A Programming Problem

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Problem Description

Gödel's T

Types:

$$\begin{array}{lcll} \tau & ::= & \mathtt{nat} & \text{naturals} \\ & \mid & \tau_1 \to \tau_2 & \text{functions} \end{array}$$

Expressions:

$$\begin{array}{lcll} e & ::= & x & \text{variable} \\ & \mid & \mathtt{z} & \text{zero} \\ & \mid & \mathtt{s}(e) & \text{successor} \\ & \mid & \mathtt{rec}[\tau](e; e_0; x\,.\,y\,.\,e_1) & \text{recursor} \\ & \mid & \lambda(x{:}\tau.\,e) & \text{lambda} \\ & \mid & e_1(e_2) & \text{application} \end{array}$$

Judgements:

$$\begin{array}{ll} \Gamma \vdash e : \tau & \text{Typing Judgement} \\ \Gamma \vdash e_1 \equiv e_2 : \tau & \text{Maximal Consistent Congruence} \end{array}$$

Definability in T

A function $F : \mathbb{N} \to \mathbb{N}$ is *definable* in **T** iff there exists a term e_F of type $\mathtt{nat} \to \mathtt{nat}$ such that $F(m) = n$ iff $e_F(\overline{m}) \equiv \overline{n}$.

Theorem 1 (Gödel)**.** *The functions definable in **T** are those provable total in **HA**.*

Proof. Normalization proof is formalizable in **HA**. Totality proofs in **HA** can be erased to terms in **T**. □

Using Gödel-numbering and diagonalization one may exhibit a function that is *not* definable in **T**.

An Undefinable Function

For an expression e of **T**, let $\ulcorner e \urcorner \in \mathbb{N}$ be the Gödel-number of e.

Let the function $E : \mathbb{N} \to \mathbb{N}$ be such that if e is a closed term of type $\mathtt{nat} \to \mathtt{nat}$, then $E(\ulcorner e \urcorner) = n$ iff $e(\overline{\ulcorner e \urcorner}) \equiv \overline{n}$.

Theorem 2. *The function E is not definable in **T**.*

Proof. Suppose e_E defines E, and let $e_D = \lambda(x{:}\mathtt{nat}.\,\mathtt{s}(e_E(x)))$. We have

$$e_D(\overline{\ulcorner e_D \urcorner}) \equiv \mathtt{s}(e_E(\overline{\ulcorner e_D \urcorner})) \tag{1}$$

$$\equiv \mathtt{s}(e_D(\overline{\ulcorner e_D \urcorner})). \tag{2}$$

This contradicts consistency of equivalence in **T**. □

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Theorem 4. *The function E is not definable in* ***T****.*

Proof. Suppose e_E defines E, and let $e_D = \lambda(x{:}\mathtt{nat}.\,\mathtt{s}(e_E(x)))$. We have

$$e_D(\overline{\ulcorner e_D \urcorner}) \equiv \mathtt{s}(e_E(\overline{\ulcorner e_D \urcorner})) \quad (1)$$

$$\equiv \mathtt{s}(e_D(\overline{\ulcorner e_D \urcorner})). \quad (2)$$

This contradicts consistency of equivalence in **T**. □

Corollary 5. *The function E is not provably total in* ***HA****.*

Definability in F

Theorem 6. *The function E is provably total in **HA**$_2$.*

Proof. Essentially, can comprehend all possible computability predicates in order to account for all possible programs. □

Definability in F

Theorem 9. *The function E is provably total in* ***HA***$_2$.

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Theorem 10 (Girard)**.** *A function on the natural numbers is definable in System* ***F*** *iff it is provably total in* ***HA***$_2$.

Corollary 11. *The function E is definable in System* ***F***.

Definability in F

Theorem 12. *The function E is provably total in **HA**$_2$.*

Proof. Essentially, can comprehend all possible computability predicates in order to account for all possible programs. □

Theorem 13 (Girard)**.** *A function on the natural numbers is definable in System **F** iff it is provably total in **HA**$_2$.*

Corollary 14. *The function E is definable in System **F**.*

This raises an interesting programming problem . . . .

The Problem

Give an explicit definition of the function E in System **F**.

In other words, define an evaluator for Gödel's **T** in Girard's **F**.

This seems to be a hard problem!

1. The evaluator must be *manifestly total*, in accordance with Girard's Theorem.
2. The implicit proof of its totality must encompass *all possible* proofs of termination formalizable in (first-order) **HA**.

Some Guidelines

You may use any sort of term representation you'd like, as long as it's obvious that it can be Church-encoded. That is, you are permitted to use inductively defined types in **F**.

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You may use a lexicographic extension of structural induction to any finite number of places. That is, may use a nested structural induction in which the outer induction dominates the inner induction.

Any characterization of equivalence in **T** sufficient for definability of computations of type `nat` is acceptable. You need not prove that it is the maximal consistent congruence.

Partial Credit

Partial credit will be awarded for solutions to any of these problems:

1. Show that E is definable in **Agda** or **Coq**, using dependent types and large eliminations to define families of types indexed by an inductive type.
2. Show that the analogue of E for simply typed λ-calculus with Booleans is definable in System **F**.

The first may or may not be "on track" for a full-credit solution, but the second definitely is.