Qubit Uncertainty Tutorial

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Outline

Plan:

- Review the origins of Measurement Uncertainty Relations (MURs)
- Review the controversy over the validity of MURs
- Survey qubit measurement uncertainty
- Scrutinise a failed attempt, interpret its experimental “confirmations”, and confront it with a viable alternative

Heisenberg 1927

Heisenberg microscope:

"Let q_1 be the precision with which the value q is known (q_1 is, say, the mean error of q), therefore here the wavelength of the light. Let p_1 be the precision with which the value p is determinable; that is, here, the discontinuous change of p in the Compton effect. Then, according to the elementary laws of the Compton effect p_1 and q_1 stand in the relation

$$p_1\, q_1 \;\sim\; h. \tag{1}$$

- Makes clear reference to error and disturbance
- Sketches proof of *preparation uncertainty relation (PUR)* for the case of a Gaussian (minimum uncertainty) wave function
- Makes informal reference to rms error as standard deviation (of Q-distribution)

Heisenberg 1927: three faces of quantum uncertainty

Preparation Uncertainty Relations (PUR), Measurement Uncertainty Relations MUR

$$\begin{aligned}
(\textsc{Width of } Q \textsc{ distribution}) \cdot (\textsc{Width of } P \textsc{ distribution}) &\sim \hbar \\
(\textsc{Error of } Q \textsc{ measurement}) \cdot (\textsc{Error of } P) &\sim \hbar \\
(\textsc{Error of } Q \textsc{ measurement}) \cdot (\textsc{Disturbance of } P) &\sim \hbar
\end{aligned}$$

Quantum uncertainty: immediate reaction

Wolfgang Pauli's expression of the quantum pioneers' worries concerning $QP - PQ = i\hbar$:

> *"One may view the world with the p-eye*
> *and one may view it with the q-eye*
> *but if one opens both eyes simultaneously*
> *then one gets crazy."*

Wolfgang Pauli in a letter to Werner Heisenberg, 19 Oct. 1926)
And on reading Heisenberg's 1927 paper:

> *"Day is dawning in quantum mechanics."*

PUR – early developments

- Kennard (1927), Weyl (1928) ($p_i = \sqrt{2}\Delta P,\ q_i = \sqrt{2}\Delta Q$)

$$p_i\, q_i \ \geq \ \frac{h}{2\pi}$$

- Robertson (1929)

$$\Delta A\, \Delta B \ \geq \ \frac{1}{2}\big|\langle [A,B]\rangle\big|$$

- Schrödinger (1931)

$$(\Delta A)^2(\Delta B)^2 \ \geq \ \left(\frac{\overline{AB+BA}}{2} - \overline{AB}\right)^2 + \left|\frac{\overline{AB-BA}}{2}\right|^2 .$$

MUR – early denials

- Popper 1934: precursor or EPR (rebuttal by von Weizsäcker)
- Einstein, Podolsky, Rosen (EPR) 1935: use correlations to infer simultaneous sharp values of Q, P
- Park, Margenau 1967: time-of-flight determination of position and momentum
- Aharonov *et al*, since 1990: definite values of incompatible observables *between pre- and post-selection*

MUR – textbook wisdom

- PUR and MUR are conflated; no reflection on how to define measurement error/disturbance
- PUR $\neq$ MUR, hence claim no limitation on joint measurements

MUR – recent challenges

Heisenberg according to Ozawa:

$$\varepsilon(A,\rho)\,\varepsilon(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle[A,B]\rangle_\rho\right| \quad (???)$$
$$\varepsilon(A,\rho)\,\eta(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle[A,B]\rangle_\rho\right| \quad (???)$$

- Heisenberg didn't actually state this ... and it is of limited validity
- Ozawa was the first to propose formal definitions of measures of error $\varepsilon(A,\rho)$ and disturbance $\eta(B,\rho)$
- Ozawa's correction of the above:

$$\varepsilon(A,\rho)\eta(B,\rho) + \varepsilon(A,\rho)\Delta_\rho B + \Delta_\rho A\eta(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle[A,B]\rangle_\rho\right|$$

- Experimentally confirmed...
- ... yet, further scrutiny is needed ...

nature physics

LETTERS

PUBLISHED ONLINE: 15 JANUARY 2012 | DOI: 10.1038/NPHYS2194

Experimental demonstration of a universally valid error–disturbance uncertainty relation in spin measurements

Jacqueline Erhart[1], Stephan Sponar[1], Georg Sulyok[1], Gerald Badurek[1], Masanao Ozawa[2] and Yuji Hasegawa[1]*

The uncertainty principle generally prohibits simultaneous measurements of certain pairs of observables and forms the basis of indeterminacy in quantum mechanics[1]. Heisenberg's original formulation, illustrated by the famous γ-ray microscope, sets a lower bound for the product of the measurement error and the disturbance[2]. Later, the uncertainty relation was reformulated in terms of standard deviations[3–5], where the focus was exclusively on the indeterminacy of predictions, whereas the unavoidable recoil in measuring devices has been ignored[6]. A correct formulation of the error–disturbance uncertainty relation, taking recoil into account, is essential for a deeper understanding of the uncertainty principle, as Heisenberg's original relation is valid only under specific circumstances[7–10]. A new error–disturbance relation, derived using the theory of general quantum measurements, has been claimed to be universally valid[11–14]. Here, we report a neutron-optical experiment that records the error of a spin-component measurement as well as the disturbance caused on another spin-component. The results confirm that both error and disturbance obey the new relation but violate the old one in a wide range of an experimental parameter.

The uncertainty relation was first proposed by Heisenberg[2] in 1927 as a limitation of simultaneous measurements of canonically conjugate variables owing to the back-action of the measurement: the measurement of the position Q of the electron with the error $\epsilon(Q)$, or 'the mean error', induces the disturbance $\eta(P)$, or 'the discontinuous change', of the momentum P so that they always satisfy the relation

$$\epsilon(Q)\eta(P) \sim \frac{\hbar}{2} \qquad (1)$$

as $\sigma(A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. Note that a positive definite covariance term can be added to the right-hand side of equation (2), if squared, as discussed by Schrödinger[5]. For our experimental setting, this term vanishes. Robertson's relation (equation (2)) for standard deviations has been confirmed by many different experiments. In a single-slit diffraction experiment[15] the uncertainty relation, as expressed in equation (2), has been confirmed. A trade-off relation appears in squeezing coherent states of radiation fields[16], and many experimental demonstrations have been carried out[17].

Robertson's relation (equation (2)) has a mathematical basis, but has no immediate implications for limitations on measurements. This relation is naturally understood as limitations on state preparation or limitations on prediction from the past. On the other hand, the proof of the reciprocal relation for the error $\epsilon(A)$ of an A measurement and the disturbance $\eta(B)$ on observable B caused by the measurement, in a general form of Heisenberg's error–disturbance relation

$$\epsilon(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (3)$$

is not straightforward, as Heisenberg's proof[2] used an unsupported assumption on the state just after the measurement[12], despite successful justifications for the Heisenberg-type relation for unbiased joint measurements[8–10]. Recently, rigorous and general theoretical treatments of quantum measurements have revealed the failure of Heisenberg's relation (equation (1)), and derived a new universally valid relation[11–14] given by

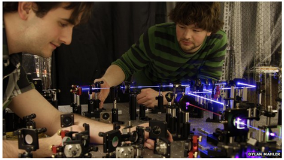
$$\epsilon(A)\eta(B)+\epsilon(A)\sigma(B)+\sigma(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (4)$$

Recent media hype: the end of quantum uncertainty?

7 September 2012 Last updated at 17:24

Heisenberg uncertainty principle stressed in new test

By Jason Palmer
Science and technology reporter, BBC News



The experiment requires preparing pairs of "entangled" photons, the particles from which light is made

Pioneering experiments have cast doubt on a founding idea of the branch of physics called quantum mechanics.

Related Stories

Quantenphysik

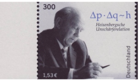
Der große Heisenberg irrte

17.11.2012 · Werner Heisenberg wollte seine berühmte Unbestimmtheitsbeziehung auch in den Störungen wiedererkennen, die ein Messung verursacht. Diesen Schluss haben kanadische Forscher widerlegt.

