The Complexity of Approximately Counting Retractions to Square-Free Graphs

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A *retraction* is a homomorphism from a graph G to an induced subgraph H of G that is the identity on H. In a long line of research, retractions have been studied under various algorithmic settings. Recently, the problem of approximately counting retractions was considered. We give a complete trichotomy for the complexity of approximately counting retractions to all square-free graphs (graphs that do not contain a cycle of length 4). It turns out there is a rich and interesting class of graphs for which this problem is complete in the class #BIS. As retractions generalise homomorphisms, our easiness results extend to the important problem of approximately counting homomorphisms. By giving new #BIS-easiness results we now settle the complexity of approximately counting homomorphisms for a whole class of non-trivial graphs which were previously unresolved.

CCS Concepts: • **Theory of computation → Approximation algorithms analysis**; **Problems, reductions and completeness**; • **Mathematics of computing → Combinatorics**.

Additional Key Words and Phrases: approximate counting, counting complexity, graph homomorphisms, retractions, square-free graphs

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1 INTRODUCTION

A function h that maps the vertices of a graph G to the vertices of a graph H is a *homomorphism* from G to H if h preserves the edges of G, i.e. if for every pair of adjacent vertices $u, v \in V(G)$ we have $\{h(u), h(v)\} \in E(H)$. It is well-known that homomorphisms represent graph-theoretic structures including proper vertex colourings and independent sets. For example, consider the graphs I, K_3 and C_4 given in Figure 1. A homomorphism from a graph G to I corresponds to an independent set
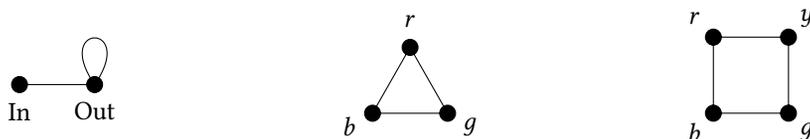

Fig. 1. The graphs I (on the left), K_3 (in the middle) and C_4 (on the right).

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in G, whereas a homomorphism from G to K_3 corresponds to a proper 3-colouring of G. Finally, a homomorphism from G to C_4 corresponds to a 4-colouring of G that uses the colours red (r), blue (b), green (g) and yellow (y), but for which $\{r, g\}$- and $\{b, y\}$-coloured edges are forbidden.

If for each vertex $v \in V(G)$ we specify a so-called "list" $S_v \subseteq V(H)$ and set $\mathbf{S} = \{S_v \mid v \in V(G)\}$, then h is a homomorphism from $(G, \mathbf{S})$ to H if it is a homomorphism from G to H such that for all $v \in V(G)$ it holds that $h(v) \in S_v$. This generalisation of a homomorphism is known as a *list homomorphism*. For example one could consider list homomorphisms from the Petersen graph (see Figure 2 on the left) to the graph K_3. These homomorphisms then correspond to proper 3-colourings of the Petersen graph, where some vertices have pre-assigned colours.
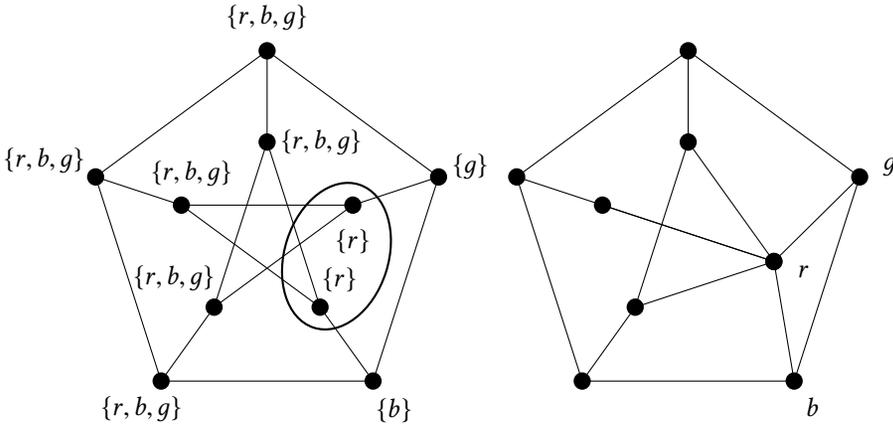

Fig. 2. Petersen graph with lists (on the left) and Petersen graph with vertices identified according to single-vertex lists (on the right).

