Exercise sheet 3 for game semantics: Sample answers

1 Proofs and terms

1a. Let $\Gamma = \{A \supset B,\ B \supset C,\ A\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A}\,\text{Id} \qquad \dfrac{}{\Gamma \vdash A \supset B}\,\text{Id}}{\Gamma \vdash B}\,{\supset}\text{E} \qquad \dfrac{}{\Gamma \vdash B \supset C}\,\text{Id}}{\Gamma \vdash C}\,{\supset}\text{E}}{A \supset B,\ B \supset C \vdash A \supset C}\,{\supset}\text{I}}{A \supset B \vdash (B \supset C) \supset (A \supset C)}\,{\supset}\text{I}}{\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))}\,{\supset}\text{I}$$

For the corresponding λ term, let $\Gamma = \{x : A \to B,\ y : B \to C,\ z : A\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash z : A}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash x : A \to B}\,\text{Var}}{\Gamma \vdash xz : B}\,\text{App} \qquad \dfrac{}{\Gamma \vdash y : B \to C}\,\text{Var}}{\Gamma \vdash y(xz) : C}\,\text{App}}{x : A \to B,\ y : B \to C \vdash \lambda z.y(xz) : A \to C}\,\text{Abs}}{x : A \to B \vdash \lambda y.\lambda z.y(xz) : (B \to C) \to (A \to C)}\,\text{Abs}}{\vdash \lambda x.\lambda y.\lambda z.y(xz) : (A \to B) \to ((B \to C) \to (A \to C))}\,\text{Abs}$$

1b. Let $\Gamma = \{C \supset A,\ C \supset B,\ C\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash C}\,\text{Id} \qquad \dfrac{}{\Gamma \vdash C \supset A}\,\text{Id}}{\Gamma \vdash A}\,{\supset}\text{E} \qquad \dfrac{\dfrac{}{\Gamma \vdash C}\,\text{Id} \qquad \dfrac{}{\Gamma \vdash C \supset B}\,\text{Id}}{\Gamma \vdash B}\,{\supset}\text{E}}{\Gamma \vdash A \wedge B}\,\wedge\text{I}}{C \supset A,\ C \supset B \vdash C \supset (A \wedge B)}\,{\supset}\text{I}}{C \supset A \vdash (C \supset B) \supset (C \supset (A \wedge B))}\,{\supset}\text{I}}{\vdash (C \supset A) \supset \big((C \supset B) \supset (C \supset (A \wedge B))\big)}\,{\supset}\text{I}$$

For the corresponding λ term let $\Gamma = \{x : C \to A,\ y : C \to B,\ z : C\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash z : C}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash x : C \to A}\,\text{Var}}{\Gamma \vdash xz : A}\,\text{App} \qquad \dfrac{\dfrac{}{\Gamma \vdash z : C}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash y : C \to B}\,\text{Var}}{\Gamma \vdash yz : B}\,\text{App}}{\Gamma \vdash \langle xz, yz\rangle : A \times B}\,\text{Prod}}{C \to A,\ C \to B \vdash \lambda z.\langle xz, yz\rangle : C \to (A \times B)}\,\text{Abs}}{C \to A \vdash \lambda y.\lambda z.\langle xz, yz\rangle : (C \to B) \to (C \to (A \times B))}\,\text{Abs}}{\vdash \lambda x.\lambda y.\lambda z.\langle xz, yz\rangle : (C \to A) \to \big((C \to B) \to (C \to (A \times B))\big)}\,\text{Abs}$$

2a.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{f : C \to A,\ x : C \vdash f : C \to A}\,\text{Var} \qquad \dfrac{}{f : C \to A,\ x : C \vdash x : C}\,\text{Var}}{f : C \to A,\ x : C \vdash fx : A}\,\text{App}}{f : C \to A \vdash \lambda x.fx : C \to A}\,\text{Abs}}{\vdash \lambda f.\lambda x.fx : (C \to A) \to C \to A}\,\text{Abs}$$

2b. Let $\Gamma = \{x : B \to C,\ y : A \to B,\ z : A\}$ in:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : B \to C}\,\text{Var} \qquad \dfrac{\dfrac{}{\Gamma \vdash y : A \to B}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash z : A}\,\text{Var}}{\Gamma \vdash yz : B}\,\text{App}}{\Gamma \vdash x(yz) : C}\,\text{App}}{x : B \to C,\ y :\vdash \lambda z.x(yz) : A \to C}\,\text{Abs}}{x : B \to C \vdash \lambda y.\lambda z.x(yz) : (A \to B) \to A \to C}\,\text{Abs}}{\vdash \lambda x.\lambda y.\lambda z.x(yz) : (B \to C) \to (A \to B) \to A \to C}\,\text{Abs}$$

2d. Let $\Gamma = \{x : A \to A \to B,\ y : A\}$ in:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : A \to A \to B}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash y : A}\,\text{Var}}{\Gamma \vdash xy : A \to B}\,\text{App} \qquad \dfrac{}{\Gamma \vdash y : A}\,\text{Var}}{\Gamma \vdash xyy : B}\,\text{App}}{x : A \to A \to B \vdash \lambda y.xyy : A \to B}\,\text{Abs}}{\vdash \lambda x.\lambda y.xyy : (A \to A \to B) \to A \to B}\,\text{Abs}$$

3. The problem in both cases is with sub-term xx, which cannot be typed. The only typing judgement we can attempt is for application:

$$\dfrac{\dfrac{}{x : T \vdash x : T} \qquad \dfrac{}{x : T \vdash x : T}}{x : T \vdash xx : A}\,\text{App}$$

The types T and A above have to satisfy the equations

$$T = B \to A \quad \text{and} \quad T = B$$

symultaneously, for some type B, i.e. $B = B \to A$. Obviously, there is no finite B that can satisfy this, so xx cannot be typed. Any terms having xx as a sub-term cannot be typed.