Exercise Sheet 2 for Categories, Proofs and Games

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Question 1

Suppose that a category $\mathcal{C}$ has a product $A \times B$ for each pair of objects A, B.

(a) Show that $A \times (B \times C)$ is the product of A, B and C (in the sense of general products defined in the notes). Extend your argument to prove by induction that $\mathcal{C}$ has products for all finite (non-empty) families of objects.

(b) Show that the definition of the product implies the following: for any morphisms $f, g : C \longrightarrow A \times B$,

$$\pi_1 \circ f = \pi_1 \circ g \text{ and } \pi_2 \circ f = \pi_2 \circ g \quad \Rightarrow \quad f = g.$$

(c) Using the previous part, show that:

$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle.$$

(d) Defining $f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$ show that this is functorial, *i.e.*

$$(f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k) \qquad \mathsf{id}_A \times \mathsf{id}_B = \mathsf{id}_{A \times B}.$$

(e) Define a family of morphisms

$$a_{A,B,C} : A \times (B \times C) \longrightarrow (A \times B) \times C$$

and show that it is a natural isomorphism.

Question 2

Let the functors $F, G, H : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$ be defined by

$$F(X,Y) = X \times Y, \qquad G(X,Y) = Y \times X \qquad H(X,Y) = X.$$

(a) Show that the only natural transformation $p : F \Longrightarrow H$ is the projection

$$p_{X,Y} : (x,y) \mapsto x.$$

(Hint: given any (X,Y) and $x \in X$, $y \in Y$, find an instance (X_0, Y_0) and maps $f : X_0 \longrightarrow X$, $g : Y_0 \longrightarrow Y$ which you can use via naturality to 'force' $p_{X,Y}(x,y)$ to be x).

(b) Similarly, show that the only natural transformation $s : F \Longrightarrow G$ is the symmetry

$$s_{X,Y} : (x,y) \mapsto (y,x).$$

A harder question - Optional

Question 3

Consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

(a) Show that if both 'small' squares A, B, D, E and B, C, E, F are pullbacks, so is the 'large' outer square A, C, D, F.

(b) Show that if the right-hand 'small' square and the large outer square are pullbacks, so is the left-hand 'small' square.