Von RAINER SCHARF

Artikel | Bilder (3) | Lesermeinungen (31)

Die von Werner Heisenberg 1927 formulierte Unschärfebeziehung ist trotz ihrer Tiefgründigkeit und Abstraktheit das wohl bekannteste Gesetz der Quantenphysik. Sie besagt vereinfacht, dass man nicht gleichzeitig die Geschwindigkeit und den Ort etwa eines Elektrons mit beliebiger Präzision bestimmen kann. Für die Popularität dieses Gesetzes hat vor allem eine ebenfalls von Heisenberg stammende bildhafte Erläuterung gesorgt,



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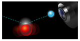
Werner Heisenberg und seine Unschärferelation sind sogar auf einer Briefmarke verewigt

Synopsis: Rescuing Heisenberg



APS/Alan Stonebraker

Proof of Heisenberg's Error-Disturbance Relation
Paul Busch, Pekka Lahti, and Reinhard F. Werner
Phys. Rev. Lett. **111**, 160405 (2013)
Published October 17, 2013

The popular conception of the Heisenberg uncertainty principle is that measurement is unavoidably invasive. We disturb an object when we observe it, thus introducing error into subsequent measurements. However, recent experiments (see 6 September 2012 Synopsis) claim to have measurement errors below the Heisenberg limit. To address this apparent contradiction, a paper in *Physical Review Letters* reports a new formulation of the uncertainty principle in which measurement disturbance depends on the performance of the measuring device, which is quantified as the maximum possible change in the state of the object.

MUR – what does the theory (QM) tell us?

(combined joint measurement errors for A, B) $\geq$ (incompatibility of A, B)

Formalising quantum measurement uncertainty: why did it take so long?

Answer (in a nutshell)

- **Cauchy-Schwarz seemed to be the "end of it"**
  (and Heisenberg (1930) endorsed Kennard's version and proof)
- **Lack of theory of approximate quantum measurements**

Quantum uncertainty: some history

Textbook wisdom elaborated:

"The uncertainty principle has nothing to do with the errors in joint measurements because ...

"The uncertainty relation is about *preparations*: spreads of distributions of separate measurements."

(and besides:)

"... joint measurements of noncommuting quantities are impossible."

(or alternatively:)

"... hence there are no limitations to joint measurements of noncommuting quantities."

Quantum uncertainty: some history

… **preciseimprecise joint measurements of noncommuting quantities are impossible. possible.**

Needed:

- notion of imprecise/approximate measurement
- measure of error

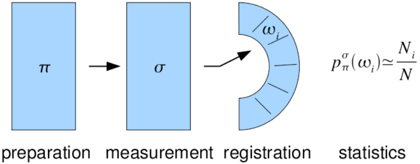
MUR – Why did it take so long: the long answer

- Heisenberg (1927): uncertainty relations *quantifying* and lifting incompatibility – explaining the *positive possibility* of approximate joint measurements and *cloud chamber trajectories*.
- von Neumann (1932): impossibility of joint measurements for noncommuting quantities
- Wigner (1932): quasi-probability distribution on phase space (Wigner function)
  concrete confirmation/illustration of von Neumann's no-go theorem
- Husimi (1940): positive phase space distributions (Q-function)
  later identified as a Heisenberg-Weyl covariant POVM on phase space
- Naimark (1940) [then still *Neumark*]: semispectral measures, POVMs general measurements

MUR – Why did it take so long: the long answer

- 1960s: Ludwig, Davies&Lewis – POVMs in physics (quantum foundations)
  noncommuting observables may be jointly measurable – if they are sufficiently *unsharp*
- Arthurs&Kelly (1965): model of joint measurement of position and momentum
  – intuitive, but no concept of approximation
- 1980s: bringing together models and concepts of joint measurements; first realisations in quantum optics
- 1990s: first attempts at systematic formulations of quantum measurement error and model-independent, *universal* measurement uncertainty relations (mainly Appleby, Ozawa)
- since 2004: Ozawa inequality; controversy over the question of the "correct" quantum version of Gauss root-mean-square (rms) error
- since 2012: new measurement uncertainty relations, new experiments

Measurement Statistics – Observables as POVMs



$$[\pi] \sim \rho, \quad [\sigma] \sim \mathsf{E} = \{\omega_i \mapsto E_i\} : \qquad p_\pi^\sigma(\omega_i) = \mathrm{tr}[\rho E_i] = p_\rho^{\mathsf{E}}(\omega_i)$$

state : $\quad \rho : \mathcal{H} \to \mathcal{H}, \quad O \le \rho \le I, \quad \mathrm{tr}[\rho] = 1$

effect : $\quad E : \mathcal{H} \to \mathcal{H}, \quad O \le E \le I$

POVM : $\quad \mathsf{E} = \{E_1, E_2, \cdots, E_n\}, \quad O \le E_i \le I, \quad \sum E_i = I$

state changes: *instrument* $\omega_i, \rho \to \mathcal{I}_i(\rho)$

measurement processes: measurement scheme $\mathcal{M} = \langle \mathcal{H}_a, \phi, U, Z_a \rangle$

General POVM

measurable space: (Ω, Σ)

observable: $\mathsf{E} : \Sigma \to \mathcal{L}(\mathcal{H})$

$O \leq \mathsf{E}(X) \leq I, \quad \mathsf{E}(\emptyset) = O, \quad \mathsf{E}(\Omega) = I,$

$\mathsf{E}(\cup_k X_k) = \sum_k \mathsf{E}(X_k)$ for any disjoint sequence $(X_k)_{k\in\mathbb{N}}$

probability: $p_\rho^{\mathsf{E}}(X) = \mathrm{tr}\left[\rho \mathsf{E}(X)\right]$

Instrument

$$\Sigma \ni X \mapsto \mathcal{I}(X), \quad \mathcal{I}(X) : \rho \mapsto \mathcal{I}(X)(\rho)$$

induced/measured observable E:

$$\mathcal{I} \rightsquigarrow \mathsf{E} : \quad X \mapsto \mathrm{tr}\left[\mathcal{I}(X)(\rho)\right] \equiv \mathrm{tr}\left[\rho \mathsf{E}(X)\right]$$

Illustration: Heisenberg Effect

Theorem: No disturbance – no information

$$\forall\rho:\ \mathcal{I}(\Omega)(\rho)=\rho \quad \Longrightarrow \forall\rho,\rho'\ \forall X\in\Sigma:\ \mathrm{tr}\left[\mathcal{I}(X)(\rho)\right]=\mathrm{tr}\left[\mathcal{I}(X)(\rho')\right]$$

Measured observable is *trivial:*

$$\forall\rho,\rho',X:\ \mathrm{tr}\left[\rho\mathsf{E}(X)\right]=\mathrm{tr}\left[\rho'\mathsf{E}(X)\right] \quad \Longleftrightarrow \quad \forall X\in\Sigma:\mathsf{E}(X)=\mu(X)\,I$$

Here μ is a fixed probability measure on (Ω,Σ).

Proof: No disturbance – no information

Assume $\mathcal{I}(\Omega)(P[\varphi]) = P[\varphi]$. Then

$$\mathcal{I}(X)(P[\varphi]) + \mathcal{I}(X^c)(P[\varphi]) = P[\varphi], \quad \text{therefore } \mathcal{I}(X)(P[\varphi]) = \mathsf{E}_\varphi(X)P[\varphi]$$

Show next: $\mathsf{E}_\varphi(X) = \mathsf{E}_\psi(X)$ for any states φ, ψ.
First, consider $\varphi \perp \psi$.
Take $\xi, \eta \in [\varphi, \psi]$, $\xi \not\perp \varphi, \psi$ and $\eta \perp \xi$
Then $P[\varphi] + P[\psi] = P[\xi] + P[\eta]$ and so

$$\mathsf{E}_\varphi(X)P[\varphi] + \mathsf{E}_\psi(X)P[\psi] \;=\; \mathsf{E}_\xi(X)P[\xi] + \mathsf{E}_\eta(X)P[\eta]\,.$$

Uniqueness of spectral decomposition $\Rightarrow$ spectrum degenerate:
$\mathsf{E}_\varphi(X) = \mathsf{E}_\psi(X) = \mathsf{E}_\xi(X) = \mathsf{E}_\eta(X)$.
Thus, given any φ, for all $\psi \perp \varphi :\quad \mathsf{E}_\psi(X) = \mathsf{E}_\varphi(X)$.
This extends to all $\xi \not\perp \varphi$.
Hence $X \mapsto \mathsf{E}_\varphi(X) \equiv \lambda(X)$ is a constant probability measure.
Also, $\mathcal{I}(X)(P[\varphi]) = \lambda(X)P[\varphi]$. $\quad \mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Example: constant channel instrument

$$\rho \;\to\; \mathcal{I}^{\mathsf{C}}_{\rho_0}(X)(\rho) = \mathrm{tr}\,[\rho \mathsf{C}(X)]\,\rho_0$$

Resulting disturbance: any observable B turned into a trivial observable B':

$$\mathrm{tr}\left[\rho \mathsf{B}'(Y)\right] = \mathrm{tr}\left[\rho \mathcal{I}^{\mathsf{C}}_{\rho_0}(\Omega)^*(\mathsf{B}(Y))\right] = \mathrm{tr}\,[\rho_0 \mathsf{B}(Y)]$$

for all Y, so that $\mathsf{B}'(Y) = \mathsf{B}_{\rho_0}(Y)\,I$.

