A categorical framework for the quantum harmonic oscillator

Jamie Vicary
Imperial College London
jamie.vicary@imperial.ac.uk

Based on arxiv.org/abs/0706.0711

Categories, Logic and the Foundations of Physics
Imperial College London
9 January 2008

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- ‘Exotic’ Fock spaces
- Where does the ‘quantumness’ come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- ‘Exotic’ Fock spaces
- Where does the ‘quantumness’ come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- ‘Exotic’ Fock spaces
- Where does the ‘quantumness’ come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- ‘Exotic’ Fock spaces
- Where does the ‘quantumness’ come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- ‘Exotic’ Fock spaces
- Where does the ‘quantumness’ come from?

Categorical quantum mechanics

Traditionally	Categorically
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$

Duals?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

$$\boxed{\textbf{Symmetric monoidal } \dagger\textbf{-category } \mathbf{C} \textbf{ with } \dagger\textbf{-biproducts}}$$

Duals?

The quantum harmonic oscillator

Particle in an n-dimensional quadratic potential

State space is symmetric Fock space:

$$F(A) := \mathbb{C} \oplus A \oplus (A \otimes_s A) \oplus (A \otimes_s A \otimes_s A) \oplus \ldots$$

Manipulated with raising and lowering operators, for $\phi : I \longrightarrow A$:

$$a_\phi : F(A) \longrightarrow F(A) \qquad a_\phi^\dagger : F(A) \longrightarrow F(A)$$

Canonical commutation relations:

$$a_\phi \circ a_\psi = a_\psi \circ a_\phi, \quad a_\phi^\dagger \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi^\dagger, \quad a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\psi^\dagger \circ \phi) \cdot \mathrm{id}_{F(A)}$$

Carries a natural commutative monoid structure
[Blute, Panangaden and Seely, 1994]

Natural isomorphisms: $F(A \oplus B) \simeq F(A) \otimes F(B)$

The quantum harmonic oscillator

Particle in an n-dimensional quadratic potential

State space is symmetric Fock space:

$$F(A) := \mathbb{C} \oplus A \oplus (A \otimes_s A) \oplus (A \otimes_s A \otimes_s A) \oplus \dots$$

Manipulated with raising and lowering operators, for $\phi : I \longrightarrow A$:

$$a_\phi : F(A) \longrightarrow F(A) \qquad a^\dagger_\phi : F(A) \longrightarrow F(A)$$

Canonical commutation relations:

$$a_\phi \circ a_\psi = a_\psi \circ a_\phi, \quad a^\dagger_\phi \circ a^\dagger_\psi = a^\dagger_\psi \circ a^\dagger_\phi, \quad a_\phi \circ a^\dagger_\psi = a^\dagger_\psi \circ a_\phi + (\psi^\dagger \circ \phi) \cdot \mathrm{id}_{F(A)}$$

Carries a natural commutative monoid structure
[Blute, Panangaden and Seely, 1994]

Natural isomorphisms: $F(A \oplus B) \simeq F(A) \otimes F(B)$

The quantum harmonic oscillator

Particle in an n-dimensional quadratic potential

State space is symmetric Fock space:

$$F(A) := \mathbb{C} \oplus A \oplus (A \otimes_s A) \oplus (A \otimes_s A \otimes_s A) \oplus \dots$$

Manipulated with raising and lowering operators, for $\phi : I \longrightarrow A$:

$$a_\phi : F(A) \longrightarrow F(A) \qquad a_\phi^\dagger : F(A) \longrightarrow F(A)$$

Canonical commutation relations:

$$a_\phi \circ a_\psi = a_\psi \circ a_\phi, \quad a_\phi^\dagger \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi^\dagger, \quad a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\psi^\dagger \circ \phi) \cdot \mathrm{id}_{F(A)}$$

Carries a natural commutative monoid structure
[Blute, Panangaden and Seely, 1994]

Natural isomorphisms: $F(A \oplus B) \simeq F(A) \otimes F(B)$

The quantum harmonic oscillator

Particle in an n-dimensional quadratic potential

State space is symmetric Fock space:

$$F(A) := \mathbb{C} \oplus A \oplus (A \otimes_s A) \oplus (A \otimes_s A \otimes_s A) \oplus \dots$$

Manipulated with raising and lowering operators, for $\phi : I \longrightarrow A$:

$$a_\phi : F(A) \longrightarrow F(A) \qquad a_\phi^\dagger : F(A) \longrightarrow F(A)$$

