Improved hardness for $H$-colourings of $G$-colourable graphs

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Abstract

We present new results on approximate colourings of graphs and, more generally, approximate $H$-colourings and promise constraint satisfaction problems.

First, we show NP-hardness of colouring $k$-colourable graphs with $\binom{k}{3} - 1$ colours for every $k \geq 4$. This improves the result of Bulín, Krokhin, and Opršal [STOC'19], who gave NP-hardness of colouring $k$-colourable graphs with $2k - 1$ colours for $k \geq 3$, and the result of Huang [APPROX-RANDOM'13], who gave NP-hardness of colouring $k$-colourable graphs with $2^{\Omega(k^{1/3})}$ colours for sufficiently large $k$. Thus, for $k \geq 4$, we improve from known linear/sub-exponential gaps to exponential gaps.

Second, we show that the topology of the box complex of $H$ alone determines whether $H$-colouring of $G$-colourable graphs is NP-hard for all (non-bipartite, $H$-colourable) $G$. This formalises the topological intuition behind the result of Krokhin and Opršal [FOCS'19] that 3-colouring of $G$-colourable graphs is NP-hard for all (3-colourable, non-bipartite) $G$. We use this technique to establish NP-hardness of $H$-colouring of $G$-colourable graphs for $H$ that include but go beyond $K_4$, including square-free graphs and circular cliques (leaving $K_4$ and larger cliques open).

Underlying all of our proofs is a very general observation that adjoint functors give reductions between promise constraint satisfaction problems.


1 Introduction

Graph colouring is one of the most fundamental and studied problems in combinatorics and computer science. A graph $G$ is called $k$-colourable if there is an assignment of colours $\{1, 2, \ldots, k\}$ to the vertices of $G$ so that any two adjacent vertices are assigned different colours. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest integer $k$ for which $G$ is $k$-colourable. Deciding whether $\chi(G) \leq k$ appeared on Karp’s original list of 21 NP-complete problems [35], and is NP-hard for every $k \geq 3$. In particular, it is NP-hard to decide whether $\chi(G) \leq 3$ or $\chi(G) > 3$. Put differently (thanks to self-reducibility of graph colouring), it is NP-hard to find a 3-colouring of $G$ even if $G$ is promised to be 3-colourable.

In the approximate graph colouring problem, we are allowed to use more colours than needed. For instance, given a 3-colourable graph $G$ on $n$ vertices, can we find a colouring of $G$ using significantly fewer than $n$ colours? On the positive side, the currently best polynomial-time algorithm of Kawarabayashi and Thorup [36] finds a colouring using $O(n^{0.19996})$ colours. Their work continues a long line of research and is based on a semidefinite relaxation. On the negative side, it is believed that finding a $c$-colouring of a $k$-colourable graph is NP-hard for all constants $3 \leq k \leq c$. Already in this regime (let alone for non-constant $c$) our understanding remains rather limited, despite lots of work and the development of complex techniques, as we will survey in Section 1.1.

A natural and studied generalisation of graph colourings is that of graph homomorphisms and, more generally, constraint satisfaction problems [32].

Given two graphs $G$ and $H$, a map $h : V(G) \to V(H)$ is a homomorphism from $G$ to $H$ if $h$ preserves edges; that is, if $\{h(u), h(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$ [31]. A celebrated result of Hell and Nešetřil established a dichotomy for the homomorphism problem with a fixed target graph $H$, also known as the $H$-colouring problem: deciding whether an input graph $G$ has a homomorphism to $H$ is solvable in polynomial time if $H$ is bipartite or if $H$ has a loop; for all other $H$ this problem is NP-hard [30]. Note that the $H$-colouring problem for $H = K_k$, the complete graph on $k$ vertices, is precisely the graph colouring problem with $k$ colours.

The constraint satisfaction problem (CSP) is a generalisation of the graph homomorphism problem from graphs to arbitrary relational structures. One type of CSP that has attracted a lot of attention is the one with a fixed target structure, also known as the non-uniform CSP; see, e.g., the work of Jeavons, Cohen, and Gyssens [34], Bulatov [14, 16], and Barto and Kozik [5, 6]. Following the above mentioned dichotomy of Hell and Nešetřil for the $H$-colouring [30] and a dichotomy result of Schaefer for Boolean CSPs [48], Feder and Vardi famously conjectured a dichotomy for all non-uniform CSPs [23]. The Feder-Vardi conjecture was

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Prior and related work  While the NP-hardness of finding a 3-colouring of a 3-colourable graph was obtained by Karp [35] in 1972, the NP-hardness of finding a 4-colouring of a 3-colourable graph was only proved in 2000 by Khanna, Linial, and Safra [37] (see also the work of Gurvits and Khanna for a different proof [27]). This result implied NP-hardness of finding a \((k + 2)\lfloor k/3 \rfloor - 1\)-colouring of a k-colourable graph for \(k \geq 3\) [37]. Early work of Garey and Johnson established NP-hardness of finding a \((2k - 5)\)-colouring of a k-colourable graph for \(k \geq 6\) [26]. In 2016, Brakensiek and Guruswami proved NP-hardness of a \((2k - 2)\)-colouring of a k-colourable graph for \(k \geq 3\) [10]. Only very recently, Bulín, Krokhin, and Opršal showed that finding a 5-colouring of a 3-colourable graph, and more generally, finding a \((2k - 1)\)-colouring of a k-colourable graph for any \(k \geq 3\), is NP-hard [18].