If every list contains either just a single vertex or all of the vertices of H, then such a list homomorphism is called a *retraction*. This definition of a retraction is equivalent to the definition from the abstract, where a retraction was defined as a homomorphism from a graph G to an induced subgraph H of G that is the identity on H. The equivalence (in the sense of parsimonious polynomial-time interreducibility) was shown by Feder and Hell [15, Theorem 4.1]. The intuition behind the equivalence is that one can identify all of the vertices in G that have the same single-vertex list $\{u\}$ with the corresponding vertex u of H. Homomorphisms from this new graph to H are retractions in the sense of the abstract. For the Petersen graph in our example, the modified graph is displayed in Figure 2 on the right and contains K_3 as an induced subgraph.

Retractions are also known under the names *one-or-all list homomorphisms* (e.g. [15, 16]) and *pre-colouring extensions* (e.g. [3, 5, 17, 41, 44, 46, 52]). Related work on retractions is described in Section 1.1.

In the study of approximate counting there are three important classes of problems [12]: (1) problems that have fully-polynomial-time randomised approximation schemes (FPRASes), (2) problems that are approximation-equivalent to #BIS, the problem of counting independent sets in a bipartite graph, and (3) problems that are approximation-equivalent to #SAT, the problem of counting satisfying assignments to a Boolean formula (these problems have no FPRAS, unless NP = RP). It is believed that these three classes are disjoint, so there are no FPRASes for the #BIS-equivalent and #SAT-equivalent problems. The problems that are interreducible with #BIS under approximation-preserving (AP-)reductions are complete in a complexity class which is sometimes called #RHΠ_1

and is sometimes just called #BIS. For convenience we say that a graph H is #BIS-easy or #BIS-hard if the problem of approximately counting retractions to H is #BIS-easy or #BIS-hard, respectively. We similarly use the terms #SAT-easy, and #SAT-hard.

In this work we give a complete complexity trichotomy for approximately counting retractions to all square-free graphs (graphs that do not contain a 4-cycle) and we show that all of these problems fall within the three given complexity classes. An interesting feature is that the class of #BIS-equivalent graphs turns out to be surprising and rich.

First we give some illustrative examples. Afterwards we describe the class of #BIS-equivalent graphs in detail. A key idea that emerges in the proofs is the role of triangles (3-cycles). It turns out that triangles in graphs can induce hardness, but they can also "turn" #SAT-hard cases into #BIS-easy ones. For example, consider Figure 3. The graph on the left was shown to be #SAT-hard [21, Lemma 2.15]. In comparison, we will show that the graph on the right is actually #BIS-easy. Note that this is not because a vertex was added: If one deletes any of the edges of the triangle, the resulting graph is #SAT-hard again. To give an even more striking example, it seems surprising
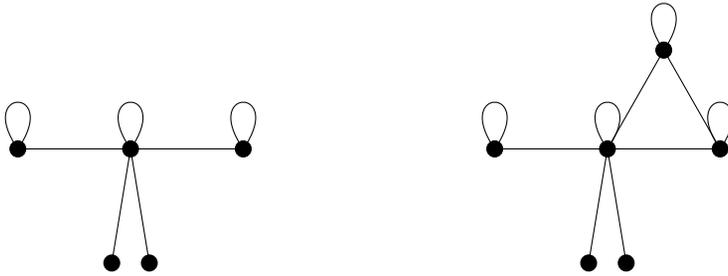
Fig. 3. Triangles can induce #BIS-easiness: The graph on the left is #SAT-hard whereas the graph on the right is #BIS-easy.

that the graph on the left in Figure 3 is #SAT-hard, but the graph depicted in Figure 4 turns out to be #BIS-easy. There exists an interesting underlying balancing process between looped cliques
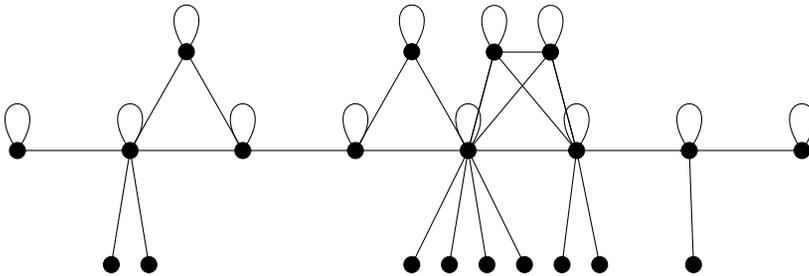
Fig. 4. Example graph which turns out to be #BIS-easy.

in the neighbourhood of a looped vertex and its number of unlooped neighbours, which decides whether a graph is #BIS-easy or #SAT-hard.

