Exercise Sheet 4 for Categories, Proofs and Games

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1. Give Natural Deduction proofs of the following sequents.

   - $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$
   - $\vdash (A \supset (A \supset B)) \supset (A \supset B)$
   - $\vdash (C \supset A) \supset ((C \supset B) \supset (C \supset (A \wedge B)))$
   - $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

   In each case, give the corresponding λ-term.

2. For each of the following λ-terms, find a type for it. Try to find the 'most general' type, built from 'type variables' α, β etc. For example, the most general type for the identity $\lambda x.\, x$ is $\alpha \to \alpha$. (It is 'most general' in the sense that any other type you could give this term would be of the form $T \to T$, and hence arise by substituting T for α. Of course, any type variable would do in place of α.) In each case, give the derivation of the type for this term in the System of Simple Types (where you may assume that types can be built up from type variables as well as base types).

   - $\lambda f.\, \lambda x.\, fx$
   - $\lambda x.\, \lambda y.\, \lambda z.\, x(yz)$
   - $\lambda x.\, \lambda y.\, \lambda z.\, xzy$
   - $\lambda x.\, \lambda y.\, xyy$
   - $\lambda x.\, \lambda y.\, x$
   - $\lambda x.\, \lambda y.\, \lambda z.\, xz(yz)$

   Reflect a little on the methods you used to do this exercise. Could they be made algorithmic?

3. Can you type the following terms?

   - $\lambda x.\, xx$
   - $\lambda f.\, (\lambda x.\, f(xx))(\lambda x.\, f(xx))$

   Discuss.

Harder: do these if you have time

Subject Reduction

Using the Lemma on the Admissibility of the Cut Rule proved in the notes, show that the *Subject Reduction property* holds: if we can derive $\Gamma \vdash t : A$, and $t \twoheadrightarrow u$ (*i.e.* u can be obtained from t by performing a number of β-reductions), then we can derive $\Gamma \vdash u : A$. Use firstly induction on the number of β-reductions used in getting from t to u, and then induction on the height of t (or equivalently, on the derivation of $\Gamma \vdash t : A$). The only case not taken care of by the induction hypothesis is when t is itself a redex — and in this case we can apply the Lemma.

Weak Normalization in simple type theory (Optional)

This exercise guides you through a proof of the important property that *every λ-term typable in Simple Types has a normal form.* Moreover, we will derive a bound on the size of the normal form. (The bound is huge, but necessarily so).

Firstly, it will be convenient to introduce a variant of the syntax of typed terms. We replace the rule for typing abstractions given in the notes by the following:

$$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x : U.\, t : U \to T}$$

Note that in this version, the type of each sub-term can be deduced unambiguously. For example, $\lambda x : T.\, x$ can only have type $T \to T$, while $\lambda x.\, x$ can have any type of the form $U \to U$.

Now recall the definition of the *height* of a term:

$$\mathsf{height}(x) = 1, \qquad \mathsf{height}(\lambda x : T.\, t) = \mathsf{height}(t)+1, \qquad \mathsf{height}(tu) = \max(\mathsf{height}(t), \mathsf{height}(u))+1.$$

Recall also the notion of *rank* of a type, $\rho(T)$, defined in the notes. The rank of a term of type T is defined to be the rank of T, while the rank of a redex $(\lambda x : T.\, t)u$ is defined to be the rank of $\lambda x : T.\, t$. The *degree* of a term, $\mathsf{degree}(t)$, is defined to be the maximum rank of any redex occuring in t. (We assume a typed term $t : T$ to be given with a derivation of $\Gamma \vdash t : T$, with free variables of t then having types as stipulated by Γ).

1. Suppose we have $\Gamma, x : T \vdash u : U$ and $\Gamma \vdash t : T$. Show that $\mathsf{height}(u[t/x]) \leq \mathsf{height}(t) + \mathsf{height}(u)$.

2. Prove that a term t such that $\mathsf{degree}(t) \leq d \geq 1$ and $\mathsf{height}(t) \leq h$ can be β-reduced to a term of degree $\leq d-1$ and height $\leq 2^h$. Use induction on $\mathsf{height}(t)$. The non-trivial case is when t is an application $t_1 t_2$. By induction hypothesis, we can reduce both t_1 (say to u_1)and t_2 (say to u_2). The non-trivial sub-case is when t_1 reduces to a λ-abstraction, $u_1 = \lambda x.\, v$, so that we have a 'new' redex. The key point is that when we substitute u_2 for y in v, the only new redexes which are formed have degree equal to that of u_1, which by induction hypothesis is at most $d-1$. We can use the previous part to estimate the height of the term resulting from the substitution.

3. Now define $e(m, n)$ by $e(m, 0) = m$, $e(m, n+1) = 2^{e(m,n)}$. Thus $e(m, n)$ is an exponential 'stack' of n 2's with an m at the top.

   Prove that a term of degree d and height h has a normal form of height bounded by $e(h, d)$. Use the previous part, and induction on the degree.