Canonical commutation relations:

$$a_\phi \circ a_\psi = a_\psi \circ a_\phi, \quad a_\phi^\dagger \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi^\dagger, \quad a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\psi^\dagger \circ \phi) \cdot \mathrm{id}_{F(A)}$$

Carries a natural commutative monoid structure

[Blute, Panangaden and Seely, 1994]

Natural isomorphisms: $F(A \oplus B) \simeq F(A) \otimes F(B)$

The quantum harmonic oscillator

Particle in an n-dimensional quadratic potential

State space is symmetric Fock space:

$$F(A) := \mathbb{C} \oplus A \oplus (A \otimes_s A) \oplus (A \otimes_s A \otimes_s A) \oplus \dots$$

Manipulated with raising and lowering operators, for $\phi : I \longrightarrow A$:

$$a_\phi : F(A) \longrightarrow F(A) \qquad a_\phi^\dagger : F(A) \longrightarrow F(A)$$

Canonical commutation relations:

$$a_\phi \circ a_\psi = a_\psi \circ a_\phi, \quad a_\phi^\dagger \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi^\dagger, \quad a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\psi^\dagger \circ \phi) \cdot \mathrm{id}_{F(A)}$$

Carries a natural commutative monoid structure

[Blute, Panangaden and Seely, 1994]

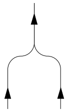
Natural isomorphisms: $F(A \oplus B) \simeq F(A) \otimes F(B)$

Internal commutative monoids

Category of internal commutative monoids $\mathbf{C}_+$

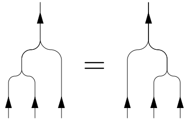
Multiplication and unit morphisms

Denoted $(A, g, u)_+$

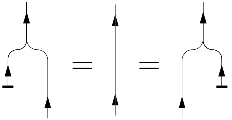


$g : A \otimes A \longrightarrow A$

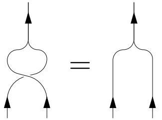
$u : I \longrightarrow A$



Associativity law



Unit laws



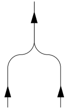
Commutativity law

Initial object	$I_+ := (I, \lambda_I, \mathrm{id}_I)_+$	Coproducts
Terminal object	$0_+ := (0, 0_{0\otimes 0,0}, \mathrm{id}_0)_+$	Products

Internal commutative monoids

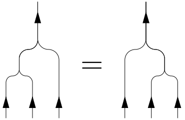
Category of internal commutative monoids $\mathbf{C}_+$

Multiplication and unit morphisms

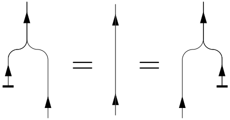


Denoted $(A, g, u)_+$

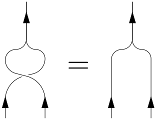
$g : A \otimes A \longrightarrow A$ $\qquad$ $u : I \longrightarrow A$



Associativity law



Unit laws



Commutativity law

Initial object $\quad I_+ := (I, \lambda_I, \mathrm{id}_I)_+ \quad$ Coproducts

Terminal object $\quad 0_+ := (0, 0_{0\otimes 0,0}, \mathrm{id}_0)_+ \quad$ Products

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, bend left, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \xrightarrow{\cdot} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \xrightarrow{\cdot} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, bend left, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \overset{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \overset{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, bend left, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \overset{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \overset{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q\text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R\text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \overset{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \overset{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left=2, "Q \text{ free}"] \arrow[r, phantom, "\top"] & \mathbf{C}_\times \arrow[l, shift left=2, "R \text{ forgetful}"] \end{tikzcd}$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Preserving products unitarily

Unique natural isomorphisms induced in $\mathbf{C}_\times$:

$$k_{A,B} : Q(A \oplus B) \longrightarrow Q(A) \times Q(B)$$
$$k_0 : Q(0) \longrightarrow I_\times$$

Require $Rk_{A,B}$, Rk_0 unitary in $\mathbf{C}$:

$$(Rk_{A,B})^\dagger = Rk_{A,B}^{-1} \qquad (Rk_0)^\dagger = Rk_0^{-1}$$

Preserving products unitarily

Unique natural isomorphisms induced in $\mathbf{C}_\times$:

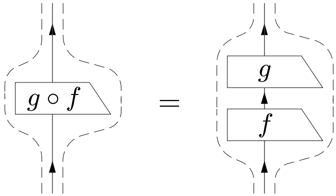
$$\begin{aligned} k_{A,B} &: Q(A \oplus B) \longrightarrow Q(A) \times Q(B) \\ k_0 &: Q(0) \longrightarrow I_\times \end{aligned}$$

Require $Rk_{A,B}$, Rk_0 unitary in $\mathbf{C}$:

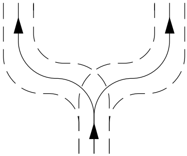
$$(Rk_{A,B})^\dagger = Rk_{A,B}^{-1} \qquad\qquad (Rk_0)^\dagger = Rk_0^{-1}$$

Graphical representations for F, e, d

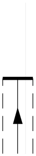
$F(A)$



$F(g) \circ F(f) \;=\; F(g \circ f)$



$d_A : F(A) \longrightarrow F(A) \otimes F(A)$

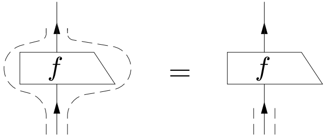


$e_A : \; F(A) \; \longrightarrow I$

Graphical representation for ϵ, η

$\epsilon_A : F(A) \longrightarrow A$

Naturality

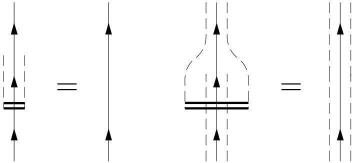


$$\epsilon_A \circ F(f) \quad = \quad f \circ \epsilon_A$$

$R\eta_{(A,g,u)_\times}$

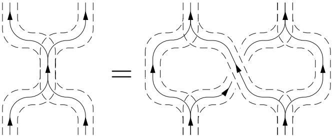
$R\eta_{(A,g,u)_\times}$:
$A \longrightarrow F(A)$

Adjunction equations

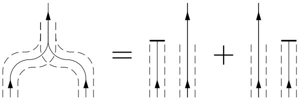


Some emergent properties...

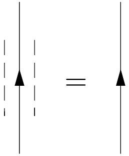
Bialgebra identity



$\epsilon \circ d^\dagger$ identity



ϵ normalised



(cheated)

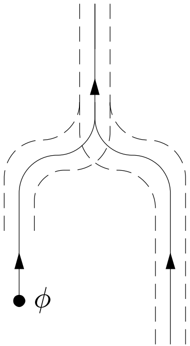
e normalised

$= \mathrm{id}_I$

e, ϵ orthogonal

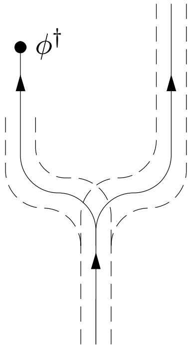
$= 0_{A,I}$

Raising and lowering operators



Raising morphism

$a_\phi^\dagger : F(A) \longrightarrow F(A)$

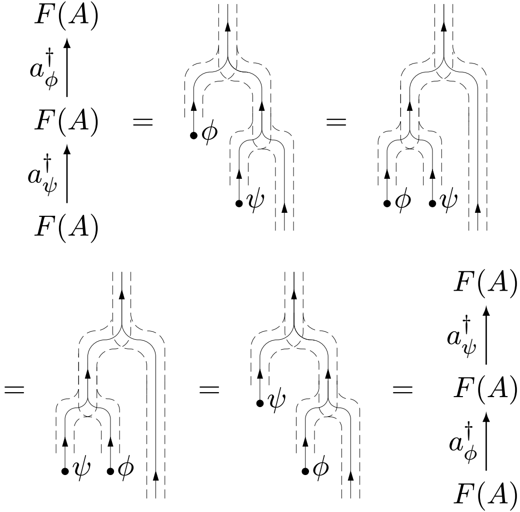


Lowering morphism

$a_\phi : F(A) \longrightarrow F(A)$

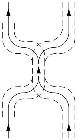
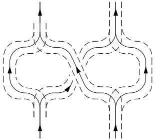
Canonical commutator

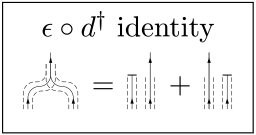
$$a_\phi^\dagger \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi^\dagger$$

