Categories and nonassociative C*-algebras in Quantum Field Theory

Keith Hannabuss (Oxford)

Categories, Logic, and Foundations of Physics,
Computing Laboratory, Oxford,

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Outline

1. Quantum mechanics and C*-algebras.
2. Gel’fand’s Theorem and the Dixmier-Douady obstruction.
3. Twisted compact operators.
4. T-duality.
5. Monoidal categories.
The observables generate an algebra of operators on a Hilbert space $\mathcal{H}$, closed under addition, multiplication, and adjoints.

One can restrict to bounded operators $B(\mathcal{H})$:

$$\|A\| = \sup\{\|A\psi\| : \|\psi\| = 1\} < \infty$$

A C*-algebra is a norm-closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. 

Example: the compact operators $\mathcal{K}(\mathcal{H})$ are the $C^*$-algebra generated by the rank-one operators in $\mathcal{B}(\mathcal{H})$, i.e. those of the form

$$\langle \xi \rangle \langle \eta \rangle : \psi \mapsto \xi \langle \eta, \psi \rangle.$$ 

If $\mathcal{H} = L^2(X)$, (normalisable wave functions on $X$, the compact operators $K$ can be represented as integral operators

$$(K \psi)(x) = \int_X K(x, y) \psi(y) \, dy$$

where the integral kernel $K(x, y)$ is a limit of separable kernels

$$K_N(x, y) = \sum_{j=1}^N \alpha_j(x) \beta_j(y).$$
The composition is then

\[(K_1 \circ K_2)(x, z) = \int_X K_1(x, y)K_2(y, z) \, dy\]

The adjoint is

\[K^*(x, y) = \overline{K(y, x)}\]

cf matrices when integral is replaced by a sum.
Example: function algebras

Take $E$ a locally compact Hausdorff topological space with a measure $\mu$, let $\mathcal{H} = L^2(E, \mu)$, multiplication by compactly supported, continuous, complex-valued functions $f \in C_K(E)$

$$(f.\psi)(x) = f(x)\psi(x)$$

for $x \in E$, $\psi \in L^2(E, \mu)$ gives a subalgebra of $B(\mathcal{H})$, with

$$(f_1 \circ f_2)(x) = f_1(x)f_2(x), \quad f^*(x) = \overline{f(x)}$$
Every commutative C*-algebra is $C_K(E)$ for some locally compact Hausdorff space $E$, and

the category of commutative C*-algebras is contravariantly equivalent to the category of locally compact Hausdorff spaces, via the functors

$$
\text{spec}(A) \rightarrow A \\
E \leftarrow C_K(E)
$$

where the spectrum of $A$

$\text{spec}(A) =$ equivalence classes of irreducible representations $\sim$ maximal ideals.
What if $\mathcal{A}$ is not commutative?

There is a broader class, the continuous trace $C^*$-algebras which are given by algebra-valued functions over the spectrum.

Continuous trace $C^*$-algebra $\mathcal{A} \sim$ sections of $K(\mathcal{H})$-bundle over spectrum $E = \text{spec} \mathcal{A}$ (equivalence classes of irreducible representations).

The bundle structure is trivial if and only if the Dixmier–Douady obstruction $\delta \in H^2(E, T) \cong H^3(E, \mathbb{Z})$ is trivial, (Brauer 1927, . . ., Dixmier–Douady 1964)
Dixmier–Douady Theorem.

For every such $E$ and $\delta \in H^3(E, \mathbb{Z})$ there is a $C^*$-algebra $\mathcal{A} = CT(E, \delta)$ with spectrum $E$ and Dixmier–Douady obstruction $\delta$, and it is unique up to Morita equivalence.
Dixmier–Douady Theorem.

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Algebras $A_1$ and $A_2$ are Morita equivalent if there is an additive equivalence between the categories of $A_1$-modules and $A_2$-modules.

Raeburn, Kumjian, Muhly, Renault, Williams: groupoid $\mathbb{C}^*$ algebra proof.
Morita Equivalence

Theorem (Morita–Rieffel) For each additive equivalence from $A_1$-modules to $A_2$-modules there exists a left $A_2$-right $A_1$-bimodule $E$ such that the equivalence is given by $E \otimes_{A_1} \cdot$.