In 2001, Khot proved an asymptotic result: for sufficiently large \(k\), finding a \(\frac{k \log k}{2}\)-colouring of a k-colourable graph is NP-hard [38]. In 2013, Huang improved the gap by showing the hardness of finding a \(2\Omega(k^{1/3})\)-colouring of a k-colourable graph [33].

The NP-hardness of colouring (k-colourable graphs) with \((2k - 1)\) colours for \(k \geq 3\) from [18] and with \(2\Omega(k^{1/3})\) colours for sufficiently large \(k\) from [33] constitute the currently strongest known NP-hardness results for approximate graph colouring.

Under stronger assumptions (Khot’s 2-to-1 Conjecture [39] for \(k \geq 4\) and its non-standard variant for \(k = 3\)), Dinur, Mossel, and Regev showed that finding a c-colouring of a k-colourable graph is NP-hard for all constants \(3 \leq k \leq c\) [20]. A variant of Khot’s 2-to-1 Conjecture with imperfect completeness has recently been proved [19, 40], which implies hardness for approximate colouring variants where most but not all of the graph is guaranteed to be k-colourable.

Hypergraphs colourings, a special case of PCSPs, is another line of work intensively studied. A k-colouring of a hypergraph is an assignment of colours \(\{1, 2, \ldots, k\}\) to its vertices that leaves no hyperedge monochromatic. Dinur, Regev, and Schnyder showed that for any constants \(2 \leq k \leq c\), it is NP-hard to find a c-colouring of given 3-uniform k-colourable hypergraph [21].

Some results are also known for colourings with a super-constant number of colours. For graphs, conditional hardness was obtained by Dinur and Shinkar [22]. For hypergraphs, NP-hardness results were obtained in recent work of Bangale [8] and Austrin, Bangale, and Potukuchi [1].
2 Results

For two graphs or digraphs $G$, $H$, we write $G \to H$ if there exists a homomorphism from $G$ to $H$. We are interested in the following computational problem.

**Definition 2.1.** Fix two graphs $G$ and $H$ with $G \to H$. The (decision variant of the) PCSP($G$, $H$) is, given an input graph $I$, output YES if $I \to G$, and NO if $I \not\to H$.

To state our results it will be convenient to use the following definition.

**Definition 2.2.** A graph $H$ is left-hard if for every non-bipartite graph $G$ with $G \to H$, PCSP($G$, $H$) is NP-hard. A graph $G$ is right-hard if for every loop-less graph $H$ with $H \to G$, PCSP($G$, $H$) is NP-hard.

If $G \to G'$ and $H' \to H$, then PCSP($G$, $H$) trivially reduces to PCSP($G'$, $H'$) (this is called homomorphic relaxation [18]; intuitively, increasing the promise gap makes the problem easier). Therefore, if $H$ is a left-hard graph, then all graphs left of $H$ (that is, $H'$ such that $H' \to H$) are trivially left-hard. If $G$ is right-hard, then all graphs right of $G$ are right-hard.

For the same reason, since every non-bipartite graph admits a homomorphism from an odd cycle, to show that $H$ is left-hard it suffices to show that PCSP($C_n$, $H$) is NP-hard for arbitrarily large odd $n$, where $C_n$ denotes the cycle on $n$ vertices. Dually, since every loop-less graph admits a homomorphism to a clique, to show that $G$ is right-hard it suffices to show that PCSP($G$, $K_k$) is NP-hard for arbitrarily large $k$.

Brakensiek and Guruswami conjectured that all non-trivial PCSPs for (undirected) graphs are NP-hard, greatly extending Hell and Nešetřil’s theorem:

**Conjecture 2.1.** ([12]) PCSP($G$, $H$) is NP-hard for every non-bipartite loop-less $G$, $H$. Equivalently, every loop-less graph is left-hard. Equivalently, every non-bipartite graph is right-hard.

In addition to the results on classical colourings discussed above (the case where $G$ and $H$ are cliques), the following result was recently obtained in a novel application of topological ideas.

**Theorem 2.1.** (Krokhin and Opršal [41]) $K_3$ is left-hard.

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2.1 Improved hardness of classical colouring

In Section 3, we focus on right-hardness. We use a simple construction called the arc digraph or line digraph, which decreases the chromatic number of a graph in a controlled way. The construction allows to conclude the following, in a surprisingly simple way:

**Proposition 2.1.** There exists a right-hard graph if and only if $K_4$ is right-hard.

More concretely, we show in particular that PCSP($K_6$, $K_{2k}$) log-space reduces to PCSP($K_4$, $K_k$), for all $k \geq 4$. This contrasts with [4, Proposition 10.3], where it is shown to be impossible to obtain such a reduction with *minion homomorphisms*: an algebraic reduction, described in the full version [55], central to the framework of [18, 4] (in particular, there exists a $k$ such that PCSP($K_4$, $K_k$) admits no minion homomorphism to any PCSP($K_{n'}$, $K_{k'}$) for $4 < n' \leq k'$).