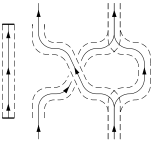
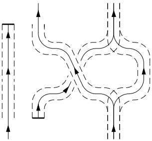
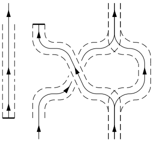
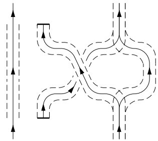


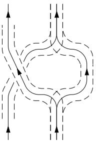
Canonical commutator

$$a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\phi^\dagger \circ \psi) \cdot \mathrm{id}_{F(A)}$$

$a_A \circ a_A^\dagger$ =  = 

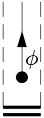


=  +  +  + 

=  + 0 + 0 +  = $a_A^\dagger \circ a_A + \mathrm{id}_A \otimes \mathrm{id}_{F(A)}$

Coherent states

single-particle state $\phi : I \longrightarrow A$ becomes multi-particle state $\mathrm{Coh}(\phi) : I \longrightarrow F(A)$



Employ $R\eta_{I_\times} : I \longrightarrow F(I)$ here

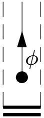
Correspond to *points* $\mathrm{Hom}_{\mathbf{C}_\times}\big(I_\times, Q(A)\big)$

Have classical properties:

- Can be copied:
  $$d_A \circ \mathrm{Coh}(\phi) = \mathrm{Coh}(\phi) \otimes \mathrm{Coh}(\phi)$$
- Can be deleted:
  $$e_A \circ \mathrm{Coh}(\phi) = \mathrm{id}_I$$
- Unchanged by lowering operator:
  $$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$

Coherent states

single-particle state $\phi : I \longrightarrow A$ becomes multi-particle state $\mathrm{Coh}(\phi) : I \longrightarrow F(A)$



Employ $R\eta_{I_\times} : I \longrightarrow F(I)$ here

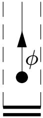
Correspond to *points* $\mathrm{Hom}_{\mathbf{C}_\times}\big(I_\times, Q(A)\big)$

Have classical properties:

- Can be copied:
  $$d_A \circ \mathrm{Coh}(\phi) = \mathrm{Coh}(\phi) \otimes \mathrm{Coh}(\phi)$$
- Can be deleted:
  $$e_A \circ \mathrm{Coh}(\phi) = \mathrm{id}_I$$
- Unchanged by lowering operator:
  $$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$

Coherent states

single-particle state $\phi : I \longrightarrow A$ becomes multi-particle state $\mathrm{Coh}(\phi) : I \longrightarrow F(A)$



Employ $R\eta_{I_\times} : I \longrightarrow F(I)$ here

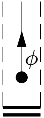
Correspond to *points* $\mathrm{Hom}_{\mathbf{C}_\times}\big(I_\times, Q(A)\big)$

Have classical properties:

- Can be copied:
  $$d_A \circ \mathrm{Coh}(\phi) = \mathrm{Coh}(\phi) \otimes \mathrm{Coh}(\phi)$$
- Can be deleted:
  $$e_A \circ \mathrm{Coh}(\phi) = \mathrm{id}_I$$
- Unchanged by lowering operator:
  $$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$

Coherent states

single-particle state $\phi : I \longrightarrow A$ becomes multi-particle state $\mathrm{Coh}(\phi) : I \longrightarrow F(A)$



Employ $R\eta_{I_\times} : I \longrightarrow F(I)$ here

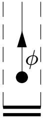
Correspond to *points* $\mathrm{Hom}_{\mathbf{C}_\times}\big(I_\times, Q(A)\big)$

Have classical properties:

- Can be copied:
  $$d_A \circ \mathrm{Coh}(\phi) = \mathrm{Coh}(\phi) \otimes \mathrm{Coh}(\phi)$$
- Can be deleted:
  $$e_A \circ \mathrm{Coh}(\phi) = \mathrm{id}_I$$
- Unchanged by lowering operator:
  $$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$

Coherent states

single-particle state $\phi : I \longrightarrow A$ becomes multi-particle state $\mathrm{Coh}(\phi) : I \longrightarrow F(A)$