The compact operators $\mathcal{K}(\mathcal{H})$ are Morita equivalent to $\mathbb{C}$ via the left $\mathcal{K}(\mathcal{H})$-right $\mathbb{C}$-bimodule $\mathcal{H}$.

(Uniquenes of the Canonical Commutation Relations)
Summary

loc. cpt Hausdorff space $\leftrightarrow$ comm. C$^*$-algebra

cpt Hausdorff space $\leftrightarrow$ comm. C$^*$-algebra with 1

$E$ $\rightarrow$ $C_0(E)$

spectrum $\text{spec}(A)$ $\leftarrow$ algebra $A$

noncommutative geometry $\leftrightarrow$ continuous trace C$^*$-algebra

flux $H \in H^3(E, \mathbb{Z})$ $\leftrightarrow$ DD class $\delta \in H^2(E, \mathbb{T})$
Twisted compact operators

Take the same integral operators $\mathcal{K}(L^2(X))$, but with a composition

$$(K_1 \ast K_2)(x, z) = \int_X \frac{K_1(x, y)K_2(y, z)}{\phi(x, y, z)} \, dy$$

for a scalar function $\phi : X \times X \times X \to U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$.

Problem this is not generally associative:

$$(K_1 \ast K_2) \ast K_3 \neq K_1 \ast (K_2 \ast K_3)$$

unless $\phi(x, y, z)\phi(x, z, w) = \phi(x, y, w)\phi(y, z, w)$.
Einstein's principle of General Covariance:

Physical theories should be completely invariant under coordinate transformations.

Quantum Field Theory and String Theory:

Symmetries extend to phase space.

configuration space $\mathbb{R}^D \leftrightarrow$ momentum space $\hat{\mathbb{R}}^D$
String theory (Hull and Townsend)

T-duality: Momentum and winding number interchange

Added ingredient: flux $H \in H^3(E)$
T-duality interchanges two principal torus bundles over the same base, and interchanges the curvature of each with the $H$-flux ($H \in \Omega^3(E)$ or $\tilde{H} \in \Omega^3(\tilde{E})$) of the other.

\[
\begin{align*}
T &: (E, H) \quad \text{dual} \\
\tilde{T} &: (\tilde{E}, \tilde{H}) \\
M &= E/T \cong \tilde{E}/\tilde{T} \\
T &= T^k = \mathbb{R}^k/\mathbb{Z}^k \cong \tilde{T}
\end{align*}
\]
It will suffice to consider $E = T^3$, $T = T^k$ ($k \leq 3$), and $M = T^{3-k}$, with $H$ $k$ times the volume 3-form.
**Known geometric dual principal torus bundles**

\[ H = H_3 + H_2 + H_1 + H_0 \text{ where } H_p \in \Omega^p(M, \wedge^{3-p} \mathfrak{k}) \]

with \( \mathfrak{k} \) the dual of the Lie algebra of \( T \).

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Example

$T^3$ with volume form: volume is generated by $a, b, c \in \mathbb{R}^3$ is $[a, b, c] = a \cdot b \times c$ for $a, b, c \in \mathbb{R}^3$.

- as a $T$-bundle over $T^2$: $H = H_2$ geometric dual;
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- as a $\mathbb{T}^2$-bundle over $\mathbb{T}$: $H = H_1$: noncommutative dual
  (Mathai–Rosenberg 2004)
- as a $\mathbb{T}^3$ bundle over a point: $H = H_0$: nonassociative dual (Bouwknegt, KCH, Mathai 2005,6).
Gel’fand’s theorem allows one to replace $E$ by a C*-algebra.

The exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

gives an isomorphism $H^3(E, \mathbb{Z}) \cong H^2(E, \mathbb{T})$:

Identify $H$ with the Dixmier–Douady class $\delta$ and replace $(E, H)$ by $CT(E, \delta)$. 
Now consider a principal $T = G/N$-bundle $E$ over $M$.

Does $G$ act as automorphisms of $CT(E, \delta)$?

Suppose $\alpha : G \to \text{Aut}(A)$, and the same subgroup $N$ stabilises each irreducible.

Then $E = \text{spec}(A)$ is a $T = G/N$-bundle over $M = \text{spec}(A)/G$.