Furthermore, we strengthen the best known asymptotic hardness: Huang [33] showed that for all sufficiently large $n$, PCSP($K_n$, $K_{2n^{1/3}}$) is NP-hard. We improve this in two ways, using Huang’s result as a black-box. First, we improve the asymptotics from sub-exponential $2^n/\sqrt{\pi n/2}$ to single-exponential $\binom{n}{\lceil n/2 \rceil} \sim 2^n\sqrt{\pi n/2}$. Second, we show the claim holds for $n$ as low as $4$.

**Theorem 2.2.** (Main Result #1) For all $n \geq 4$, PCSP($K_n$, $K_{\binom{n}{\lfloor n/2 \rfloor}-1}$) is NP-hard.

In comparison, the previous best result relevant for all integers $n$ was proved in [18]: PCSP($K_n$, $K_{2n-1}$) is NP-hard for all $n \geq 3$. For $n = 3$ we are unable to obtain any results; for $n = 4$ the new bound $\binom{n}{\lfloor n/2 \rfloor} - 1 = 5$ is worse than $2n - 1 = 7$, while for $n = 5$ the two bounds coincide at $9$. However, already for $n = 6$ we improve the bound from $2n - 1 = 11$ to $\binom{n}{\lfloor n/2 \rfloor} - 1 = 19$.

2.2 Left-hardness and topology

In Section 4, we focus on left-hardness. The main idea behind Krokhin and Opršal’s [41] proof that $K_3$ is left-hard is simple to state. To prove that PCSP($C_n$, $H$) is NP-hard for all odd $n$, the algebraic framework of [18] shows that it is sufficient to establish certain properties of polymorphisms: homomorphisms $f: C_n^L \to H$ for $L \in \mathbb{N}$.

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5 Jakub Opršal and Andrei Krokhin realised that in this Proposition, 4 can be improved to 3 by using the fact that $\delta(K_4)$ is 3-colourable, as proved by Rorabaugh, Tardif, Wehlau, and Zaguia [46]. Details will appear in a future journal version.

6 [4] is a full version of [18]. Proposition 10.3 in [4] is Proposition 5.31 in the previous two versions of [4].
(where $G^L = G \times \cdots \times G$ is the $L$-fold tensor product\(^\text{7}\)).

For large $n$ the graph $C_{n}^L$ looks like an $L$-torus: an $L$-fold product of circles, so the pertinent information about $f$ seems to be subsumed by its topological properties (such as winding numbers, when $H$ is a cycle). We refer to [41] for further details, but this general principle applies to any $H$ and in fact we prove (in Theorem 2.3 below) that whether $H$ is left-hard or not depends only on its topology.

The topology we associate with a graph is its box complex. See the full version [55] for formal definitions and statements. Intuitively, the box complex $\Box(\cdot)$ is a topological space built from $H$ by taking the tensor product $H \times K_2$ and then gluing faces to each four-cycle and more generally, gluing higher-dimensional faces to complete bipartite subgraphs. The added faces ensure that the box complex of a product of graphs is the same as the product space of their box complexes: thanks to this, $\Box(C_n^L)$ is indeed equivalent to the $L$-torus. The product with $K_2$ equips the box complex with a symmetry that swaps the two sides of $H \times K_2$. This makes the resulting space a $\mathbb{Z}_2$-space: a topological space together with a continuous involution from the space to itself, which we denote simply as $\sim$. A $\mathbb{Z}_2$-map between two $\mathbb{Z}_2$-spaces is a continuous function which preserves this symmetry: $f(-x) = -f(x)$. This allows to concisely state that a given map is “non-trivial” (in contrast, there is always some continuous function from one space to another: just map everything to a single point). The main use of the box complex is then the statement that every graph homomorphism $G \to H$ induces a $\mathbb{Z}_2$-map from $\Box(G)$ to $\Box(H)$. Graph homomorphisms can thus be studied with tools from algebraic topology.

The classical example of this is an application of the Borsuk-Ulam theorem: there is no $\mathbb{Z}_2$-map from $S^n$ to $S^m$ for $n > m$, where $S^n$ denotes the $n$-dimensional sphere with antipodal symmetry. Hence if $G$ and $H$ are graphs such that $\Box(G)$ and $\Box(H)$ are equivalent to $S^n$ and $S^m$, respectively, then there can be no graph homomorphism $G \to H$. See Figure 1.

This is essentially the idea in Lovász’ proof [42] of Kneser’s conjecture that the chromatic number of Kneser graphs $K_G(n,k)$ is $n - 2k + 2$. In the language of box complexes, the proof amounts to showing that the box complex of a clique $K_k$ is equivalent to $S^{n-2}$, while the box complex of a Kneser graph contains $S^{n-2k}$. We refer to [43] for an in-depth, yet accessible reference.

We show that the left-hardness of a graph depends only on the topology of its box complex (in fact, it is only important what $\mathbb{Z}_2$-maps it admits, which is significantly coarser than $\mathbb{Z}_2$-homotopy equivalence):

**Theorem 2.3.** (Main Result #2) If $H$ is left-hard and $H'$ is a graph such that $\Box(H')$ admits a $\mathbb{Z}_2$-map to $\Box(H)$, then $H'$ is left-hard.

