisms $A \to \mathbf{M}_{sa}$, $\mathbf{M}$ a complex $*$-algebra, factor uniquely through $*$-homomorphisms $C^*(A) \to \mathbf{M}$.

Universal and standard embeddings agree *except:*

- $C^*(C_n) = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$
- $C^*(Q_2) = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
- $C^*(V_n) = M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$ for $n = 2k + 1$

Notation: $A \widetilde{\otimes} B = A \odot B$ for the universal embedding (Studied by Hanche-Olsen, 1983).

Standard and Universal Tensor Products

For $A, B \in \{R_n, C_n, Q_n | n \geq 2\}$, $A \widetilde{\otimes} B$ can be computed explicitly (Hanche-Olsen 83):

$A \widetilde{\otimes} B$	R_k	C_k	Q_k
R_n	R_{nk}	C_{nk}	Q_{nk}
C_n	C_{nk}	$C_{nk} \oplus C_{nk}$	C_{nk}
$Q_{n>2}$	Q_{nk}	C_{nk}	R_{nk}

The case of two quaternionic bits or quabits is a bit special:

$$Q_2 \widetilde{\otimes} Q_2 = R_{16} \oplus R_{16} \oplus R_{16} \oplus R_{16}.$$

Theorem 4: *A, B simple $\Rightarrow$ AB a direct summand of $A \widetilde{\otimes} B$.*

So the only possible composites are standard ones, except for $C_n C_k$ (two candidates) and $Q_2 Q_2$ (four candidates).

III. Categories of EJCs

CJP Mappings

Let $\mathcal{C}$ be any class of EJCs closed under $\odot$. For $A, B \in \mathcal{C}$, a CP mapping $\phi : \mathbf{M}_A \to \mathbf{M}_B$ is *completely Jordan preserving (CJP)* (rel. $\mathcal{C}$) iff, for all $C \in \mathcal{C}$, $\phi \otimes \mathrm{id}_{\mathbf{M}_C}(A \odot C) \subseteq B \odot C$.

Theorem 5: *With relatively CJP mappings as morphisms, $(\mathcal{C}, \odot)$ is a SMC.*

If $\mathcal{C}$ contains any universally embedded V_n with $n \neq 2, 3$, then $\mathcal{C}(A, I) = \{0\}$. Two nicer examples:

- **RSE**: Reversible, Standardly Embedded EJCs (no non-quantum V_n);
- **URUE**: Universally Reversible, Universally Embedded EJCs (no V_n, $n > 3$ — so, no Q_2!)

Compact structure

The category of finite-dimensional matrix $*$-algebras and CP maps has a natural compact structure: dual of $\mathbf{M}$ is $\overline{\mathbf{M}}$ (conjugate algebra); unit and counit given by

$$f_{\mathbf{M}} = \sum_i e_i \otimes \overline{e}_i \text{ and } \eta_{\mathbf{M}}(a \otimes \overline{b}) = \mathrm{Tr}(ab^*),$$

where $\{e_i\}$ is any Tr-orthonormal basis for $\mathbf{M}$.

Theorem 6 *Both* **RSE** *and* **URUE** *inherit this compact structure.*

Conclusion and Speculation

- **RSE** unifies real, complex and quaternionic QM. **URUE** *almost* does so, except that it omits Q_2 and gives the wrong composite for two complex QM systems.
- Real and quaternionic QM = a compact closed subcategory of **RSE**. Pure-state version: Baez (2011). Guess: Former = CPM(latter).
- A striking feature of both **RSE** and **URUE**:

$$\text{anything} \otimes \text{complex} = \text{complex}.$$

Maybe a universe initially containing all three types of quantum systems would evolve into one in which complex systems predominate?

References

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