If $G = \mathbb{R}$ every principal $G/N$-bundle arises in this way, but for general groups $G$ this is not always true.
Crossed product $\mathcal{A} \rtimes G = C_0(G, \mathcal{A})$

$$(f \ast g)(x) = \int_G f(y)\alpha_y[g(y^{-1}x)] \, dy, \quad f^*(x) = \alpha_x[f(x^{-1})]^*$$

**Facts.**

1. Under suitable assumptions $\hat{\mathcal{A}} = \mathcal{A} \rtimes G$ is also a continuous trace algebra with an action of the dual group $\hat{G}$;

2. (Takai-Takesaki duality) $\hat{\mathcal{A}} \rtimes \hat{G} \cong \mathcal{A} \otimes K(L^2(G)) \sim_{M} \mathcal{A}$. 
\( \hat{T} \) is isomorphic to the group-theoretic dual of \( N \).

Connes’ Thom isomorphism theorem: \( K_*(A \rtimes \mathbb{R}^D) \cong K_{*+D}(A) \).
\[ \hat{T} \text{ is isomorphic to the group-theoretic dual of } N. \]

Connes' Thom isomorphism theorem: \[ K_*(\mathcal{A} \rtimes \mathbb{R}^D) \cong K_{*+D}(\mathcal{A}). \]
Puzzle:

$H_0 \neq 0$ never seems to show up in $C^*$-algebra literature.

There are spaces with any $H$, so problem must lie with group action.
Inner automorphisms act trivially on spectrum.

\[ G \to \text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A}) \]

Lift to \( \alpha : G \to \text{Aut}(\mathcal{A}) \): \( \alpha_x \alpha_y = \text{ad}(u(x, y)) \alpha_{xy} \)
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**Lift to** \( \alpha : G \rightarrow \text{Aut}(A) \): \( \alpha_x \alpha_y = \text{ad}(u(x, y)) \alpha_{xy} \)

\[ \text{ad}(u(x, y)) \text{ad}(u(xy, z)) \alpha_{(xy)z} = \text{ad}(\alpha_x [u(y, z)]) \text{ad}(u(x, yz)) \alpha_{xz} \]
Nonassociative case

Inner automorphisms act trivially on spectrum.

\[ G \to \text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A}) \]

Lift to \( \alpha : G \to \text{Aut}(\mathcal{A}) \): \( \alpha_x \alpha_y = \text{ad}(u(x, y))\alpha_{xy} \)

\[
\text{ad}(u(x, y))\text{ad}(u(xy, z))\alpha_{(xy)z} = \text{ad}(\alpha_x[u(y, z)])\text{ad}(u(x, yz))\alpha_{x(yz)}
\]

\[
\phi(x, y, z)u(x, y)u(xy, z) = \alpha_x[u(y, z)]u(x, yz)
\]
Properties of $\phi$

Central, and satisfies pentagonal cocycle identity

$$\phi(x, y, z)\phi(x, yz, w)\phi(y, z, w) = \phi(xy, z, w)\phi(x, y, zw)$$

$\phi$ is independent of liftings up to coboundaries

$$\eta(x, y)\eta(xy, z)/\eta(y, z)\eta(x, yz)$$

so only $H^3(G, T)$ class of $\phi$ matters (but cf. Majid)

$$\phi(\exp(X), \exp(Y), \exp(Z)) = \exp(iH_0(\xi_X, \xi_Y, \xi_Z))$$

where $\xi_X$ is vector field generated by $X$.

For $T^3$ with $k \times \text{vol}$: $\phi(a, b, c) = \exp(2\pi ik[a, b, c])$
†-Category $\mathcal{C}_G$ of $\hat{G}$-modules $\sim C_0(G)$-modules with $G$-morphisms, and

- module tensor product ($f \in C_0(G)$ acting via comultiplication
  $(\Delta f)(x, y) = f(xy)$), † action multiplies by $f^*(x) = f(x)$;

- identity object: trivial module

- associator $\Phi : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C \sim \text{the action of } \phi \in C(G \times G \times G) = C(G) \otimes C(G) \otimes C(G)$.
\[ C_\hat{G} \text{-modules} \sim C_0(G)\text{-modules with } G\text{-morphisms, and} \]

- module tensor product \((f \in C_0(G))\) acting via comultiplication
  \((\Delta f)(x, y) = f(xy)\), \(\hat{\cdot}\) action multiplies by \(f^* = f\);

- identity object: trivial module \(\mathbb{C}\) (action by the counit \(\epsilon(f) = f(1)\)).
\[\dagger\]-Category \(C_G\) of \(\hat{G}\)-modules \(\sim C_0(G)\)-modules with \(G\)-morphisms, and