Using Krokhin and Oporś’s result that $K_3$ is left-hard (Theorem 2.1), since $\Box(K_3)$ is the circle $S^1$ (up to $\mathbb{Z}_2$-homotopy equivalence), we immediately obtain the following:

**Corollary 2.1.** Every graph $H$ for which $\Box(H)$ admits a $\mathbb{Z}_2$-map to $S^1$ is left-hard.

Two examples of such graphs (other than 3-colourable graphs) are loopless square-free graphs and circular cliques $K_{p/q}$ with $2 < \frac{p}{q} < 4$ (see the full version [55] for proofs), which we introduce next. Square-free graphs are graphs with no cycle of length exactly 4. In particular, this includes all graphs of girth at least 5 and hence graphs of arbitrarily high chromatic number (but incomparable to $K_4$ and larger cliques, in terms of the homomorphism → relation). The circular clique $K_{p/q}$ (for $p,q \in \mathbb{N}$, $\frac{p}{q} > 2$) is the graph with vertex set $\mathbb{Z}_p$ and an edge from $i$ to every integer at least $q$ apart: $i + q, i + q + 1, \ldots, i + p - q$. They generalise cliques $K_n = K_{n/1}$ and odd cycles $C_{2n+1} \simeq K_{(2k+1)/2}$. Their basic property is that $K_{p/q} \to K_{p'/q'}$ if and only if $\frac{p}{q} < \frac{p'}{q'}$. Thus circular cliques refine the chain of cliques and odd cycles, corresponding to rational numbers between integers. For example:

$$\cdots \to C_7 \to C_5 \to C_3 = K_3 \to K_{7/2} \to K_4 \to K_{9/2} \to K_5 \to \cdots$$

The circular chromatic number $\chi_c(G)$ is the infimum over $\frac{p}{q}$ such that $G \to K_{p/q}$. Therefore:

**Corollary 2.2.** For every $2 < r \leq r' < 4$, it is $NP$-hard to distinguish graphs $G$ with $\chi_c(G) \leq r$ from those with $\chi_c(G) > r'$.

In this sense, we conclude that $K_{4-\varepsilon}$ is left-hard, thus extending the result for $K_3$. However, the closeness to $K_4$ is only deceptive and no conclusions on 4-colourings follow. For $K_4$, since the box complex is equivalent to the standard 2-dimensional sphere, we can at least conclude that to prove left-hardness of $K_4$ it would be enough to prove left-hardness of any other graph with the same topology: these include all non-bipartite quadrangulations of the projective plane, in

\(\text{Footnote:} 7\) The tensor (or categorical) product $G \times H$ of graphs $G, H$ has pairs $(g,h) \in V(G) \times V(H)$ as vertices and $(g,h)$ is adjacent to $(g',h')$ whenever $g$ is adjacent to $g'$ (in $G$) and $h$ is adjacent to $h'$ (in $H$).
In this sense, the exact geometry of Mycielskians, and 4-chromatic Schrijver graphs [43, 9]. In particular the Grötzsch graph, 4-chromatic generalised Mycielskians, and 4-chromatic Schrijver graphs [43, 9]. In this sense, the exact geometry of $K_4$ is irrelevant. However, the fact that it is a finite graph, with only finitely many possible maps from $C_n^L$ for any fixed $n, L$ should still be relevant, as it is for $K_3$. It is also quite probable that any proof for a “spherical” graph would apply just as well to $K_4$, where the proof could be just notationally much simpler.

In the full version [55] we rephrase Krokhin and Opršal’s [41] proof of Theorem 2.1 in terms of the box complex. In particular, left-hardness of $K_4$ follows from some general principles and the fact that $|Box(K_4)|$ is a circle. The proof also extends to all graphs $H$ such that $|Box(H)|$ admits a $\mathbb{Z}_2$-map to $S^1$, giving an independent, self-contained proof of Corollary 2.1 (and Theorem 2.1 in particular).

The general principle is that a homomorphism $C_n^L \to H$ induces a $\mathbb{Z}_2$-map $(S^1)^L \to |Box(H)|$, in a way that preserves minors (identifications within the $L$ variables) and automorphisms. (In the language of category theory, the box complex is a functor from the category of graphs to that of $\mathbb{Z}_2$-spaces, and the functor preserves products). In turn, the $Z_2$-map induces a group homomorphism between the fundamental group of $(S^1)^L$, which is just $Z^L$, and that of $|Box(H)|$. This is essentially the map $\mathbb{Z}^L \to \mathbb{Z}$ obtained in [41]. While this rephrasing requires a bit more technical definitions, the main advantage is that it allows to replace a tedious combinatorial argument (about winding numbers preserving minors) with straightforward statements about preserving products.