We now informally define the #BIS-easy class $\mathcal{H}_{\mathrm{BIS}}$ (it is defined formally in Definition 10). This class also contains graphs with squares and is more general than what we will need to classify all square-free graphs. A graph in $\mathcal{H}_{\mathrm{BIS}}$ is a path P of looped vertices with some attached unlooped

degree-1 vertices (bristles, depicted below the path in Figure 4) and some attached looped vertices forming cliques with two consecutive vertices of P (depicted above the path in Figure 4). For each vertex p of the path P the number of attached bristles satisfies the following properties:

- If p is an endpoint of P then it does not have a bristle.
- If p is not an endpoint of P then it is part of exactly two reflexive cliques K_L ("to the left" of p) and K_R ("to the right" of p). Then p has at most $(|K_L| - 1) \cdot (|K_R| - 1)$ bristles.

Consider again the graph in Figure 4. Note that the graph is not square-free as this is not required by the definition of $\mathcal{H}_{\text{BIS}}$. The third vertex from the right on the path is part of a reflexive 4-clique to the left and a reflexive 2-clique to the right. Hence the vertex can have at most $(4 - 1) \cdot (2 - 1) = 3$ bristles (and has in fact only 2 bristles).

Theorem 1. *Let H be a graph in $\mathcal{H}_{\text{BIS}}$. Then approximately counting retractions to H is #BIS-equivalent under approximation-preserving reductions.*

$\mathcal{H}_{\text{BIS}}$ is a fairly broad class of graphs but it turns out that approximately counting retractions to any member of $\mathcal{H}_{\text{BIS}}$ is #BIS-easy. For the class of square-free graphs, $\mathcal{H}_{\text{BIS}}$ completely captures the truth — together with the class of non-trivial irreflexive caterpillars (defined momentarily) they form precisely the class of #BIS-equivalent square-free graphs. Here (following a couple of necessary definitions) is the full complexity classification.

A graph H is *reflexive* if every vertex of H is looped and it is *irreflexive* if it contains only unlooped vertices. A *square* is a cycle of length 4. A connected irreflexive graph is a *star* if it contains at most one vertex of degree greater than 1, and it is a *caterpillar* if it contains a path P such that all vertices outside of P have degree 1. A graph is *trivial* if it is a reflexive clique or an irreflexive complete bipartite graph.

Theorem 2. *Let H be a square-free graph.*

i) *If every connected component of H is trivial then approximately counting retractions to H is in FP.*

ii) *Otherwise, if every connected component of H is*
- *trivial,*
- *in the class $\mathcal{H}_{\text{BIS}}$, or*
- *is an irreflexive caterpillar*

then approximately counting retractions to H is #BIS-equivalent.

iii) *Otherwise, approximately counting retractions to H is #SAT-equivalent.*

As a second central contribution of this work we emphasise the implications of Theorem 1 on the complexity of approximately counting homomorphisms. This problem has been studied intensively in the past [12, 13, 21, 24, 25, 29, 31, 43], but despite these efforts its complexity is still mostly open. For example, there are graphs with as few as four vertices for which its complexity is unresolved. In a partial result, Galanis, Goldberg and Jerrum [24] proved that approximately counting homomorphisms to a connected graph H is #BIS-hard unless H is an irreflexive complete bipartite graph or a reflexive clique. However, the knowledge about #BIS-easiness to complement this result is very fragmented. Since approximately counting homomorphisms reduces to approximately counting retractions ([21, Observation 1.2]), our #BIS-easiness results from Theorem 1 carry over to the homomorphism-counting domain. Thus, we are able to resolve the complexity of approximately counting homomorphisms to graphs in $\mathcal{H}_{\text{BIS}}$.

Corollary 3. *Let H be a graph in $\mathcal{H}_{\text{BIS}}$. Then approximately counting homomorphisms to H is #BIS-equivalent under approximation-preserving reductions.*

Remark 4. As a side note, the #BIS-easiness result given in Corollary 3 also extends to the problems of approximately counting surjective homomorphisms and approximately counting compactions. This follows from the reductions given in [21, Theorem 1.5].