Employ $R\eta_{I_\times} : I \longrightarrow F(I)$ here

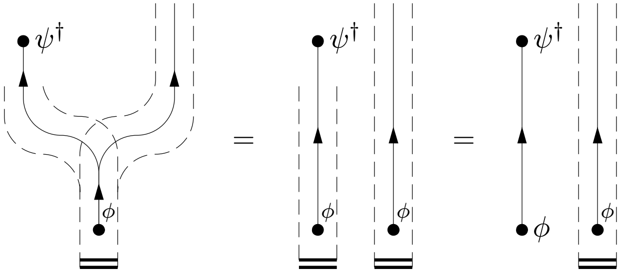
Correspond to *points* $\mathrm{Hom}_{\mathbf{C}_\times}\big(I_\times, Q(A)\big)$

Have classical properties:

- Can be copied:
  $$d_A \circ \mathrm{Coh}(\phi) = \mathrm{Coh}(\phi) \otimes \mathrm{Coh}(\phi)$$
- Can be deleted:
  $$e_A \circ \mathrm{Coh}(\phi) = \mathrm{id}_I$$
- Unchanged by lowering operator:
  $$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$

Coherent state is eigenstate of a_ψ

$$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$



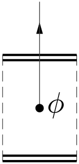
Morphism exponentials

Construct an exponential

$$\exp_{(A,g,u)_+}(\phi)$$

from a state $\phi : I \longrightarrow A$ and a commutative monoid $(A, g, u)_+$

Intuition: $\exp_{(A,g,u)_+}(\phi) = \frac{1}{0!} \cdot u + \frac{1}{1!} \cdot \phi + \frac{1}{2!} \cdot g \circ (\phi \otimes \phi) + \cdots$



employing

$$(R\eta_{(A,g^\dagger,u^\dagger)_\times})^\dagger : F(A) \longrightarrow A$$
$$R\eta_{I_\times} : I \longrightarrow F(I)$$

Has the following familiar properties:

- Additivity: $g \circ \big(\exp(\phi) \otimes \exp(\psi)\big) = \exp(\phi + \psi)$
- Unit: $\exp_{(A,g,u)_+}(0_{I,A}) = u$

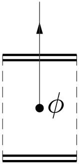
Morphism exponentials

Construct an exponential

$$\exp_{(A,g,u)_+}(\phi)$$

from a state $\phi : I \longrightarrow A$ and a commutative monoid $(A, g, u)_+$

Intuition: $\exp_{(A,g,u)_+}(\phi) = \frac{1}{0!} \cdot u + \frac{1}{1!} \cdot \phi + \frac{1}{2!} \cdot g \circ (\phi \otimes \phi) + \cdots$



employing

$$(R\eta_{(A,g^\dagger,u^\dagger)_\times})^\dagger : F(A) \longrightarrow A$$
$$R\eta_{I_\times} : I \longrightarrow F(I)$$

Has the following familiar properties:

- Additivity: $g \circ \big(\exp(\phi) \otimes \exp(\psi)\big) = \exp(\phi + \psi)$
- Unit: $\exp_{(A,g,u)_+}(0_{I,A}) = u$

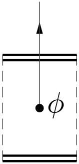
Morphism exponentials

Construct an exponential

$$\exp_{(A,g,u)_+}(\phi)$$

from a state $\phi : I \longrightarrow A$ and a commutative monoid $(A, g, u)_+$

Intuition: $\exp_{(A,g,u)_+}(\phi) = \frac{1}{0!} \cdot u + \frac{1}{1!} \cdot \phi + \frac{1}{2!} \cdot g \circ (\phi \otimes \phi) + \cdots$



employing

$$(R\eta_{(A,g^\dagger,u^\dagger)_\times})^\dagger : F(A) \longrightarrow A$$
$$R\eta_{I_\times} : I \longrightarrow F(I)$$

Has the following familiar properties:

- Additivity: $g \circ \big(\exp(\phi) \otimes \exp(\psi)\big) = \exp(\phi + \psi)$
- Unit: $\exp_{(A,g,u)_+}(0_{I,A}) = u$

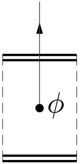
Morphism exponentials

Construct an exponential

$$\exp_{(A,g,u)_+}(\phi)$$

from a state $\phi : I \longrightarrow A$ and a commutative monoid $(A, g, u)_+$

Intuition: $\exp_{(A,g,u)_+}(\phi) = \frac{1}{0!} \cdot u + \frac{1}{1!} \cdot \phi + \frac{1}{2!} \cdot g \circ (\phi \otimes \phi) + \cdots$