2.3 Methodology – adjoint functors While the proof of the first main result is given elementarily in Section 3, it fits together with the second main result in a much more general pattern. The underlying principle is that pairs of graph constructions satisfying a simple duality condition give reductions between PCSPs. To introduce them, let us consider a concrete example. For a graph $G$ and an odd integer $k$, $\Lambda_kG$ is the graph obtained by subdividing each edge into a path of $k$ edges; $\Gamma_kG$ is the graph obtained by taking the $k$-th power of the adjacency matrix (with zeroes on the diagonal); equivalently, the vertex set remains unchanged and two vertices are adjacent if and only if there is a walk of length exactly $k$ in $G$. (For example $\Gamma_3G$ has loops if $G$ has triangles).

We say a graph construction $\Lambda$ (a function from graphs to graphs) is a thin (graph) functor if $G \to H$ implies $\Lambda G \to \Lambda H$ (for all $G, H$). A pair of thin functors $(\Lambda, \Gamma)$ is a thin adjoint pair if

$$\Lambda G \to H \text{ if and only if } G \to \Gamma H.$$  

We call $\Lambda$ the left adjoint of $\Gamma$ and $\Gamma$ the right adjoint of $\Lambda$.

For all odd $k$, $(\Lambda_k, \Gamma_k)$ are a thin adjoint pair. For example, since $\Gamma_3C_5 = K_5$, we have $G \to K_5$ if and only if $\Lambda_kG \to C_5$. This is a basic reduction that shows the NP-hardness of $C_5$-colouring; in fact adjointness of
various graph construction is the principal tool behind
the original proof of Hell and Neˇ setˇ ril’s theorem (char-
acterising the complexity of $H$-colouring) [30].

In category theory, there is a stronger and more
technical notion of (non-thin) functors and adjoint
pairs. A thin graph functor is in fact a functor in the
thin category of graphs, that is, the category whose
objects are graphs, and with at most one morphism
from one graph to another, indicating whether a ho-
omorphism exists or not. In other words, we are
only interested in the existence of homomorphisms, and
not in their identity and how they compose. Equiva-
ently, we look only at the preorder of graphs by the
$G \rightarrow H$ relation (we can also make this a poset by con-
sidering graphs up to homomorphic equivalence).
In order-theoretic language, thin functors are just order-
preseving maps, while thin adjoint functors are known
as Galois connections. We prefer the categorical lan-
guage as most of the constructions we consider are in
fact functors (in the non-thin category of graphs), which
is important for constructions to the algebraic framework
of [18], as we discuss in the full version [55]. While
unnecessary for our main results, we believe it may be
important to understand these deeper connections to
resolve the conjectures completely.

Thin adjoint functors give us a way to reduce one
PCSP to another. We say that a graph functor $\Gamma$ is
log-space computable if, given a graph $G$, $\Gamma G$ can be
computed in logarithmic space in the size of $G$.

**Observation 2.1.** Let $\Lambda, \Gamma$ be thin adjoint graph func-
tors and suppose $\Lambda$ is log-space computable. Then
PCSP($\Gamma G, \Omega H$) reduces to PCSP($\Lambda G, \Omega H$) in log-space,
for all graphs $G, H$.

**Proof.** Let $F$ be an instance of PCSP($\Gamma G, \Omega H$). Then $\Lambda F$ is
an appropriate instance of PCSP($\Lambda G, \Omega H$). Indeed, if
$F \rightarrow G$, then $\Lambda F \rightarrow \Lambda G$ (because $\Lambda$ is a thin functor).
If $\Lambda F \rightarrow H$, then $F \rightarrow \Gamma H$ by adjointness. $\square$

In some cases, a thin functor $\Gamma$ that is a thin right
adjoint in a pair $(\Lambda, \Gamma)$ is also a thin left adjoint in a pair
$(\Gamma, \Omega)$. This allows to get a reduction in the opposite
direction:

**Observation 2.2.** Let $(\Lambda, \Gamma)$ and $(\Gamma, \Omega)$ be thin ad-
joint pairs of functors. Then PCSP($\Gamma G, H$) and
PCSP($G, \Omega H$) are log-space equivalent (assuming $\Lambda$ and
$\Gamma$ are log-space computable).

**Proof.** The previous observation gives a reduction from
PCSP($G, \Omega H$) to PCSP($\Gamma G, H$). For the other direc-
tion, let $F$ be an instance of PCSP($\Gamma G, H$). Then $\Lambda F$
is an appropriate instance of PCSP($\Gamma G, \Omega H$). Indeed,
if $F \rightarrow \Gamma G$, then $\Lambda F \rightarrow G$. If $\Lambda F \rightarrow \Omega H$, then
$F \rightarrow \Gamma \Omega H \rightarrow H$. The last arrow follows from the trivial
$\Omega H \rightarrow \Omega H$. $\square$

The proofs of Observations 2.1 and 2.2 of course extend to
digraphs and general relational structures. Note that the above proofs reduce decision problems;
they work just as well for search problems: all the thin
adjoint pairs $(\Lambda, \Gamma)$ we consider with $\Lambda$ log-space com-
putable also have the property that a homomorphism
$\Lambda F \rightarrow H$ can be computed from a homomorphism
$F \rightarrow \Gamma H$ and vice versa, in space logarithmic in the
size of $F$.

