Axiomatizing complete positivity

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Introduction

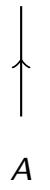
- Model quantum mechanics by **FdHilb** (Dagger Compact Symmetric Monoidal Category)
- Can model mixed states in two ways
  - CPM[**FdHilb**]: Same objects, morphisms are completely positive maps.
  - CP*[**FdHilb**]: Category of C*-algebras and completly positive maps.
- Generalises to functors $\mathrm{CPM}, \mathrm{CP}^* : \mathbf{DCSMC} \to \mathbf{DCSMC}$.
- Canonical inclusions $\mathbf{C} \hookrightarrow \mathrm{CPM}[\mathbf{C}] \hookrightarrow \mathrm{CP}^*[\mathbf{C}]$
- Question: When is $\mathbf{D}$ equal to some $\mathrm{CPM}[\mathbf{C}]$ or $\mathrm{CP}^*[\mathbf{C}]$?

Categories

Diagram

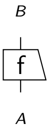
Example: **FdHilb**

Object:



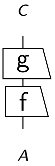
Finite-dim Hilbert space A

Morphism:



Linear map $f : A \to B$

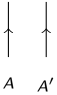
Composition:



Composition of linear maps

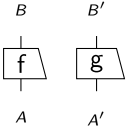
Symmetric Monoidal Categories

Monoidal product:



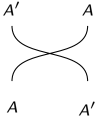
Tensor product $A \otimes A'$

On morphisms:



$f \otimes g$

Swap:



Canonical map $A \otimes A' \to A' \otimes A$

Compact Monoidal Categories

Dual object: 

Dual vector space A^*

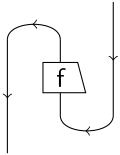
Cup: 

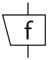
$\mathbb{C} \to End(A) \cong A \otimes A^*$

$z \mapsto zI$

Cap: 

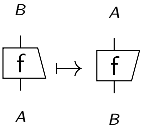
Trace: $A^* \otimes A \cong End(A) \to \mathbb{C}$

Can define transpose: 

denoted as 

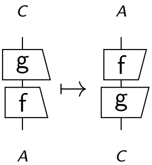
Dagger Categories

Dagger:



$f^\dagger$ s.t. $\langle fv, w\rangle = \langle v, f^\dagger w\rangle$

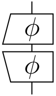
Gives a contravariant functor:

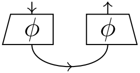


Hence have all four orientations:

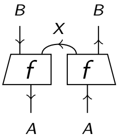
Original Transpose Adjoint Conjugate

Density Matrices

In QM, mixed states are given by positive matrices: 

These correspond to states of $A^* \otimes A$: 

Generalising this, the Choi-Jamiolkowski isomorphism says that the completely positive maps are precisely the ones of the form:

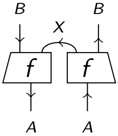


The CPM construction

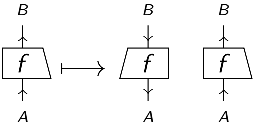
(P. Selinger **Dagger Compact Closed Categories and Completely Positive Maps**)

CPM[**D**] has objects $ob(\mathbf{D})$

Morphisms $A \to B$ are



D embeds into $CPM[\mathbf{D}]$ by

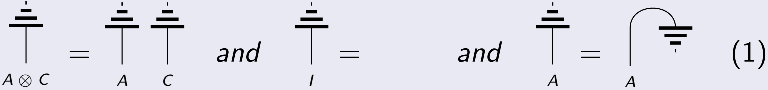


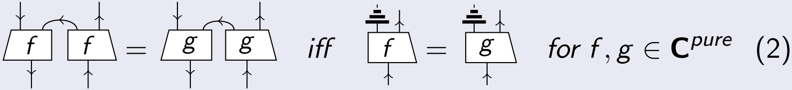
Question: Given **C**, when is $\mathbf{C} \sim CPM[\mathbf{D}]$ for some **D**?