Measurement Scheme

$\mathcal{M} = \langle \mathcal{H}_a, \phi, U, Z_a \rangle$

$$\mathrm{tr}\left[(\rho \otimes \sigma) U^* (B \otimes \mathsf{Z}(X)) U\right] = \mathrm{tr}\left[\mathcal{I}(X)(\rho) B\right]$$

$$\mathrm{tr}\left[(\rho \otimes \sigma) U^* (I \otimes \mathsf{Z}(X)) U\right] = \mathrm{tr}\left[\rho\, \mathsf{E}(X)\right]$$

Hence:

$$\mathcal{M} \rightsquigarrow \mathcal{I} \rightsquigarrow \mathsf{E}$$

Exampe: SWAP

$$\mathcal{H}_a = \mathcal{H}, \quad U = \mathrm{SWAP}, \quad \mathsf{Z} = \mathsf{E}$$

$$\mathrm{tr}\,[(\rho \otimes \sigma) U^*(B \otimes \mathsf{E}(X))U] \;=\; \mathrm{tr}\,[\mathcal{I}(X)(\rho)B] \;=\; \mathrm{tr}\,[\sigma B]\,\mathrm{tr}\,[\rho \mathsf{E}(X)]$$

$$i.e.,\ \mathcal{I}(X)(\rho) \;=\; \mathrm{tr}\,[\rho \mathsf{E}(X)]\;\sigma$$

$$\mathrm{tr}\,[(\rho \otimes \sigma) U^*(I \otimes \mathsf{E}(X))U] \;=\; \mathrm{tr}\,[\rho\,\mathsf{E}(X)]$$

Signature of an observable: its statistics

$$p_\rho^{\mathsf{C}} = p_\rho^{\mathsf{A}} \quad \text{for all } \rho \quad \Longleftrightarrow \quad \mathsf{C} = \mathsf{A}$$

Minimal indicator for a measurement of C *to be a good approximate measurement of* A*:*

$$p_\rho^{\mathsf{C}} \simeq p_\rho^{\mathsf{A}} \quad \text{for all } \rho$$

Unbiased approximation – absence of systematic error:

$$\mathsf{C}[1] = \textstyle\sum_j c_j C_j = \mathsf{A}[1] = \sum_i a_i A_i = A$$

... $\mathsf{C}[1] = \mathsf{A}[1]$ is often taken as sole criterion for a good measurement
... but equality of all moments required for exact measurement: $\mathsf{C}[k] = \mathsf{A}[k]$

Joint Measurability/Compatibility

Definition: joint measurability (compatibility)

Observables $\mathsf{C} = \{C_1, C_2, \ldots, C_m\}, \quad \mathsf{D} = \{D_1, D_2, \ldots, D_n\}$ are *jointly measurable*
if they are margins of an observable $\mathsf{G} = \{G_{k\ell}\}$:

$$C_k = \sum_\ell G_{k\ell}, \quad D_\ell = \sum_k G_{k\ell}$$

$$\mathsf{C}(X) = \mathsf{G}(X \times \Omega_2), \quad \mathsf{D}(Y) = \mathsf{G}(\Omega_1 \times Y)$$

Compatibility

Theorem

If one of C, D *is sharp (projection valued), then these observables are jointly measurable iff they commute:*

$$[C_k, D_\ell] = 0$$

and the joint observable G is uniquely determined:

$$G_{k\ell} = C_k D_\ell$$

Joint measurability in general

Pairs of **unsharp** observables may be jointly measurable
– **even when they do not commute!**

Example: compatibility by smearing

C, D – discrete observables, $\Omega_j = \{1, 2, \ldots, N_j\}$, $\Sigma_j = \mathbf{2}^{\Omega_j}$, $j = 1, 2$

$\mathsf{C}(\{k\}) \equiv C_k$, $\mathsf{D}(\{\ell\}) \equiv D_\ell$ $\qquad \sum_k C_k = I = \sum_\ell D_\ell$

$p_k \geq 0$, $q_\ell \geq 0$, $\quad \sum_k p_k = 1 = \sum_\ell q_\ell$

$\mathsf{C}^{(\lambda)}$, $\mathsf{D}^{(\mu)}$ $\quad (\lambda, \mu \in [0, 1])$:

$$C_k^{(\lambda)} = \lambda C_k + (1 - \lambda) p_k I, \quad D_\ell^{(\mu)} = \mu D_\ell + (1 - \mu) q_\ell I$$

$\mathsf{C}^{(\lambda)}$, $\mathsf{D}^{(\mu)}$ are jointly measurable if $\lambda + \mu \leq 1$.

Proof: $\mathsf{G} = \{G_{k\ell}\}$ is a joint observable, where

$$G_{k\ell} = \lambda C_k q_\ell + \mu D_\ell p_k + (1 - \lambda - \mu) p_k q_\ell$$

Compatibility - some results

Proposition

If C, D and C', D' are compatible pairs of observables, then $\widetilde{\mathsf{C}} = \lambda\mathsf{C} + (1-\lambda)\mathsf{C}'$ and $\widetilde{\mathsf{D}} = \lambda\mathsf{D} + (1-\lambda)\mathsf{D}'$ are compatible for any $\lambda \in [0, 1]$.

Proof: if G, G' are joint observables for C, D and C', D', respectively, then $\widetilde{\mathsf{G}} = \lambda\mathsf{G} + (1-\lambda)\mathsf{G}'$ is a joint observable for $\widetilde{\mathsf{C}}, \widetilde{\mathsf{D}}$.

Compatibility - some results

Proposition

If C, D are compatible, then C', D' are compatible, where $\mathsf{C}'(X) = V\mathsf{C}(X)V^*$, $\mathsf{D}'(X) = V\mathsf{D}(X)V^*$ and V is (anti-)unitary.

Proof: If G is a joint observable for C, D, then G' is a joint observable for C', D', where $\mathsf{G}'(Z) = V\mathsf{G}(Z)V^*$.

Approximate joint measurement: concept

$$
\begin{tikzcd}
 & G \arrow[dl] \arrow[dr] & \\
C \arrow[d, squiggly] & & D \arrow[d, squiggly] \\
A & & B
\end{tikzcd}
$$

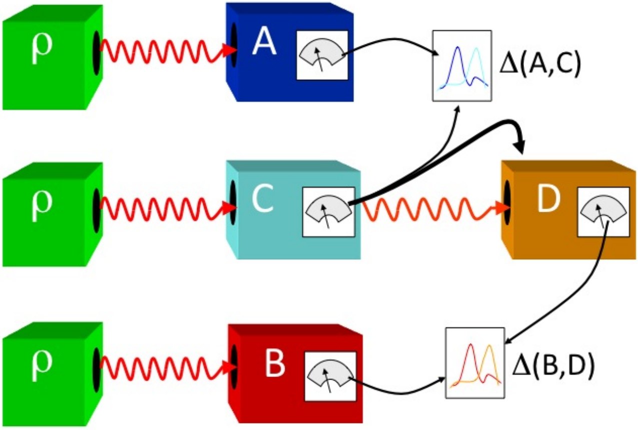
joint observable

approximator observables (compatible)

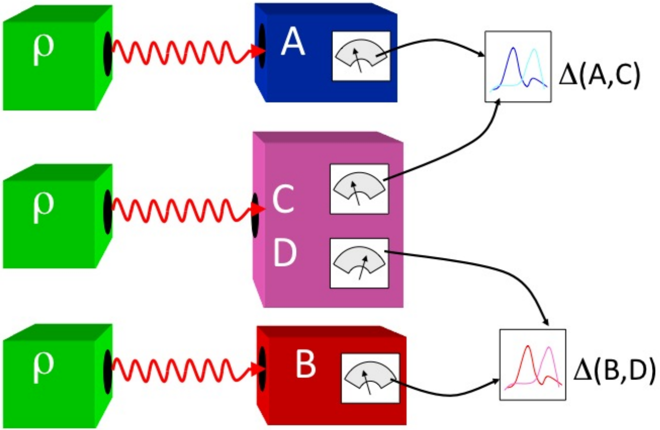
target observable

Task: find suitable measures of *approximation errors*

Disturbance



Disturbance quantified as approximation error



PUR

Preparation Uncertainty Relations

PUR for Q, P

$$\begin{aligned} x_0 > 0: \quad \tfrac{4\hbar^2}{x_0^2}(\Delta_\rho Q)^2 + x_0^2(\Delta_\rho P)^2 &\ \geq\ 2\hbar^2 \\ \text{(ground state of harmonic oscillator)} &\ \Updownarrow \\ (\Delta_\rho Q)^2\,(\Delta_\rho P)^2 &\ \geq\ \tfrac{\hbar^2}{4} \\ &\ \Updownarrow \\ \tfrac{2\hbar}{x_0}\Delta_\rho Q + x_0 \Delta_\rho P &\ \geq\ 2\hbar \end{aligned}$$

Proof: use $\quad \xi^2 + \dfrac{1}{\xi^2} = \left(\xi - \dfrac{1}{\xi}\right)^2 + 2 \geq 2 \quad (\xi > 0)$

$$\Delta Q \Delta P \geq \tfrac{\hbar}{2} \ \Leftrightarrow \left(\tfrac{2\hbar}{x_0}\Delta Q - x_0 \Delta P\right)^2 + 4\hbar \Delta Q \Delta P \geq 2\hbar^2$$

for "$\Leftarrow$" use $\quad Q \ \to\ \lambda Q, \quad P \to \dfrac{1}{\lambda} P$

PUR in general

Important observation

For bounded observables A, B, the standard PUR $\Delta_\rho A \Delta_\rho B \ \geq \ \frac{1}{2}\left|\langle [A, B] \rangle_\rho\right|$ is **not** a strong constraint: the lower bound vanishes for (near) eigenstates.