As we discuss in Section 4, all of our results follow
from reductions that are either trivial (homomorphic
relaxations) or instantiations of Observation 2.1. While
for the first main result we prefer to first give a direct
proof that avoids this formalism (in Section 3), it will be
significantly more convenient for the second main result
(in Section 4), where we use a certain thin right adjoint
$\Omega_k$ to the $k$-th power $\Gamma_k$.

### 2.4 Hedetniemi’s conjecture

Another leitmotif of this paper is the application of various tools developed in
research around Hedetniemi’s conjecture. A graph
$K_n$ is multiplicative if $G \times H \rightarrow K_n$ implies $G \rightarrow K_n$
or $H \rightarrow K_n$. The conjecture states that all cliques
$K_n$ are multiplicative. Equivalently, $\chi(G \times H) =
\min(\chi(G), \chi(H))$; see [56, 47, 51] for surveys. In a
very recent breakthrough, Shitov [49] proved that the
conjecture is false (for large $n$).

The arc digraph construction, which we will use in
Section 3 to prove Theorem 2.2, was originally used by
Poljak and Rödl [45] to show certain asymptotic bounds on chromatic numbers of products. The
functors $\Lambda_k, \Gamma_k, \Omega_k$ were applied by Tardif [50] to show that
colourings to circular cliques $K_{p/q}$ ($2 < \frac{p}{q} < 4$) satisfy
the conjecture. Matsushita [44] used the box complex to
show that Hedetniemi’s conjecture would imply an
analogous conjecture in topology. This was indepen-
dently proved by the first author [54] using $\Omega_k$ functors,
while the box complex was used to show that square-
free graphs are multiplicative [53]. See [25] for a survey
on applications of adjoint functors to the conjecture.

The refutation of Hedetniemi’s conjecture and the
fact that methods for proving the multiplicativity of
$K_3$ extend to $K_{4-\epsilon}$ and square-free graphs, but fail
to extend to $K_4$, might suggest that the Conjecture 2.1 is
doomed to the same fate. However, it now seems clear
that proving multiplicativity requires more than just
topology [52]: known methods do not even extend to all
graphs $H$ such that $|\text{Box}(H)|$ is a circle. This contrasts
with Theorem 2.3: topological tools work much more
3 The arc digraph construction

Let $D$ be a digraph. The arc digraph (or line digraph) of $D$, denoted $δD$, is the digraph whose vertices are arcs (directed edges) of $D$ and whose arcs are pairs of the form $((u, v), (v, w))$. We think of undirected graphs as symmetric relations: digraphs in which for every arc $(u, v)$ there is an arc $(v, u)$. So for an undirected graph $G$, $δ(G)$ has $|E(G)|$ vertices and is a directed graph: the directions will not be important in this section, but will be in the full version [55]. The chromatic number of a digraph is the chromatic number of the underlying undirected graph (obtained by symmetrising each arc; so $χ(D) ≤ n$ if and only if $D → K_n$).

The crucial property of the arc digraph construction is that it decreases the chromatic number in a controlled way (even though it is computable in log-space). We include a short proof for completeness. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

**Lemma 3.1. (Harner and Entringer [29])** For any graph $G$:

- if $χ(δ(G)) ≤ n$, then $χ(G) ≤ 2^n$,
- if $χ(G) ≤ (n/n/2)$, then $χ(δ(G)) ≤ n$.

**Proof.** Suppose $δG$ has an $n$-colouring. Recall that we think of $G$ as a digraph with two arcs $(u, v)$ and $(v, u)$ for each edge $\{u, v\} \in E(G)$; thus $δG$ contains two vertices $(u, v)$ and $(v, u)$, as well as (by definition of $δ$) two arcs from one pair to the other. In particular, an $n$-colouring of $δG$ gives distinct colours to $(u, v)$ and $(v, u)$. Define a $2^n$-colouring $ϕ$ of $G$ by assigning to each vertex $v$ the set $ϕ(v)$ of colours of incoming arcs. For any edge $\{u, v\}$ of $G$, $ϕ(v)$ contains the colour $c$ of the arc $(u, v)$. Since every arc incoming to $u$ gets a different colour from $(u, v)$, the set $ϕ(u)$ does not contain $c$. Hence $ϕ(u) ≠ ϕ(v)$, so $ϕ$ is a proper colouring.

Suppose $G$ has a $(n/n/2)$-colouring $ϕ$. We interpret colours $ϕ(v)$ as $[n/2]$-element subsets of $[n]$. Define an $n$-colouring of $δG$ by assigning to each arc $(u, v)$ an arbitrary colour in $ϕ(u) \setminus ϕ(v)$ (the minimum, say). Such a colour exists because $ϕ(u) ≠ ϕ(v)$. For arcs $(u, v)$, $(v, w)$ clearly $ϕ(u) \setminus ϕ(v)$ is disjoint from $ϕ(v) \setminus ϕ(w)$, so this is a proper colouring of $δ(G)$. ☐

The proofs in fact works for digraphs as well. For graphs, it is not much harder to show an exact correspondence (we note however that most conclusions only require the above approximate correspondence). Let us denote $b(n) := (n/n/2)$.

