Exercise Sheet 6 for Categories, Proofs and Games

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Given games A, B define $A\&B$ by

$$\begin{array}{rcl} M_{A\&B} & = & M_A + M_B \\ \lambda_{A\&B} & = & [\lambda_A, \lambda_B] \\ P_{A\&B} & = & \{\mathtt{inl}^*(s) \mid s \in P_A\} \cup \{\mathtt{inr}^*(t) \mid t \in P_B\}. \end{array}$$

(Draw a picture of the game tree of $A\&B$; it is formed by gluing together the trees for A and B at the root. There is no overlap because we take the disjoint union of the alphabets.)

Also, define $A \oslash B$ (the 'left biassed tensor') which differs from $A \otimes B$ only in that the positions are restricted to those in which the *first* move is made in A. And define $A \multimap_s B$ (the 'strict linear function type') which differs from $A \multimap B$ only in that the positions are constrained so that the *second* move must be made in A.

1. Let A, B be games. Define a *game isomorphism* to be a function

$$\psi : P_A \longrightarrow P_B$$

such that:

- ψ is a bijection
- ψ is length-preserving: $|\psi(s)| = |s|$
- ψ is prefix-preserving: $s \sqsubseteq t \;\Rightarrow\; \psi(s) \sqsubseteq \psi(t)$.

A *strict game isomorphism* is a function

$$\phi : M_A \longrightarrow M_B$$

such that:

- ϕ is a bijection
- $\lambda_A = \lambda_B \circ \phi$
- For all $s \in M_A^*$: $s \in P_A \iff \phi^*(s) \in P_B$.

We say that A and B are isomorphic if there is an isomorphism between them, and strictly isomorphic if there is a strict isomorphism between them.

- Show that if ϕ is a strict game isomorphism, then ϕ^* is a game isomorphism.
- Show that two games may be isomorphic without being strictly isomorphic.
- Show that each of the following pairs of games is isomorphic. In each case, determine whether they are strictly isomorphic.
  (a) $A \cong A \otimes I$
  (b) $A \otimes B \cong B \otimes A$
  (c) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$
  (d) $I \multimap A \cong A$
  (e) $A \multimap I \cong I$
  (f) $(A \otimes B) \multimap C \cong A \multimap (B \multimap C)$
  (g) $A \multimap (B \& C) \cong (A \multimap B) \,\&\, (A \multimap C)$
  (h) $A \otimes B \cong (A \oslash B) \,\&\, (B \oslash A)$
  (i) $(A \oslash B) \oslash C \cong A \oslash (B \otimes C)$
  (j) $(A \oslash B) \multimap_s C \cong A \multimap_s (B \multimap C)$
  (k) $(A \multimap B) \multimap_s C \cong B \multimap_s (C \oslash A)$
- Do we have $A \cong A \otimes A$? $A \cong A \& A$?

2. Define a strategy for the game $(\mathbb{B} \otimes \mathbb{B}) \multimap \mathbb{B}$ which implements the operation of conjunction ('and') on truth-values. Is there more than one reasonable way of doing this? Discuss.

3. Now consider the 'second-order' type

$$(\mathbb{B} \multimap \mathbb{B}) \multimap \mathbb{B}$$

Define a strategy which returns `true` if its 'argument' (a strategy of type $\mathbb{B} \multimap \mathbb{B}$) returns an answer without inspecting its input; and `false` if its argument does inspect its input. Is there a mathematical function (mapping Boolean functions to Booleans) which this strategy represents? Discuss.