Exercise Sheet 5 for Categories, Proofs and Games

Samson Abramsky
Oxford University Computing Laboratory

1. Give proofs of the following sequents in Linear Logic:
   - $\vdash A \multimap A$
   - $A \multimap B, B \multimap C \vdash A \multimap C$
   - $\vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C)$
   - $A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C$
   - $A \otimes B \vdash B \otimes A$
2. Show carefully that **Rel**, the category of sets and *relations*, is a symmetric monoidal closed category, with the tensor given by cartesian product. Show that this is *not* the categorical product in **Rel**. Identify the categorical product and coproduct in **Rel**.
3. For each of the Linear Logic proofs which you constructed in Exercise 1:
   - Give the term corresponding to the proof.
   - Give the interpretation in **Rel** of this proof.
4. Can you construct proofs in Linear Logic of the following sequents?
   - $A \vdash A \otimes A$
   - $(A \otimes A) \multimap B \vdash A \multimap B$
   - $\vdash A \multimap (B \multimap A)$

   Discuss.
5. Show that the proof rules given for $\otimes$, $\multimap$ are *admissible* as rules for $\wedge$ and $\supset$ in the Natural Deduction system for these connectives. That is, show that, whenever we have proofs of the sequents $\Gamma \vdash A$ and $\Delta \vdash B$, we can construct a proof of the sequent $\Gamma, \Delta \vdash A \wedge B$ in the Natural Deduction system, and similarly for the other rules for $\otimes$ and $\multimap$. (Hint: it is useful firstly to show that Weakening is admissible in the Natural Deduction system; i.e. that whenever we have a proof of a sequent $\Gamma \vdash A$, we can construct a proof of the sequent $\Gamma, \Delta \vdash A$. To prove this, use complete induction on the 'size' of the proof of $\Gamma \vdash A$ (i.e. how many times rules were used to build the proof).)

Additional Exercises – if you have time

1. Give the categorical interpretation of the $\multimap$-left rule:

$$\frac{\Gamma \vdash A \qquad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}$$

2. Prove that, using the interpretation given on p. 47 of the notes on Curry-Howard and Linear Logic, each Natural Deduction proof can be interpreted by a Linear Logic proof.