The Chocolate Game

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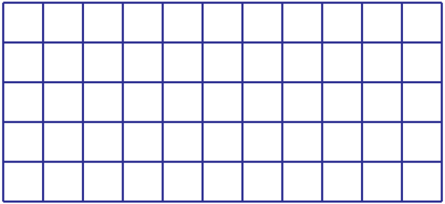

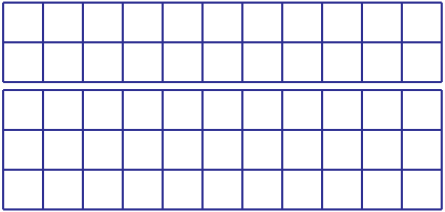

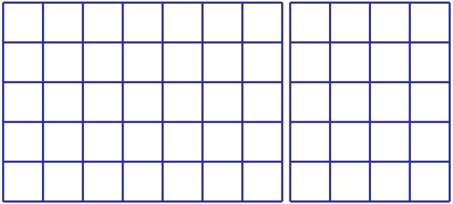

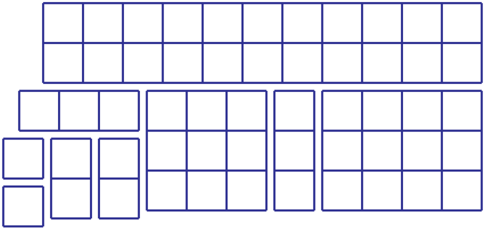

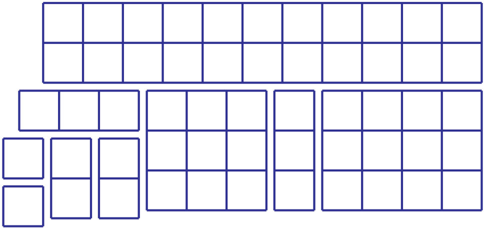

$$\begin{array}{ccccccccc} (11,5) & \mapsto & (11,3) & \mapsto & (7,3) & \mapsto & (6,3) & \mapsto & (3,3) \\ & \mapsto & (3,2) & \mapsto & (2,2) & \mapsto & (1,2) & \mapsto & (1,1) \end{array}$$

Impartial Two-Person Games

Winning and losing

- The chocolate game is an example of an

  impartial two-person game.

- A game is fixed by a set of positions and a set of moves.

$$\text{move} :: \text{Pos} \longrightarrow [\text{Pos}]$$

- The two players take it in turn to make a move.
- The games ends when it is not possible to make a move.
- The player whose turn it is loses.

Winning and losing positions

- A position is a *losing position* iff every move leads to a winning position.
- From a *winning position* there is at least one move to a losing position.

Sum games

- The chocolate game is an example of a *sum game*.
- It consists of two components: the left and the right game.
- A move consists in making a move
  - either in the left or
  - in the right game.
- A position in the combined game is a pair of positions.

Sprague-Grundy numbers

- *Idea:* assign a natural number to each component position so that (i, j) is a losing position iff $sg\ i = sg\ j$.
- Every move from a losing position makes the numbers unequal.
- For every winning position there is a move that makes them equal.

$$\mathrm{sg}\ p \quad = \mathrm{mex}\ \{\mathrm{sg}\ q \mid q \leftarrow \mathrm{move}\ p\}$$

$$\mathrm{mex}\ x = \mathrm{head}\ \langle n \mid n \leftarrow \mathrm{nat}, n \notin x \rangle$$

$$\text{sg } i = (\text{frac})_i \quad \textbf{where} \quad \text{frac} = \text{nat} \curlyvee \text{frac}$$

References

- R. Sprague. *Über mathematische Kampfspiele*. Tohoku Mathematical Journal 41:438–444.
- P.M. Grundy. *Mathematics and games*. Eureka 2:6–8, 1939. Reprinted, 27:9–11, 1964, see also `http://www.archim.org.uk/eureka/27/games.html`.
- P.M. Grundy and C.A.B Smith. *Disjunctive games with the last player losing*. Proc. Cambridge Philos. Soc., 52:527–533, 1956.
- R. Backhouse and D. Michaelis. *A Calculational Presentation of Impartial Two-Person Games*. Abstract presented at RelMiCS8, 2005.