**Lemma 3.2. (Poljak and Rödl [45])** For a (symmetric) graph $G$, $χ(δ(G)) = \min\{n \mid χ(G) ≤ b(n)\}$. In other words, $δG → K_n$ if and only if $G → K_{b(n)}$.

This immediately gives the following implication for approximate colouring:

**Lemma 3.3.** PCSP($K_{b(n)}, K_{b(k)}$) log-space reduces to PCSP($K_n, K_k$), for all $n, k \in \mathbb{N}$.

**Proof.** Let $G$ be an instance of the first problem. Then $δG$ is a suitable instance of PCSP($K_n, K_k$): if $G → K_{b(n)}$, then $δG → K_n$. If $δG → K_k$, then $G → K_{b(k)}$.

**Remark 3.1.** As a side note, adding a universal vertex gives the following obvious reduction: PCSP($K_n, K_k$) log-space reduces to PCSP($K_{n+1}, K_{k+1}$), for $n, k \in \mathbb{N}$.

Recall also that if $n ≤ n' ≤ k' ≤ k$, then PCSP($K_n, K_k$) trivially reduces to PCSP($K_{n'}, K_{k'}$). One corollary of Lemma 3.3 is that if any clique of size at least 4 is right-hard, then all of them are:

**Proposition 3.1.** For all integers $n, n' ≥ 4$, PCSP($K_n, K_k$) is NP-hard for all $k ≥ n$ if and only if PCSP($K_{n'}, K_{k'}$) is NP-hard for all $k' ≥ n'$.

**Proof.** Let $n ≤ n'$. For one direction, right-hardness of $K_n$ trivially implies right-hardness of $K_{n'}$.

On the other hand, we claim that if $K_{b(n)}$ is right-hard, then so is $K_n$. Indeed, suppose PCSP($K_{b(n)}, K_k$) is hard for all $k ≥ b(n)$. In particular it is hard for all $k$ of the form $k → b(k')$ for an integer $k' ≥ n$. Hence by Lemma 3.3, PCSP($K_{n}, K_{k'}$) is hard for all $k' ≥ n$.

Suppose $K_n$ is not right-hard. Then $K_{b(n)}$ is not right-hard, $K_{b(b(n))}$ is not right-hard and so on. Since starting with $n ≥ 4$, the sequence $b(b(\ldots b(\ldots n\ldots )))$ grows to infinity, we conclude that $K_{n''}$ is not right-hard for some $n'' ≥ n'$. Therefore, trivially $K_{n'}$ is not right-hard.

In other words if any loop-less graph $H$ is right-hard, then trivially some large enough clique $K_{χ(H)}$ is right-hard; by the above, $K_3$ and all graphs right of it are right-hard. This proves Proposition 2.1. The proof fails to extend to $K_3$ because $b(3) = (3/2)$ is not strictly greater than 3.

The other consequence we derive from Lemma 3.3 is a strengthening of Huang’s result:

**Theorem 3.1. (Huang [33])** For all sufficiently large $n$, PCSP($K_n, K_{2(n/n/2)}$) is NP-hard.

**Theorem 3.2. (Main Result #1)** For all $n ≥ 4$, PCSP($K_n, K_{n/n/2}$) is NP-hard.
We thus improve the asymptotics from sub-exponential $f(n) := 2^{n^{1/3}}$ to single-exponential $b(n) = \left(\frac{n}{\ln n}\right)^{2/\sqrt{3}}$. The informal idea of the proof (delayed to the full version [55]) is as follows. Suppose $n$ vs $f(n)$-colouring is hard (that is, PCSP($K_n, K_f(n)$) is NP-hard for all sufficiently large $n$). Substituting $n = b(m)$, we conclude that $b(m)$ vs $f(b(m))$ is hard (for sufficiently large $m$). Trivially, we can then replace $f(b(m))$ with any smaller number of the form $b(k)$. So $b(m) vs f(b(m))$ is hard for $k = b^{-1}(f(b(m)))$ where $b^{-1}(x) := \max\{k: b(k) \leq x\}$. By Lemma 3.3, we conclude that $m vs k$ is hard. That is, $m vs b^{-1}(f(b(m)))$ is hard for all sufficiently large $m$.

Thus any $f(n)$ can be improved to $b^{-1}(f(b(n)))$. Since $b(n)$ is roughly exponential and $b^{-1}(n)$ is roughly logarithmic, starting from a function $f(n)$ of order $\exp(i+1)(\alpha \cdot \log^{i}(n))$ with $i$-fold compositions and a constant $\alpha > 0$, such as $f(n) = 2^{n^{1/3}} = 2^{\frac{2}{3}\log n}$ from Huang’s hardness, results in

$$b^{-1}(f(b(n))) \approx \log \left( \exp^{(i+1)}(\alpha \cdot \log^{i}(\exp(n))) \right) = \exp^{(i)}(\alpha \cdot \log^{(i-1)}(n)),$$

so a similar composition but with $i$ decreased. In a constant number of steps, this results in a single-exponential function. In fact using one more step, but without approximating the function $b(n)$, this results in exactly $b(n) - 1$. We note it would not be sufficient to start from a quasi-polynomial $f(n)$, like $n^{O(\log n)}$ in Khot’s [38] result. To relax the requirement for “for sufficiently large $n$” to “for all $n \geq 4$”, the argument is similar as in Proposition 3.1.