Qubits

$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ (Pauli matrices acting on $\mathbb{C}^2$)

- *States:* $\rho = \frac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}), \quad |\boldsymbol{r}| \leq 1$
- *Effects:* $A = \frac{1}{2}(a_0 I + \boldsymbol{a} \cdot \boldsymbol{\sigma}) \in [O, I], \quad 0 \leq \frac{1}{2}(a_0 \pm |\boldsymbol{a}|) \leq 1$
- *observables:* $(\Omega = \{+1, -1\})$

$$
\begin{aligned}
\mathrm{A}: \ \pm 1 &\mapsto A_\pm = \tfrac{1}{2}(I \pm \boldsymbol{a} \cdot \boldsymbol{\sigma}) \quad |\boldsymbol{a}| = 1 \\
\mathrm{B}: \ \pm 1 &\mapsto B_\pm = \tfrac{1}{2}(I \pm \mathbf{b} \cdot \boldsymbol{\sigma}) \quad |\mathbf{b}| = 1 \\
\mathrm{C}: \ \pm 1 &\mapsto C_\pm = \tfrac{1}{2}(1 \pm \gamma)\, I \pm \tfrac{1}{2}\boldsymbol{c} \cdot \boldsymbol{\sigma} \quad |\gamma| + |\boldsymbol{c}| \leq 1 \\
\mathrm{D}: \ \pm 1 &\mapsto D_\pm = \tfrac{1}{2}(1 \pm \delta)\, I \pm \tfrac{1}{2}\boldsymbol{d} \cdot \boldsymbol{\sigma} \quad |\delta| + |\boldsymbol{d}| \leq 1
\end{aligned}
$$

C symmetric (unbiased): $\gamma = 0$

C sharp: $\gamma = 0$, $|\boldsymbol{c}| = 1$; $\quad \rightarrow \quad$ *unsharpness:* $U(\mathrm{C})^2 = 1 - |\boldsymbol{c}|^2$

PUR for qubit observables

$$\sigma_k^2 = I, \quad \langle \sigma_k \rangle_\rho = r_k, \quad \rho = \tfrac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}), \quad \boldsymbol{r} = (r_1, r_2, r_3)$$

$$(\Delta_\rho \sigma_1)^2 \, (\Delta_\rho \sigma_2)^2 \ \geq \ \tfrac{1}{4} \left| \langle [\sigma_1, \sigma_2] \rangle_\rho \right|^2 \ + \ \tfrac{1}{4} \left(\langle \sigma_1 \sigma_2 + \sigma_2 \sigma_1 \rangle_\rho - 2 \langle \sigma_1 \rangle \langle \sigma_2 \rangle_\rho \right)^2$$

$$\Updownarrow$$

$$(1 - \langle \sigma_1 \rangle_\rho)^2 \, (1 - \langle \sigma_2 \rangle_\rho)^2 \ \geq \ \langle \sigma_3 \rangle_\rho^2 + \langle \sigma_1 \rangle_\rho^2 \langle \sigma_2 \rangle_\rho^2$$

$$\Updownarrow$$

$$\langle \sigma_1 \rangle_\rho^2 + \langle \sigma_2 \rangle_\rho^2 + \langle \sigma_3 \rangle_\rho^2 \ = \ |\boldsymbol{r}|^2 \ \leq 1 \quad (\rho \ \geq \ 0)$$

$$\Updownarrow$$

$$(\Delta_\rho \sigma_1)^2 \ + \ (\Delta_\rho \sigma_2)^2 \ + \ (\Delta_\rho \sigma_3)^2 \ \geq 2$$

$$\Downarrow$$

$$(\Delta_\rho \sigma_1)^2 \ + \ (\Delta_\rho \sigma_2)^2 \ \geq 1$$

Preparation uncertainty for qubits – continued

$$A = \boldsymbol{a} \cdot \boldsymbol{\sigma}, \quad B = \mathbf{b} \cdot \boldsymbol{\sigma}, \quad |\boldsymbol{a}| = |\mathbf{b}| = 1$$

$$(\Delta_\rho A)^2 + (\Delta_\rho B)^2 \geq 1 - |\boldsymbol{a} \cdot \mathbf{b}| = 1 - \sqrt{1 - |\boldsymbol{a} \times \mathbf{b}|^2}$$
$$= 1 - \sqrt{1 - \|[A, B]\|^2}$$

L.H.S.$\geq |\hat{\boldsymbol{r}} \times \boldsymbol{a}|^2 + |\hat{\boldsymbol{r}} \times \mathbf{b}|^2, \quad \hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}| \quad (\boldsymbol{r} \neq \mathbf{0})$

Tight bound, attained at $\boldsymbol{r} = (\boldsymbol{a} \pm \mathbf{b})/|\boldsymbol{a} \pm \mathbf{b}|$ if $\boldsymbol{a} \cdot \mathbf{b} \geq 0$ and ≤ 0, resp.

$$\Delta_\rho A \, + \, \Delta_\rho B \; \geq \; |\boldsymbol{a} \times \mathbf{b}| \; = \; \big\|[A, B]\big\|.$$

L.H.S.$\geq |\hat{\boldsymbol{r}} \times \boldsymbol{a}| + |\hat{\boldsymbol{r}} \times \mathbf{b}|, \quad \hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}| \quad (\boldsymbol{r} \neq \mathbf{0})$

Tight bound, attained at $\boldsymbol{r} = \pm\boldsymbol{a}$ or $\boldsymbol{r} = \pm\mathbf{b}$.

Preparation uncertainty for qubits – which inequality?

Problem: Uncertainty region

Characterise the region of points with coordinates $(\Delta A, \Delta B) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\Delta A = \Delta_\rho A$ and $\Delta B = \Delta_\rho B$: the *uncertainty region of A, B*.

Particularly: find the "lower boundary curve" of the uncertainty region.

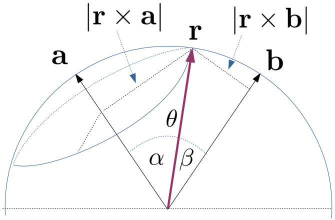
Thus, given $\Delta A = \Delta_\rho A$, find ρ^* such that

$$\Delta_{\rho^*} B = \min\{\, \Delta_{\rho'} B \ :|\, \Delta_{\rho'} A = \Delta_\rho A \,\}$$

Note: $\Delta_\rho A \, \Delta_\rho B \geq \frac{1}{2} \big| \langle [A, B] \rangle_\rho \big|$ doesn't help...

