Exercise Sheet 1 for Game Semantics: HT 2003
Model Answers

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Question 1

Consider the following properties of an arrow f in a category $\mathcal{C}$.

- f is *monic* (*i.e.* a monomorphism) if for all arrows g and h (with domaims and codomains such that the following equations make sense)

$$f \circ g = f \circ h \implies g = h.$$

- f is *epic* if

$$g \circ f = h \circ f \implies g = h.$$

- f is *iso* if for some g, both $f \circ g$ and $g \circ f$ are identity arrows.
- f is *split monic* if for some g, $g \circ f$ is an identity arrow.
- f is *split epic* if for some g, $f \circ g$ is an identity arrow.

(a) Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.

(b) Prove that, if f is split epic, then it is epic.

(c) Prove that, if f and $g \circ f$ are iso, then g is iso.

(d) Prove that, if f is monic and split epic, then it is iso.

Question 2

Show that $i : A \longrightarrow B$ has at most one inverse $j : B \longrightarrow A$ making it an isomorphism. This justifies writing i^{-1} for *the* inverse of an isomorphism i.

Answer 1.a

$$f \circ x = f \circ y \Rightarrow g \circ f \circ x = g \circ f \circ y \Rightarrow x = y.$$

Answer 1.b

Suppose $f \circ s = 1$. Then

$$g \circ f = h \circ f \Rightarrow g = g \circ 1 = g \circ f \circ s = h \circ f \circ s = h \circ 1 = h.$$

Answer 1.c

Define $g^{-1} = f \circ (g \circ f)^{-1}$. Then

$$g \circ g^{-1} = g \circ f \circ (g \circ f)^{-1} = 1.$$

Now

$$(g \circ f)^{-1} \circ g = ((g \circ f)^{-1} \circ g) \circ f \circ f^{-1} = ((g \circ f)^{-1} \circ g \circ f) \circ f^{-1} = f^{-1}.$$

Hence

$$g^{-1} \circ g = f \circ (g \circ f)^{-1} \circ g = f \circ f^{-1} = 1.$$

Answer 1.d

Suppose $f \circ s = 1$. Then

$$f \circ s \circ f = 1 \circ f = f = f \circ 1.$$

Since f is monic, this implies that $s \circ f = 1$, and hence $s = f^{-1}$.

Answer 2

Suppose that $f : A \longrightarrow B$ has inverses $g, h : B \longrightarrow A$, satisfying

$$g \circ f = h \circ f = \mathsf{id}_A \qquad f \circ g = f \circ h = \mathsf{id}_B$$

Then

$$g = g \circ \mathsf{id}_B = g \circ (f \circ h) = (g \circ f) \circ h = \mathsf{id}_A \circ h = h.$$