4 Adjoint functors and topology

Recall that $\Lambda_k$ and $\Gamma_k$ denote $k$-subdivision and the $k$-th power of a graph; they are thin adjoints:

$$\Lambda_k G \rightarrow H \text{ if and only if } G \rightarrow \Gamma_k H.$$ 

(for all odd $k$). More surprisingly, $\Gamma_k$ is itself the thin left adjoint of a certain thin functor $\Omega_k$:

$$\Gamma_k G \rightarrow H \text{ if and only if } G \rightarrow \Omega_k H.$$ 

This characterizes $\Omega_k G$ up to homomorphic equivalence; the exact definition will be irrelevant (we give it in the full version [55]). We note that $\Lambda_k$ and $\Gamma_k$ are log-space computable, for all odd $k$; however, $\Omega_k$ is not: $\Omega_k G$ is exponentially larger than $G$. See [54] for more about the thin functors $\Lambda_k, \Gamma_k, \Omega_k$ and their properties.

Observation 2.1 tells us that PCSP($G, \Omega_k H$) log-space reduces to PCSP($\Gamma_k G, H$). To give conclusions on left-hardness, we will need only two more facts about these functors. First, $\Omega_k G \rightarrow G$; second, $\Lambda_k G \rightarrow \Omega_k G$ for all $G$ and odd $k$ (see Lemma 2.3(iv) and (vi) in [54]).

**Lemma 4.1.** For every odd $k$, $\Omega_k H$ is left-hard if and only if $H$ is left-hard.

**Proof.** If $H$ is left-hard, then trivially so is $\Omega_k H$ because $\Omega_k H \rightarrow H$. For the other implication, suppose $\Omega_k H$ is left-hard, that is, PCSP($G, \Omega_k H$) is hard for every non-bipartite $G$ such that $G \rightarrow \Omega_k H$. By Observation 2.1, this implies PCSP($\Gamma_k G, H$) is hard. Let $G'$ be any non-bipartite graph such that $G' \rightarrow H$.

We want to show that PCSP($G', H$) is hard. Observe that $\Omega_k G'$ is non-bipartite, because $\Lambda_k G' \rightarrow \Omega_k G'$ and $\Lambda_k$ subdivides each edge of $G'$ an odd number of times. Since $\Omega_k G' \rightarrow \Omega_k H$, using $G := \Omega_k G'$ we conclude that PCSP($\Gamma_k \Omega_k G', H$) is hard. Since $\Gamma_k \Omega_k G' \rightarrow G'$, this implies PCSP($G', H$) is hard. \[\square\]

As an example, consider the circular clique $K_{7/2}$ (we have $K_3 \rightarrow K_{7/2} \rightarrow K_4$). Knowing that $K_3$ is left-hard, one could check that $\Omega_3(K_{7/2})$ is 3-colorable and hence left-hard as well; the above lemma then allows to conclude that $K_{7/2}$ is left-hard. What other graphs could one use in place of $K_{7/2}$? The answer turns out to be topological. Intuitively, while the operation $\Gamma_k$ gives a “thicker” graph, the operation $\Omega_k$ gives a “thinner” one. In fact, $\Omega_k$ behaves like barycentric subdivision in topology: it preserves the topology of a graph (formally: its box complex is $\mathbb{Z}_2$-homotopy equivalent to the original graph’s box complex) but refines its geometry. With increasing $k$, this eventually allows to model any continuous map with a graph homomorphism; in particular:

**Theorem 4.1.** ([54]) There exists a $\mathbb{Z}_2$-map $[\text{Box}(G)] \rightarrow [\text{Box}(H)]$ if and only if for some odd $k$, $\Omega_k G \rightarrow H$.

This concludes our second main result:

**Proof.** [Proof of Theorem 2.3] Let $H$ be left-hard and let $H'$ be a graph s.t. $[\text{Box}(H')] \rightarrow [\text{Box}(H)]$. By Theorem 4.1, $\Omega_k H' \rightarrow H$ for some odd $k$. Hence $\Omega_k H'$ is left-hard. By Lemma 4.1, $H'$ is left-hard. \[\square\]

**Conclusions** In the full version [55], we consider other examples of thin adjoint functors. In particular we discuss how results of Section 3 follow from Observation 2.1 by considering a thin right adjoint $\delta_R$ of $\delta$. We also introduce the algebraic framework of [18] and contemplate how Observation 2.1 and 2.2 could fit into it. A few of the thin adjoint functors we considered are in fact adjoint functors, in the category-theoretic sense. This does allow to deduce some instances of Obs. 2.1 and 2.2.
from algebraic reductions known as minion homomorphisms. However, the most useful examples: \((\delta, \delta_R)\) and \((\Gamma_k, \Omega_k)\), are only thin adjoint. Hence the question remains whether the notion of minion homomorphisms can be extended to account for them. It could also be interesting to consider how \(\delta\) or \(\delta_R\) affect the topology of a graph. Another direction could be to look at Huang’s Theorem 3.1 not as a black-box: could constructions like \(\delta\) be useful to say something directly about PCPs?

References