Preparation uncertainty for qubits – uncertainty region

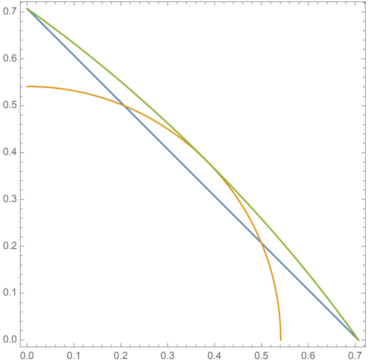
Solution



$$\begin{aligned}\theta &= \alpha+\beta\\ \arcsin|\mathbf{a}\times\mathbf{b}| &= \arcsin|\mathbf{r}\times\mathbf{a}| + \arcsin|\mathbf{r}\times\mathbf{b}|\end{aligned}$$

$$\Delta_\rho A \geq |\hat{\boldsymbol{r}}\times\boldsymbol{a}|, \quad \Delta_\rho B \geq |\hat{\boldsymbol{r}}\times\mathbf{b}|$$

$$\Delta_\rho A\sqrt{1-(\Delta_\rho B)^2} + \Delta_\rho B\sqrt{1-(\Delta_\rho A)^2} \;\geq\; \big\|[A,B]\big\|$$



$$|\boldsymbol{a}\cdot\mathbf{b}| = \frac{1}{\sqrt{2}}: \qquad x + y \ \geq \ \frac{1}{\sqrt{2}} \ \left(= \ |\boldsymbol{a}\times\mathbf{b}|\right)$$

$$x^2 + y^2 \ \geq \ 1 - \frac{1}{\sqrt{2}} \ \left(= \ 1 - |\boldsymbol{a}\cdot\mathbf{b}|\right)$$

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} \ \geq \ \frac{1}{\sqrt{2}} \ \left(= \ |\boldsymbol{a}\times\mathbf{b}|\right)$$

Compatibility

Compatibility of Qubit Effects

Compatibility of C, D

Symmetric case (sufficient for optimal compatible approximations):
$C_\pm = \frac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma}),\ D_\pm = \frac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})$

Proposition

$\mathrm{C} = \{C_\pm = \frac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma})\}$, $\mathrm{D} = \{D_\pm = \frac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})\}$ are compatible if and only if

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2.$$

Interpretation: *unsharpness* $U(\mathrm{C})^2 = 1 - |\boldsymbol{c}|^2$; $|\boldsymbol{c}\times\boldsymbol{d}| = 2\big\|[C_+, D_+]\big\|$

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2 \ \Leftrightarrow\ \left(1-|\boldsymbol{c}|^2\right)\left(1-|\boldsymbol{d}|^2\right) \ge |\boldsymbol{c}\times\boldsymbol{d}|^2$$

$$\mathrm{C}, \mathrm{D} \text{ compatible } \ \Leftrightarrow\ U(\mathrm{C})^2 \times U(\mathrm{D})^2 \ge 4\big\|[C_+, D_+]\big\|^2$$

Qubit compatibility: example

Take $\boldsymbol{c} \perp \boldsymbol{d}$:

$$\mathrm{C}, \mathrm{D} \text{ compatible } \iff |\boldsymbol{d}|^2 + |\boldsymbol{d}|^2 \leq 1 \iff U(\mathrm{C})^2 + U(\mathrm{D})^2 \geq 1$$

$|\boldsymbol{c}| = |\boldsymbol{d}| = \lambda$:
$C_\pm = \frac{1}{2}(I \pm \boldsymbol{c} \cdot \boldsymbol{\sigma}) = \lambda \frac{1}{2}(I \pm \hat{\boldsymbol{c}} \cdot \boldsymbol{\sigma}) + (1 - \lambda)\frac{1}{2} I$
$D_\pm = \frac{1}{2}(I \pm \boldsymbol{d} \cdot \boldsymbol{\sigma}) = \lambda \frac{1}{2}(I \pm \hat{\boldsymbol{d}} \cdot \boldsymbol{\sigma}) + (1 - \lambda)\frac{1}{2} I$

C, D compatible iff $\lambda \leq 1/\sqrt{2}$: *degree of incompatibility*

Qubit compatibility: proof

C, D with $C_+ = \frac{1}{2}(c_0 I + \mathbf{c} \cdot \boldsymbol{\sigma})$, $D_+ = \frac{1}{2}(d_0 I + \mathbf{d} \cdot \boldsymbol{\sigma})$ are jointly measurable iff $\exists$ observable: $\mathsf{G} = \{G_{++}, G_{+-}, G_{-+}, G_{--}\}$ such that

$$C_k = G_{k+} + G_{k-}, \quad D_\ell = G_{+\ell} + G_{-\ell}$$

iff $\exists G = \frac{1}{2}(gI + \mathbf{g} \cdot \boldsymbol{\sigma})$:

$$\begin{aligned}
(G_{++} =)\ G &\geq O \\
(G_{+-} =)\ C_+ - G &\geq O, \\
(G_{-+} =)\ D_+ - G &\geq O, \\
(G_{--} =)\ I - C_+ - D_+ + G &\geq O
\end{aligned}$$

Qubit compatibility: proof continued

Thus:

C, D compatible

iff $\exists G = \frac{1}{2}(g_0 I + \boldsymbol{g}\cdot\boldsymbol{\sigma})\ (= G_{++})$:

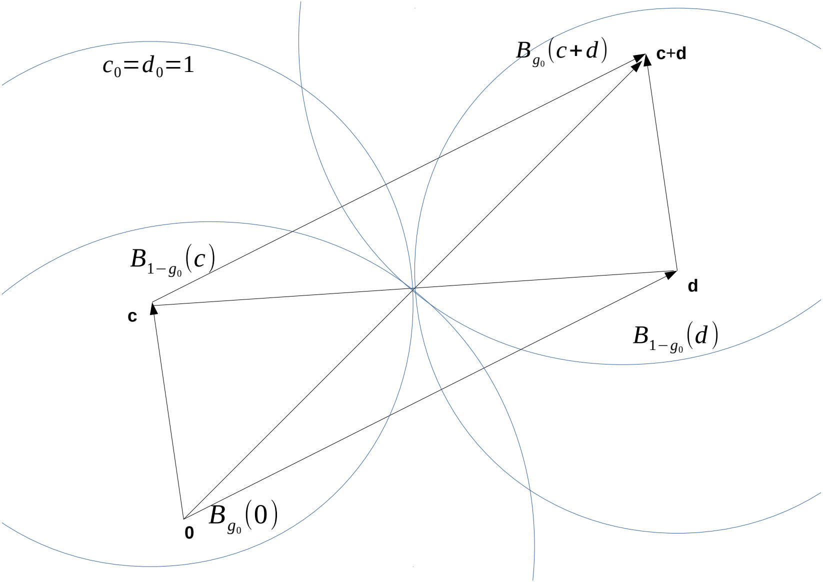
$$G \geq O, \quad C_+ - G\ (= G_{+-})\ \geq O, \quad D_+ - G\ (= G_{-+})\ \geq O,$$
$$I - C_+ - D_+ + G\ (= G_{--})\ \geq O$$

iff $\exists\ g_0,\ \boldsymbol{g}$:

$$|\boldsymbol{g}| \leq g_0, \quad |\boldsymbol{c} - \boldsymbol{g}| \leq c_0 - g_0, \quad |\boldsymbol{d} - \boldsymbol{g}| \leq d_0 - g_0,$$
$$|\boldsymbol{c} + \boldsymbol{d} - \boldsymbol{g}| \leq 2 + g_0 - c_0 - d_0$$

iff $\exists\ g_0$:

$$B_{g_0}(\mathbf{0})\ \cap\ B_{c_0-g_0}(\boldsymbol{c})\ \cap\ B_{d_0-g_0}(\boldsymbol{d})\ \cap\ B_{2+g_0-c_0-d_0}(\boldsymbol{c}+\boldsymbol{d})\ \neq\ \emptyset$$


$c_0 = d_0 = 1$
$B_{g_0}(c+d)$
c+d
$B_{1-g_0}(c)$
c
d
$B_{1-g_0}(d)$
0
$B_{g_0}(0)$

Qubit compatibility: proof continued

Necessary: diagonally opposite balls must intersect

$$|\boldsymbol{c}+\boldsymbol{d}| \leq 2+2g_0-c_0-d_0, \quad |\boldsymbol{c}-\boldsymbol{d}| \leq c_0+d_0-2g_0$$

and therefore

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \ \leq \ 2 \qquad (\star)$$

... which is in fact equivalent to

$$|\boldsymbol{c}+\boldsymbol{d}| \ \leq \ 1+\boldsymbol{c}\cdot\boldsymbol{d} \ \leq \ 2-|\boldsymbol{c}-\boldsymbol{d}| \qquad (\star\star)$$

(as well as)

$$|\boldsymbol{c}-\boldsymbol{d}| \ \leq \ 1-\boldsymbol{c}\cdot\boldsymbol{d} \ \leq \ 2-|\boldsymbol{c}+\boldsymbol{d}| \qquad (\star\star)$$

Qubit compatibility: proof completed

Special case $C_{\pm} = \frac{1}{2}(I \pm \boldsymbol{c} \cdot \boldsymbol{\sigma})$, $D_{\pm} = \frac{1}{2}(I \pm \boldsymbol{d} \cdot \boldsymbol{\sigma})$, $c_0 = d_0 = 1$:

Given $(\star)$, choose $g_0 = \frac{1}{2}(1 + \boldsymbol{c} \cdot \boldsymbol{d})$, $\boldsymbol{g} = \frac{1}{2}(\boldsymbol{c} + \boldsymbol{d})$, then $\mathsf{G} = \{G_{k\ell}\}$ is a joint observable, where

$$G_{k\ell} \;=\; \frac{1}{4}\left[(1 + k\ell \boldsymbol{c} \cdot \boldsymbol{d})\, I + (k\boldsymbol{c} + \ell \boldsymbol{d}) \cdot \boldsymbol{\sigma}\right]$$

Positivity: $G_{k\ell} \geq O \iff (\star\star)$.

