Exercise Sheet 7 for Game Semantics

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1 Composition examples

1. Do Exercise 1.5 from the Lecture Notes on Game Semantics (available on the course web page).
2. Prove that $\mathsf{not};\mathsf{not} = \mathsf{id}_{\mathbb{B}}$. (Here not is as in the previous exercise).

2 Parallel composition of strategies

Recall that composition of strategies

$$\sigma : A \longrightarrow B, \qquad\qquad \tau : B \longrightarrow C$$

is defined in terms of parallel composition and hiding:

$$\sigma;\tau \quad = \quad (\sigma \parallel \tau)/B$$

where

$$\sigma \parallel \tau = \{s \in (M_A + M_B + M_C)^* \mid s{\upharpoonright}A, B \in \sigma \ \wedge \ s{\upharpoonright}B, C \in \tau\}.$$

The purpose of these exercises is to study the structure of the parallel composition.

1. Give a transition diagram for $\sigma \parallel \tau$, similar to those given for the tensor product and linear implication. The 'states' will be of the form

   $$(\pi_1, \pi_2, \pi_3, \pi_4)$$

   where the π_i are polarities (P or O), and $\pi_2 = \bar{\pi}_3$. The idea is that (π_1, π_2) describes the situation from σ's point of view (playing in $A \multimap B$), while (π_3, π_4) describes the situation from τ's point of view (playing in $B \multimap C$). Thus the initial state will be (P, O, P, O). You should find that 4 distinct states are accessible in all.

2. Prove the following *Locality Property* of parallel composition: for all $s \in \sigma \parallel \tau$, for all $i : 1 \leq i < |s|$,

   $$|\mathrm{out}(s_i) - \mathrm{out}(s_{i+1})| \leq 1$$

   where

   $$\mathrm{out} : M_A + M_B + M_C \longrightarrow \{0, 1, 2\}$$

   maps moves in M_A to 0, moves in M_B to 1, and moves in M_C to 2.

3. Prove the following *parity property*: for all $smtn \in \sigma \parallel \tau$,
   - If m, n are in the same visible component (A or C), then the parity of $|t{\restriction}B|$ is even
   - If m, n are in different visible components, then the parity of $|t{\restriction}B|$ is odd.
4. The map $s \mapsto s{\restriction}A, C$ induces a surjective map

$$\psi : \sigma \parallel \tau \longrightarrow \sigma ; \tau$$

Covering Lemma. ψ is injective (and hence bijective) so for each $t \in \sigma;\tau$ there is a unique $s \in \sigma \parallel \tau$ such that $s{\restriction}A, C = t$.

If $t = m_1 m_2 .... m_k$, then s has the form

$$m_1 u_1 m_2 u_2 .... u_{k-1} m_k$$

where $u_i \in M_B^*$, $1 \leq i < k$.

Prove the Covering lemma.

Optional: try these if you have time

3 Verifying the categorical structure of $\mathcal{G}$

Do as many as possible of Exercises 1.3, 1.4, 1.7, 1.8 from Section 1 of the Lecture Notes on Game semantics.