Unordered Tuples in Quantum Computation

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What we did

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(After discussing paper of Pagani, Selinger, Valiron with Sam Staton.)

The heavy lifting

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Schur

Weyl

Quantum types as algebras

type	algebra

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qubit	

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   (Pauli exclusion principle)

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$$\frac{t \otimes t \xrightarrow{f} s \qquad (f \circ \mathsf{swap} = f)}{\mathsf{CoEq}(\mathsf{id}, \mathsf{swap}) \xrightarrow[f']{} s}$$

$\mathrm{CoEq}(\mathrm{id}, \mathrm{swap})$

CoEq(id, swap)

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(In $\text{fd-CStar}^{\text{op}}_{\text{cPsU}}$)

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(In Selinger's Q)

CoEq(id, swap)

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(In CPM_s)

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M_3 comes from $|00\rangle$, $|11\rangle$ and $|10\rangle + |01\rangle$.
$\mathbb{C}$ corresponds to $|01\rangle - |10\rangle$, which is symmetric up to global phase.

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Has simple 1/2-page proof, which led to ...

Remainder of this talk

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    - Sketch of proof
2. Cycles
3. Unordered words

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Examples

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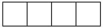
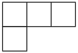
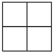
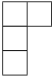
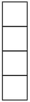
d					
2	M_5	M_3	$\mathbb{C}$		
3	M_{15}	M_{15}	M_6	M_3	
4	M_{35}	M_{45}	M_{20}	M_{15}	$\mathbb{C}$
5	M_{70}	M_{105}	M_{50}	M_{45}	M_5
6	M_{126}	M_{210}	M_{105}	M_{105}	M_{15}
7	M_{210}	M_{378}	M_{196}	M_{210}	M_{35}
8	M_{330}	M_{630}	M_{336}	M_{378}	M_{70}

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$$
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&= \mathrm{Rep}_{S_n}(H, H)
\end{aligned}
$$

The equalizer coincides with the representation endomorphisms of H!

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Schur's lemma:

$$\mathrm{Rep}(U_\lambda, U_\mu) = \begin{cases} \mathbb{C} & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases}$$

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What are the irreducible representations U_λ and their multiplicities m_λ?

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What are the irreducible representations U_λ and their multiplicities m_λ?
Answer is given by Schur-Weyl duality.

1. Unordered tuples
    - Sketch of proof
2. Cycles
3. Unordered words

3-cycle

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A 3-cycle of bits is a 4dit:

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$\{000,$

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What about a 3-cycle of qubits?

(= coequalizer of obvious action of C_3 on $B(\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2)$.)

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How to compute multiplicities?
By computing the character table.

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With some number theory:

$$m_k = \frac{1}{n} \sum_{\ell | n} d^{\frac{n}{\ell}} \mu\left(\frac{\ell}{\gcd(\ell, k)}\right) \frac{\phi(\ell)}{\phi\left(\frac{\ell}{\gcd(\ell,k)}\right)}.$$

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$$B(\ell^2) \oplus \prod_{\lambda \in Y^*} M_{m_\lambda}.$$

Y^*: Young diagrams of height at least 2.

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1. Algebras for unordered types are given by coequalizers.
2. They are more interesting than expected.
3. Representation theory of finite groups is a perfect fit to study them.

Thanks!

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Questions?