This proves sufficiency of $(\star)$ for the special case. $\mathcal{Q.E.D.}$

Approximate Joint Measurements

Approximation error for qubits: probabilistic distance

Consider observables A, C on (Ω, Σ).
Idea:
C is a good approximation to A if the probability distributions p_ρ^{C}, p_ρ^{A} are similar for all states ρ.
Quantify this with some choice of metric
or any other suitable measure of error.
Take here:

$$d_p(\mathsf{C}, \mathsf{A}) = \sup_\rho \sup_X \left|\mathrm{tr}[\rho \mathsf{C}(X)] - \mathrm{tr}[\rho \mathsf{A}(X)]\right| = \sup_X \left\|\mathsf{C}(X) - \mathsf{A}(X)\right\|$$

Qubit case: $C_+ = \frac{1}{2}\left(c_0 I + \boldsymbol{c} \cdot \boldsymbol{\sigma}\right)$, $A_+ = \frac{1}{2}\left(a_0 I + \boldsymbol{a} \cdot \boldsymbol{\sigma}\right)$

$$d_p(\mathsf{C}, \mathsf{A}) = \left\|C_+ - A_+\right\| = \tfrac{1}{2}|c_0 - a_0| + \tfrac{1}{2}|\boldsymbol{c} - \boldsymbol{a}| \ \in\ [0, 1].$$

Comparison : Measurement noise (Ozawa *et al*)

$$\begin{aligned}\varepsilon(\mathsf{C},\mathsf{A};\varphi)^2 &= \langle \varphi\otimes\phi \,|\, (Z_\tau - A)^2 \varphi\otimes\phi\rangle \\ &= \left\langle \mathsf{C}[2]-\mathsf{C}[1]^2\right\rangle_\rho + \left\langle (\mathsf{C}[1]-A)^2\right\rangle_\rho \equiv \varepsilon_a^2\end{aligned}$$

Qubit observables, symmetric case:

$$\varepsilon_a^2 = 1 - |\boldsymbol{c}|^2 + |\boldsymbol{a}-\boldsymbol{c}|^2 = U(\mathsf{C})^2 + 4d_a^2$$

$\varepsilon(\mathsf{A};\rho)$ double counts contribution from unsharpness.

(More on Measurement Noise ("Ozawa's error") tomorrow ...)

Measurement Uncertainty Relations

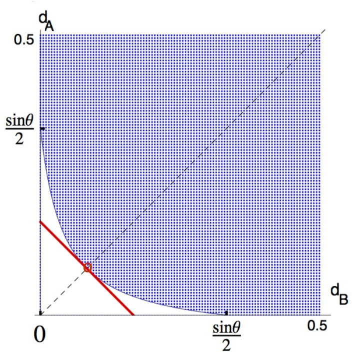
Optimising approximate joint measurements

$$
\begin{tikzcd}
 & \{G_{k\ell}\} \arrow[dl, "\sum_\ell"'] \arrow[dr, "\sum_k"] & \\
\{C_k\} \arrow[d, squiggly, "d_p(\mathrm{C{,}A})"'] & & \{D_\ell\} \arrow[d, squiggly, "d_p(\mathrm{D{,}B})"] \\
\{A_k\} & & \{B_\ell\}
\end{tikzcd}
$$

Goal

To make errors $d_\mathrm{A} = d_p(\mathrm{C},\mathrm{A})$, $d_\mathrm{B} = d_p(\mathrm{D},\mathrm{B})$ simultaneously as small as possible
subject to the constraint that C, D are compatible.

Admissible error region



$\sin\theta = |\boldsymbol{a} \times \mathbf{b}|$

$(d_A, d_B) = (d_p(C, A), d_p(D, B)) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ with C, D compatible

trivial approximations: $C_+ = \gamma I,\ D_+ = \delta I$;
then $d_A = \max(\gamma, 1-\gamma) \geq \frac{1}{2},\ d_B = \max(\delta, 1-\delta) \geq \frac{1}{2}$

Optimisation – Step 1: symmetric approximators

Given: C, D, $\quad C_\pm = \frac{1}{2}(1 \pm \gamma) I \pm \frac{1}{2}\boldsymbol{c}\cdot\boldsymbol{\sigma}$, $D_\pm = \dots$
Take T antiunitary: $T\sigma_k T^* = -\sigma_k$

$$C'_\pm = TC_\mp T^* = \tfrac{1}{2}(1 \mp \gamma) I \pm \tfrac{1}{2}\boldsymbol{c}\cdot\boldsymbol{\sigma},\ D'_\pm = \dots$$

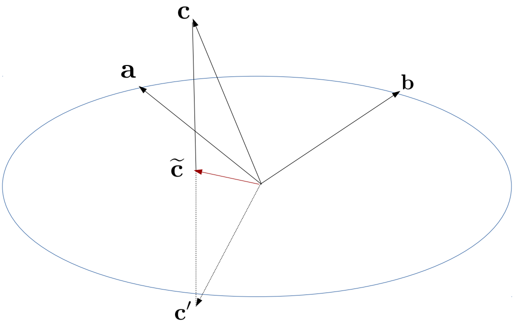
C, D compatible, with joint observable $\{G_{k\ell}\}$
$\implies$ C', D' compatible, with joint observable $\{G'_{k\ell} = TG_{-k,-\ell}T^*\}$
$\implies$ $\widetilde{\mathsf{C}}$, $\widetilde{\mathsf{D}}$ compatible, where

$$\widetilde{C}_\pm = \tfrac{1}{2}(C_\pm + C'_\pm) = \tfrac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma}), \quad \widetilde{D}_\pm = \tfrac{1}{2}(D_\pm + D'_\pm) = \tfrac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})$$

$$d_p(\mathsf{C},\mathsf{A}) \ \geq\ d_p(\widetilde{\mathsf{C}},\widetilde{\mathsf{A}}) \qquad d_p(\mathsf{D},\mathsf{B}) \ \geq\ d_p(\widetilde{\mathsf{D}},\widetilde{\mathsf{B}})$$

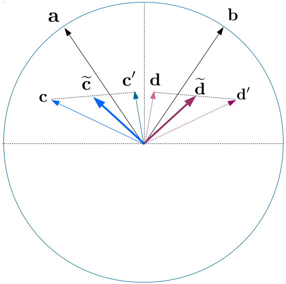
$$\left[\ \tfrac{1}{2}|\gamma| + \tfrac{1}{2}|\boldsymbol{c}-\boldsymbol{a}| \ \geq\ \tfrac{1}{2}|\boldsymbol{c}-\boldsymbol{a}|\ \right]$$

Optimisation – Step 2: planar approximators



$$d_p(\widetilde{\mathsf{C}},\mathsf{A}) \;=\; \tfrac{1}{2}\left|\tfrac{1}{2}(\boldsymbol{c}+\boldsymbol{c}')-\boldsymbol{a}\right| \le \tfrac{1}{2}\left[d_p(\mathsf{C},\mathsf{A})+d_p(\mathsf{C}',\mathsf{A})\right] \;=\; d_p(\mathsf{C},\mathsf{A})$$

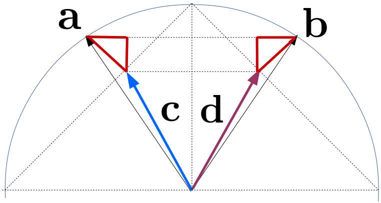
Optimisation – Step 3: symmetric constellation



$$\begin{aligned} d_p(\widetilde{\mathsf{C}},\mathsf{A}) + d_p(\widetilde{\mathsf{D}},\mathsf{B}) &\le \tfrac{1}{2}\left[d_p(\mathsf{C},\mathsf{A}) + d_p(\mathsf{C}',\mathsf{A})\right] + \tfrac{1}{2}\left[d_p(\mathsf{D},\mathsf{B}) + d_p(\mathsf{D}',\mathsf{B})\right] \\ &= d_p(\mathsf{C},\mathsf{A}) + d_p(\mathsf{D},\mathsf{B}) \end{aligned}$$

since $d_p(\mathsf{C}',\mathsf{A}) = d_p(\mathsf{D},\mathsf{B})$ and $d_p(\mathsf{D}',\mathsf{B}) = d_p(\mathsf{C},\mathsf{A})$

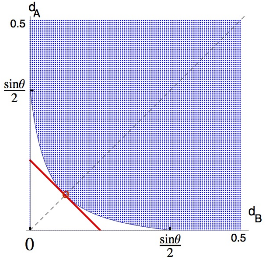
Optimisation – Step 4: optimal constellation