Answer: B. Coecke **Axiomatic Description of Mixed States From Selinger's CPM-construction**

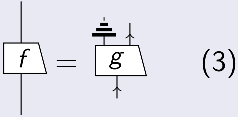
Theorem

Suppose $\mathbf{C}$ *has a subcategory* $\mathbf{C}^{pure}$ *(with the same objects) and for each object a map* ⏚ *satisfying*





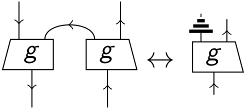
For all $f \in \mathbf{C}$ *we have* $g \in \mathbf{C}^{pure}$ *such that*



Then $\mathbf{C} \sim CPM[\mathbf{C}^{pure}]$

Proof Sketch

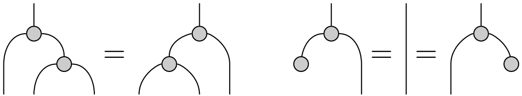
Consider functors between **C** and CPM[$\mathbf{C}^{\text{pure}}$] given by

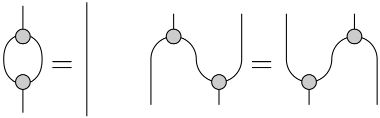


("Folding in half")
The axioms are precisely what is needed to prove that these are well defined DCSM-functors. They are clearly inverse to each other.

C*-algebras

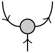
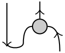
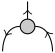
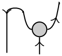
A *special dagger Frobenius structure* is an object A equipped with morphisms : $A \otimes A \to A$ and : $I \to A$ satisfying:



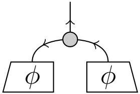


The special dagger Frobenius structures in **FdHilb** correspond precisely to the finite dimensional C*-algebras. (B. Coecke et al. **Categories of Quantum and Classical Channels**)

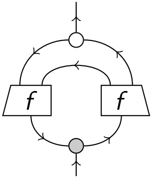
Completely positive maps

(Shorthand: Write  for , and  for .)

A state is positive when it is of the form:

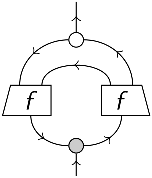


Generalising this, a morphism is completely positive when it is of the form:



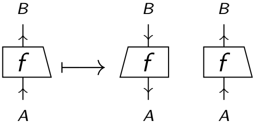
The CP* construction

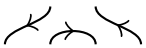
CP*[**D**] has as objects the special dagger Frobenius algebras $(A, \;, \;)$ in **D**
Morphisms $(A, \;, \;) \to (B, \;, \;)$ are



D embeds into CP*[**D**] by

$$A \longmapsto (A \otimes A^*, \;, \;)$$



Problem: We can't talk about  in CP*[**D**] because it isn't itself a completely positive map (from ($A \otimes A$, , ) to (A, , ))
Idea:  is completely positive when considered as going from ($A \otimes A^*$, , ) to (A, , ).

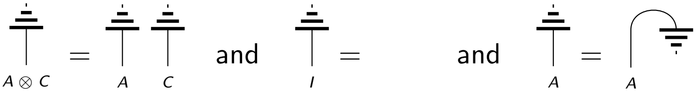
Answer:

Theorem

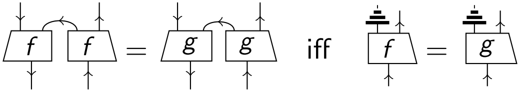
Suppose **C** *has a subcategory* $\mathbf{C}^{pure}$ *and*

- *for each object* $A \in \mathbf{C}^{pure}$ *we have specified a morphism*  $: A \to I$,
- *for each special dagger Frobenius algebra* $\mathcal{A} = (A,$ , $)$ *in* $\mathbf{C}^{pure}$ *we have specified an object* $F_{\mathcal{A}} \in \mathbf{C}$ *and a map*  $: A \to F_{\mathcal{A}}$.

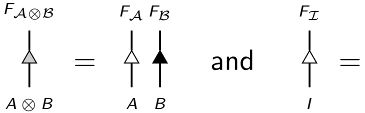
If the following axioms are satisfied then $\mathbf{C} \sim \mathrm{CP}^*[\mathbf{C}^{pure}]$



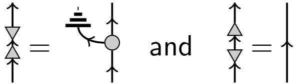
(1)



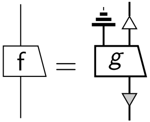
for $f, g \in \mathbf{C}^{\text{pure}}$ (2)



(3)



(4)

For all $f \in \mathbf{C}$ we have $g \in \mathbf{C}^{\text{pure}}$ such that 

(5)

Proof Sketch

The isomorphism is