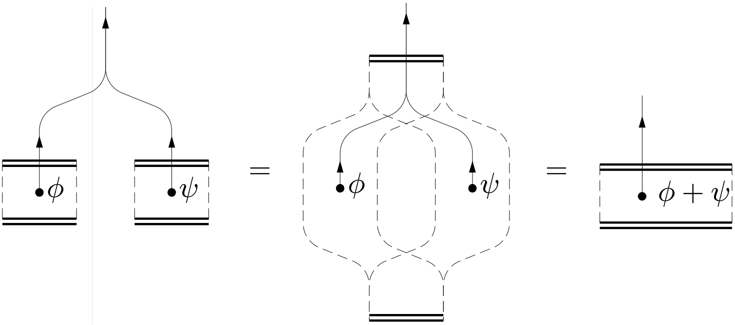
employing

$$(R\eta_{(A,g^\dagger,u^\dagger)_\times})^\dagger : F(A) \longrightarrow A$$
$$R\eta_{I_\times} : I \longrightarrow F(I)$$

Has the following familiar properties:

- Additivity: $g \circ \big(\exp(\phi) \otimes \exp(\psi)\big) = \exp(\phi + \psi)$
- Unit: $\exp_{(A,g,u)_+}(0_{I,A}) = u$

Proof of additivity



$\mathbf{C} = \mathbf{Hilb}$?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
Unbounded operators don't always compose!

Suggested solution:
Use inner-product spaces, not Hilbert spaces
Allows a well-behaved set of unbounded operators

Duals? Needed for *operator exponentials.*

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

$\mathbf{C} = \mathbf{Hilb}$?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
Unbounded operators don't always compose!

Suggested solution:
Use inner-product spaces, not Hilbert spaces
Allows a well-behaved set of unbounded operators

Duals? Needed for *operator exponentials.*

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

C = Hilb?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
Unbounded operators don't always compose!

Suggested solution:
Use inner-product spaces, not Hilbert spaces
Allows a well-behaved set of unbounded operators

Duals? Needed for *operator exponentials.*

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

C = Hilb?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
Unbounded operators don't always compose!

Suggested solution:
Use inner-product spaces, not Hilbert spaces
Allows a well-behaved set of unbounded operators

Duals? Needed for *operator exponentials*.

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

C = Hilb?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
Unbounded operators don't always compose!

Suggested solution:
Use inner-product spaces, not Hilbert spaces
Allows a well-behaved set of unbounded operators

Duals? Needed for *operator exponentials*.

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

$\mathbf{C} = \mathbf{Hilb}$?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
 Unbounded operators don't always compose!

Suggested solution:
 Use inner-product spaces, not Hilbert spaces
 Allows a well-behaved set of unbounded operators

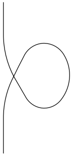
Duals? Needed for *operator exponentials*.

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

What about antisymmetric Fock space?

Need to work in the correct category:

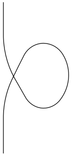


$$= \quad (-1)$$

The category encodes the statistics!

What about antisymmetric Fock space?

Need to work in the correct category:



$$= \quad (-1)$$

The category encodes the statistics!

Exotic Fock spaces

Assume all objects built from 'building blocks' $A, B, \ldots, C$
(semisimple category, other conditions)

Typical example: supergroupoid representation category
(almost)

Then $K := A \oplus B \oplus \cdots \oplus C$ is the *generating object*
Canonical commutative comonoid structure $K_\times$ on K!

Vectors in an object X correspond to $\mathrm{Hom}_{\mathbf{C}}(K, X)$

Using adjunction,

$$\mathrm{Hom}_{\mathbf{C}}\big(K, X\big) \simeq \mathrm{Hom}_{\mathbf{C}_\times}\big(K_\times, F(X)\big)$$

i.e., states of $X \simeq$ coherent states of $F(X)$

Where does the ‘quantumness’ come from?

Philosophy:

Make everything †-compatible

Nontrivial fact: the

bialgebra structure of the biproduct

gives rise to the

nontrivial commutation relation!

Where does the 'quantumness' come from?

Philosophy:

Make everything †-compatible

Nontrivial fact: the

bialgebra structure of the biproduct

gives rise to the

nontrivial commutation relation!

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- ‘Exotic’ Fock spaces
- Where does the ‘quantumness’ come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)

Summary

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states and exponentials
- A category of Hilbert spaces?
- 'Exotic' Fock spaces
- Where does the 'quantumness' come from?

(Online at `arxiv.org/abs/0706.0711`)