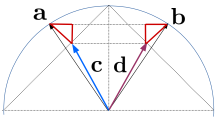


Constraint: compatibility $\frac{1}{2}|\boldsymbol{c}+\boldsymbol{d}|+\frac{1}{2}|\boldsymbol{c}-\boldsymbol{d}|=1$

$$\begin{aligned} d_p(\mathsf{C},\mathsf{A})+d_p(\mathsf{D},\mathsf{B}) &= \tfrac{1}{2}|\boldsymbol{c}-\boldsymbol{a}|+\tfrac{1}{2}|\boldsymbol{d}-\mathbf{b}| \\ &= \sqrt{2}\left[\tfrac{1}{2}|\boldsymbol{a}+\mathbf{b}|-\tfrac{1}{2}|\boldsymbol{c}+\boldsymbol{d}|\right] = \sqrt{2}\left[\tfrac{1}{2}|\boldsymbol{a}-\mathbf{b}|-\tfrac{1}{2}|\boldsymbol{c}-\boldsymbol{d}|\right] \\ &= \frac{1}{2\sqrt{2}}\left[\,|\boldsymbol{a}+\mathbf{b}|+|\boldsymbol{a}-\mathbf{b}|-2\,\right] \end{aligned}$$

Main Result 1: A Simple Qubit MUR



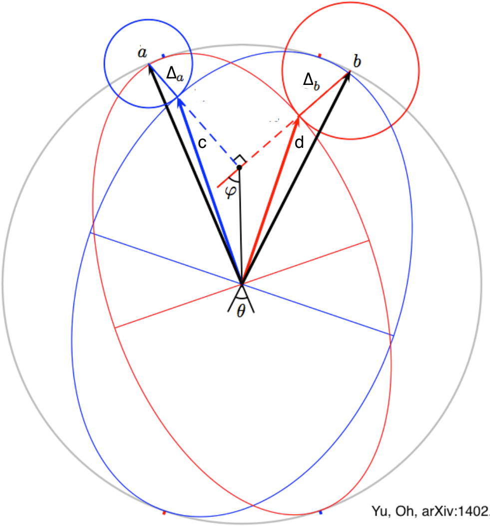


$$\sin\theta = |\boldsymbol{a} \times \mathbf{b}|$$

$$\begin{aligned} |\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| &\leq 2 \\ U(\mathrm{C})^2 \times U(\mathrm{D})^2 &\geq 4\|[C_+, D_+]\|^2 \\ d_p(\mathrm{C},\mathrm{A}) + d_p(\mathrm{D},\mathrm{B}) &\geq \tfrac{1}{2\sqrt{2}}\left[\,|\boldsymbol{a}+\mathbf{b}| + |\boldsymbol{a}-\mathbf{b}| - 2\,\right] \end{aligned}$$

$$|\boldsymbol{a}+\mathbf{b}| + |\boldsymbol{a}-\mathbf{b}| = 2\sqrt{1+|\boldsymbol{a}\times\mathbf{b}|} = 2\sqrt{1+2\|[A_+, B_+]\|}$$

Qubit Measurement Uncertainty: Tight Boundary Curve



Yu, Oh, arXiv:1402.3785

Qubit Measurement Uncertainty: Boundary Curve

PB & T Heinosaari (2008), S Yu and CH Oh (2014)

Optimiser, case $\boldsymbol{a} \perp \mathbf{b}$:

$\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}, \quad \boldsymbol{d} = |\boldsymbol{d}|\mathbf{b},$

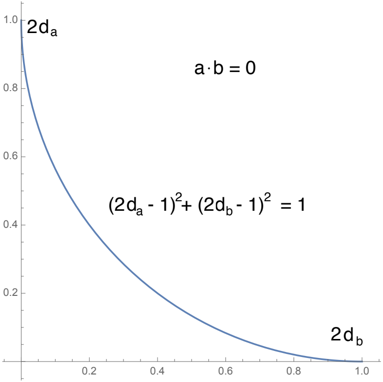
$2d_a = |\boldsymbol{a} - \boldsymbol{c}| = 1 - |\boldsymbol{c}|,$

$2d_b = |\mathbf{b} - \boldsymbol{d}| = 1 - |\boldsymbol{d}|,$

Compatibility constraint:

$|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1, \quad i.e., \; U(\mathrm{C})^2 + U(\mathrm{D})^2 = 1$

$(1 - 2d_a)^2 + (1 - 2d_b)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$



Optimal Qubit Measurement Uncertainty: Proof

Task: minimise $|\mathbf{b} - \boldsymbol{d}|$
subject to $|\boldsymbol{a} - \boldsymbol{c}|$ fixed and $|\boldsymbol{c} + \boldsymbol{d}| + |\boldsymbol{c} - \boldsymbol{d}| = 2$

$$\nabla_{\boldsymbol{c}} F = 0, \quad \nabla_{\boldsymbol{d}} F = 0$$

where $F = |\mathbf{b} - \boldsymbol{d}| + \lambda |\boldsymbol{a} - \boldsymbol{c}| + \mu \left(|\boldsymbol{c} + \boldsymbol{d}| + |\boldsymbol{c} - \boldsymbol{d}| \right)$

$$\begin{aligned}
\boldsymbol{a} - \boldsymbol{c} &\sim \frac{\boldsymbol{c} + \boldsymbol{d}}{|\boldsymbol{c} + \boldsymbol{d}|} + \frac{\boldsymbol{c} - \boldsymbol{d}}{|\boldsymbol{c} - \boldsymbol{d}|} \\
\mathbf{b} - \boldsymbol{d} &\sim \frac{\boldsymbol{c} + \boldsymbol{d}}{|\boldsymbol{c} + \boldsymbol{d}|} - \frac{\boldsymbol{c} - \boldsymbol{d}}{|\boldsymbol{c} - \boldsymbol{d}|} \\
&\implies (\boldsymbol{a} - \boldsymbol{c}) \cdot (\mathbf{b} - \boldsymbol{d}) = 0 \\
(\text{Case } \boldsymbol{a} \perp \mathbf{b} :) \quad &\implies \boldsymbol{c} = |\boldsymbol{c}| \boldsymbol{a} \ \perp \ \boldsymbol{d} = |\boldsymbol{d}| \mathbf{b}
\end{aligned}$$

Optimal Qubit Measurement Uncertainty: Proof completed

Hence, optimising joint measurement is given by

$$\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a} \;\perp\; \boldsymbol{d} = |\boldsymbol{d}|\mathbf{b}$$

and the compatibility constraint becomes: $|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$.

Considering that

$$2d_a = |\boldsymbol{a} - \boldsymbol{c}| = 1 - |\boldsymbol{c}|, \quad 2d_b = |\mathbf{b} - \boldsymbol{d}| = 1 - |\boldsymbol{d}|$$

this translates into the following equation for the optimal boundary curve of the admissible joint measurement error region:

$$(2d_a - 1)^2 + (2d_b - 1)^2 = 1$$

and the region itself is given by

$$(2d_a - 1)^2 + (2d_b - 1)^2 \leq 1 \quad \text{or } d_a \geq \tfrac{1}{2} \quad \text{or } d_b \geq \tfrac{1}{2}$$

Result: Qubit Measurement Uncertainty – Admissible Region

PB & T Heinosaari (2008), S Yu and CH Oh (2014)

Optimiser, case $\boldsymbol{a} \perp \mathbf{b}$:

$\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}, \quad \boldsymbol{d} = |\boldsymbol{d}|\mathbf{b},$

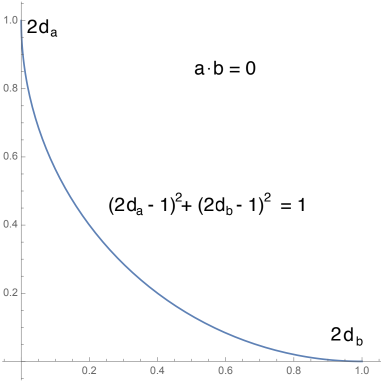
$2d_a = |\boldsymbol{a} - \boldsymbol{c}| = 1 - |\boldsymbol{c}|,$

$2d_b = |\mathbf{b} - \boldsymbol{d}| = 1 - |\boldsymbol{d}|,$

Compatibility constraint:

$|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1, \quad i.e., \; U(\mathrm{C})^2 + U(\mathrm{D})^2 = 1$

$(1 - 2d_a)^2 + (1 - 2d_b)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$



A lucky (?) accident

- The same optimiser configuration for $\boldsymbol{c}, \boldsymbol{d}$ also realises the tight boundary of an inequality due to C Branciard (2013), which is a refinement of Ozawa's inequality.
- Branciard's inequality has been confirmed experimentally.
- Hence these tests also confirm our qubit joint measurement error region.