- module tensor product \((f \in C_0(G)\) acting via comultiplication
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- associator \(\Phi : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \sim \text{the action of} \)
  \[\phi \in C(G \times G \times G) = C(G) \otimes C(G) \otimes C(G).\]
The pentagonal identity for $\phi$ gives

$$\xymatrix{ A \otimes (B \otimes (C \otimes D)) & \rightarrow & (A \otimes B) \otimes (C \otimes D) \\
A \otimes ((B \otimes C) \otimes D) & & ((A \otimes B) \otimes C) \otimes D \\
& (A \otimes (B \otimes C)) \otimes D }$$
**Algebras**

**Def.** An algebra in \( C_G \) is an object \( \mathcal{A} \) with a morphism \( \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) consistent with \( \Phi \):

\[
\begin{array}{ccc}
\mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) & \longrightarrow & \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \\
\Phi & \downarrow & & \downarrow \\
(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A}
\end{array}
\]

The action of \( \hat{G} \) is automatically by automorphisms.

\( C_G \) is a star/bar/dagger category and so one can also define \( C^* \)-algebras and Hilbert spaces in \( C_G \).
• Torus bundle $\mathbb{T}^3$ over a point, with $H_0 = k\text{vol}$  
Associated antisymmetric form on $a, b, c \in t = \mathbb{R}^3$ is then given by  
\[
\phi(a, b, c) = \exp(-2\pi i f(a, b, c)) = \exp(-2k\pi i [a, b, c])
\]
• Torus bundle $\mathbb{T}^3$ over a point, with $H_0 = k\text{vol}$
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• $A_0 = \mathbb{C}, G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \sim \{0, 1\}^3 \subseteq \mathbb{R}^3$
  $\phi(a, b, c) = (-1)^{[a, b, c]}$
  suitable $u$ gives the octonions (cf Albuquerque and Majid).
Magnetic translations $T_a = \exp(2\pi i a \cdot \nabla)$

$T_a$ and $T_b$ fail to commute by a factor $\exp(\pi i \Phi)$, where $\Phi$ is the flux through face spanned by $a$ and $b$.

$T_a$, $T_b$, and $T_c$ fail to associate by a factor $\exp(\pi i \Phi)$, where $\Phi$ is the flux out of the tetrahedron spanned by $a$, $b$, and $c$.

Dirac’s monopole argument.

But cf. Carey/Mickelsson
DEF. An $\mathcal{A}$-module in $\mathcal{C}_G$ is an object $M$ with a morphism $\mathcal{A} \otimes M \rightarrow M$ consistent with $\Phi$:

\[
\begin{array}{ccc}
\mathcal{A} \otimes (A \otimes M) & \rightarrow & A \otimes M & \rightarrow & M \\
\Phi \downarrow & & \downarrow \\
(A \otimes A) \otimes M & \rightarrow & A \otimes M & \rightarrow & M
\end{array}
\]

The actions of $\mathcal{A}$ and $\hat{G}$ on $M$ are automatically consistent in that $g[am] = (g[a])(g[m])$, for all $a \in \mathcal{A}$ and $m \in M$, that is one has a covariant representation of $(\hat{G}, \mathcal{A})$, which is really a representation of $\mathcal{A} \rtimes \hat{G}$. 
Let $\mathcal{H}$ be a Hilbert space, as right $\mathbb{C}$-module (with $\mathbb{C}$ the identity object).

Use the Rieffel construction to obtain rank-one operators

$$|\xi\rangle\langle\eta|\zeta = \Phi(\xi, \eta, \zeta)$$

These rank-one operators generate the twisted compact operators having $\mathcal{H}$ as a module.
When $\mathcal{H} = L^2(X)$ these twisted compact operators can be represented as integral operators with product

$$(K_1 * K_2)(x, z) = \int_X \frac{K_1(x, y)K_2(y, z)}{\phi(x, y, z)} \, dy$$
The twisted bounded operators

In the usual case the bounded operators can be characterised as the adjointable operators.

Now an adjointable operator $A$ is one for which there exists

$A^* : \xi \rightarrow A^*\xi \equiv \xi A$ satisfying

$$\langle A^*\xi, \eta \rangle \equiv \langle \xi A, \eta \rangle = \Phi(\langle \xi, A\eta \rangle)$$