Exercise Sheet 1 for Categories, Proofs and Games: HT 2004

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Question 1

Consider the following properties of an arrow f in a category $\mathcal{C}$.

- f is *monic* (*i.e.* a monomorphism) if for all arrows g and h (with domaims and codomains such that the following equations make sense)

$$f \circ g = f \circ h \implies g = h.$$

- f is *epic* if

$$g \circ f = h \circ f \implies g = h.$$

- f is *iso* if for some g, both $f \circ g$ and $g \circ f$ are identity arrows.
- f is *split monic* if for some g, $g \circ f$ is an identity arrow.
- f is *split epic* if for some g, $f \circ g$ is an identity arrow.

(a) Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.

(b) Prove that, if f is split epic, then it is epic.

(c) Prove that, if f and $g \circ f$ are iso, then g is iso.

(d) Prove that, if f is monic and split epic, then it is iso.

(e) Show that, in the category **Set**, an arrow is epic if and only if it is surjective.

Question 2

Show that $i : A \longrightarrow B$ has at most one inverse $j : B \longrightarrow A$ making it an isomorphism. This justifies writing i^{-1} for *the* inverse of an isomorphism i.

Question 3

Identify initial and terminal objects in the categories **Set**, **Mon**, $\mathbf{Vect}_k$, **Pos**. What does it mean to be an initial/terminal object in a preorder?

Question 4

Repeat the previous exercise with respect to the notion of product.

Question 5

Show that an A, B–pairing $(A \times B, \pi_1, \pi_2)$ is a product if and only if for every A, B–pairing (C, f, g) there is a morphism

$$\langle f, g \rangle : C \longrightarrow A \times B$$

such that

$$\pi_1 \circ \langle f, g \rangle = f, \qquad \pi_2 \circ \langle f, g \rangle = g$$

and for all $h : C \longrightarrow A \times B$:

$$\langle \pi_1 \circ h, \pi_2 \circ h \rangle = h.$$

(The main point is to show that the last equation is equivalent to asking for uniqueness explicitly).

Question 6

Explicitly dualize the notion of product (the dual notion is called *coproduct*). Repeat exercise 3 with respect to coproducts.