More on this in tomorrow's lecture ...

Connection between MUR and PUR (for Qubits)

analogy with position-momentum case

Case $\boldsymbol{a} \perp \mathbf{b}$, optimal approximators $\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}$, $\boldsymbol{d} = |\boldsymbol{d}|\mathbf{b}$, $|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$:

Heisenberg-Weyl covariant joint observable

$$
\begin{aligned}
G_{k,\ell} &= \tfrac{1}{4}\left[I + (k\boldsymbol{c} + \ell\boldsymbol{d})\cdot\boldsymbol{\sigma}\right] \\
G_{++} &= I\, G_{++} I^*, \\
G_{+-} &= (\boldsymbol{a}\cdot\boldsymbol{\sigma})\, G_{++}\, (\boldsymbol{a}\cdot\boldsymbol{\sigma})^*, \\
G_{-+} &= (\mathbf{b}\cdot\boldsymbol{\sigma})\, G_{++}\, (\mathbf{b}\cdot\boldsymbol{\sigma})^*, \\
G_{--} &= (\boldsymbol{a}\times\mathbf{b}\cdot\boldsymbol{\sigma})\, G_{++}\, (\boldsymbol{a}\times\mathbf{b}\cdot\boldsymbol{\sigma})^* \\
G_{++} &= \tfrac{1}{4}\left[I + (\boldsymbol{c} + \boldsymbol{d})\cdot\boldsymbol{\sigma}\right] = \tfrac{1}{2}\rho_0 = \tfrac{1}{4}(I + \boldsymbol{r}_0\cdot\boldsymbol{\sigma})
\end{aligned}
$$

$$U_g\, G_{k,\ell}\, U_g^* = G_{(k,\ell)g} \quad (\textit{covariance})$$

Connection between MUR and PUR (for Qubits)

Margins of covariant G

$$
\begin{aligned}
G_{k,\ell} &= \tfrac{1}{4}\left[I + (k\boldsymbol{c} + \ell\boldsymbol{d})\cdot\boldsymbol{\sigma}\right] \\
C_k &= G_{k,+} + G_{k,-} = \tfrac{1}{2}(I + k\boldsymbol{c}\cdot\boldsymbol{\sigma}) \\
D_\ell &= G_{+,\ell} + G_{-,\ell} = \tfrac{1}{2}(I + \ell\boldsymbol{d}\cdot\boldsymbol{\sigma})
\end{aligned}
$$

These realise the **optimal error bound** since

$$2d_a = |\boldsymbol{a} - \boldsymbol{c}| = 1 - |\boldsymbol{c}|, \quad 2d_b = |\mathbf{b} - \boldsymbol{d}| = 1 - |\boldsymbol{d}|$$

and therefore

$$(2d_a - 1)^2 + (2d_b - 1)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$$

Connection between MUR and PUR

$$
\begin{aligned}
(2d_a-1)^2+(2d_b-1)^2 &= 1 \quad \text{minimal errors}\\
&\Updownarrow\\
|\boldsymbol{c}|^2+|\boldsymbol{d}|^2 &= 1 \quad \text{compatibility bound}\\
&\Updownarrow\\
\left(1-|\boldsymbol{c}|^2\right)\ +\ \left(1-|\boldsymbol{d}|^2\right) &= 1 \quad \text{minimal unsharpness}\\
&\Updownarrow\ \ G_{++}=\tfrac{1}{2}\rho_0, \quad \boldsymbol{r}_0=\boldsymbol{c}+\boldsymbol{d}\\
\left(\Delta_{\rho_0}A\right)^2\ +\ \left(\Delta_{\rho_0}B\right)^2 &= 1 \quad \text{minimal uncertainty}
\end{aligned}
$$

Summary so far

Main Result 2: Qubit MUR according to QM

$$(\text{joint measurement errors for } \mathsf{A}, \mathsf{B}) \geq (\text{incompatibility of } \mathsf{A}, \mathsf{B})$$
$$(\text{unsharpness of } \textit{compatible } \mathsf{C}, \mathsf{D}) \geq (\text{noncommutativity of } \mathsf{C}, \mathsf{D})$$

Preparation uncertainty enforces measurement uncertainty!

Shown here for qubit observables.

MUR – Experiments

Measurement Noise and Weak Valued Probabilities

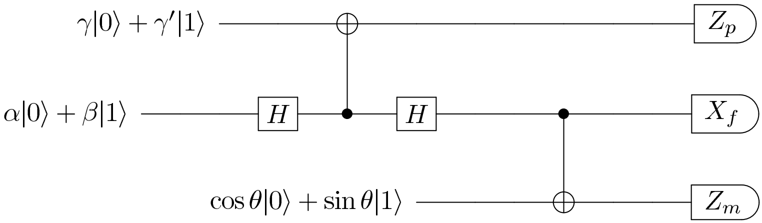
Lund & Wiseman (NJP 2010)

$$
\begin{aligned}
\varepsilon(\mathsf{C},\mathsf{A};\rho)^2 &= \left\langle (Z_\tau - A)^2 \right\rangle_{\rho\otimes\sigma} \\
&= \ \left\langle \mathsf{C}[2] - \mathsf{C}[1]^2 \right\rangle_\rho + \left\langle (\mathsf{C}[1] - A)^2 \right\rangle_\rho \\
&= \iint (x-y)^2 \mathrm{Re}\,\mathrm{tr}\left[\rho \mathsf{A}(dx)\mathsf{C}(dy)\right] \\
&=_{com,disc} \ \sum_{k,\ell} (a_k - a_\ell)^2 \mathrm{tr}\left[\rho A_k C_\ell\right]
\end{aligned}
$$

bona fide probability if A, C commute

Lund-Wiseman weak measurement scheme

$A = Z = \sigma_3$, C =a smeared version of $A = Z$,
$B = X = \sigma_1$, B$'$ = D =a smeared version of $B = X$



Determination of $\eta(X)$.
Top wire: probe; bottom wire: measuring system;
middle wire: observed qubit.
The value of $\eta(X)$ can be extracted from the joint distribution of the initial and final X measurements, obtained by reading the outputs Z_p and X_f.

First realised by Toronto group (Rozema *et al*, PRL 2012).

Weak measurement scheme (continued)

Marginal observables:

$$Z_p: \quad P_\pm = \tfrac{1}{2}\left[\, I \pm (2\gamma^2 - 1)X \,\right]$$

$$\gamma \to \tfrac{1}{\sqrt{2}} : \text{weak measurement limit}$$

$$\gamma = 1: \quad P_\pm = B_\pm = \tfrac{1}{2}(I \pm X)$$

$$X_f: \quad D_\pm = \tfrac{1}{2}\left[\, \mathbb{I} \pm \sin(2\theta)X \,\right]$$

$$Z_m: \quad C_\pm = \tfrac{1}{2}\left[\, \mathbb{I} \pm 2\gamma\gamma' \cos(2\theta)Z \,\right]$$

Use operational probabilities $P_{k,\ell} = P(Z_p = k, X_f = \ell)$ to determine "weak-valued probabilities" a la Lund, Wiseman:

$$2P_{WV}(\delta X = \pm 2) = P_{1,\pm 1} + P_{-1,\pm 1} \mp \frac{P_{1,\pm 1} - P_{-1,\pm 1}}{2\gamma^2 - 1}$$

$$\eta(X)^2 = \textstyle\sum_{\delta x} (\delta x)^2 P_{WV}(\delta x) = 2 - 2\sin(2\theta)$$

Weak measurement vs strong measurement

But: No need to use weak valued probabilities as B, D commute!

$$\sum_{k,\ell}(x_k - x_\ell)^2 P_{k,\ell} = 4P(Z_p = +1, X_f = -1) + 4P(Z_p = -1, X_f = +1)$$
$$= 2 - 2\sin(2\theta)(2\gamma^2 - 1)$$

$\gamma = 1$ (strong measurement):

$$\eta(X)^2 = 4P(Z_p = +1, X_f = -1) + 4P(Z_p = -1, X_f = +1) = 2 - 2\sin(2\theta)$$

value comparison error

Conclusion

Summary

- MURs can be rigorously formalised and proven
- Care has to be taken with the definition of error measure to ensure reliable identification of optimal joint measurements
- In qubit case, measurement noise $(\varepsilon_a, \varepsilon_b)$ and probabilistic distances (d_a, d_b) give almost consistent descriptions of optimal joint measurements
- Experimental tests of MURs for $(\varepsilon_a, \varepsilon_b)$ also confirm MURs for (d_a, d_b)

Outlook

- Much remains to be investigated, *e.g.*, alternative measures of error
- Largely outstanding: generic MURs (obtained by T Miyadera (PRA 2011) for finite-dimensional systems, finite observables)
- Possible applications of MURs: *e.g.*, quantum metrology

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