1.1 Related Work

The concept of a retraction has been studied as early as the 1930's and originates in work on continuous functions from topological spaces into subspaces [7]. Subsequently, retractions between discrete structures, and graphs in particular, have received a lot of research attention [19, 32, 33, 36, 47]. See [35] for an overview. The algorithmic problem of deciding whether a retraction exists can be expressed naturally as a constraint satisfaction problem (CSP) and has been studied in e.g., [18, 34, 53–55]. Retractions have also been studied in the context of directed graphs. A work by Larose [45] surveys CSPs with a digraph as the fixed template. The survey has a special focus on CSPs with additional unary constraints, such as the digraph retraction problem. Applications of retractions reach from classical results such as Brouwer's fixed-point theorem and its equivalent no-retraction theorem [38, pp. 272-273] to specific problems such as solving Sudoku puzzles [37].

The complexity of deciding whether there is a retraction to a fixed graph H has been studied in different contexts, such as CSPs and list homomorphisms [15, 16, 18, 34, 35], surjective homomorphisms [4, 53–55], as well as pre-colouring extensions and scheduling [3, 5, 17, 39–41, 44, 46, 52]. A complete complexity dichotomy for the retraction decision problem is now known as a consequence of the CSP dichotomy [8, 58] (assuming P $\neq$ NP). However, a corresponding graph-theoretical characterisation is not known. A characterisation is known [18] for pseudoforests, which are graphs in which each connected component has at most one cycle.

The complexity of exactly counting retractions to a fixed graph H is also classified completely [14] (assuming FP $\neq$ #P). In this case, there is a characterisation – if every connected component of H is trivial, then counting retractions is in FP. Otherwise, counting retractions is #P-complete.

The first result on approximately counting retractions is the following classification for graphs which are both square-free and triangle-free.

Theorem 5 ([21, Theorem 1.1]). *Let H be a graph of girth at least 5.*
  i) *If every connected component of H is an irreflexive star, a single looped vertex, or an edge with two loops, then approximately counting retractions to H is in* FP.
  ii) *Otherwise, if every connected component of H is an irreflexive caterpillar or a partially bristled reflexive path, then approximately counting retractions to H is approximation-equivalent to* #BIS.
  iii) *Otherwise, approximately counting retractions to H is approximation-equivalent to* #SAT.

Counting homomorphisms to square-free graphs has been studied before [27, 42] though those results apply to counting modulo a prime number, and so they cannot be applied here.

In the past, graphs without squares have been studied in various other combinatorial settings: For example, [2] investigates the complexity of finding 101-colourings of square-free graphs. Polarity graphs are another natural class of square-free graphs that have been studied, e.g. [1, 6]. Wrochna showed that square-free graphs are multiplicative [56, 57], solving a problem related to Hedetniemi's conjecture[1] [51, Problem 7.1] and improving earlier results about the multiplicativity of square-free graphs that contain triangles [11]. Furthermore, square-free graphs play an important role in extremal graph theory and the study of the Turán number, e.g. [10, 20, 22, 23, 26, 48]. Recently, it has been shown that certain bounds on the chromatic number with respect to distance-two colouring of planar graphs are unique to square-free graphs [9].

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[1]This conjecture has been refuted lately [50].

1.2 Preliminaries

For a non-negative integer k we use $[k]$ to denote the set $\{1, \ldots, k\}$. For sets X and Y we define $X \times Y = \{\{x, y\} \mid x \in X, y \in Y\}$ as an unordered version of the Cartesian product. The elements of $X \times Y$ are *multisets* of size exactly 2. Using this notation the set of edges $E(H)$ of a graph $H = (V(H), E(H))$ is a subset of $V(H) \times V(H)$. An edge with two identical elements is a *loop*. Correspondingly, a vertex $v \in V(H)$ is called *looped* if $\{v, v\} \in E(H)$ and unlooped otherwise. The *girth* of a graph H is the length of a shortest cycle in H. All cycles have length at least 3.

We have already defined reflexive and irreflexive graphs. A graph H is a *mixed* graph if it contains both looped and unlooped vertices, i.e. if it is neither reflexive nor irreflexive. Given a graph H and a subset U of $V(H)$, $H[U]$ is the *subgraph of H induced by U*.

Given graphs G and H, $\mathcal{H}(G, H)$ is the set of homomorphisms from G to H and $N(G \to H)$ denotes its size. Analogously, given a corresponding set of lists $\mathbf{S}$, $\mathcal{H}((G, \mathbf{S}), H)$ is the set of homomorphisms from $(G, \mathbf{S})$ to H and $N((G, \mathbf{S}) \to H)$ denotes its size.

We use #RET(H) to denote the problem of approximately counting retractions to H (for a fixed graph H which may have loops but does not have multi-edges). We use #Hom(H) to denote the problem of approximately counting homomorphisms to H. Formally, these problems are defined as follows.

Name: #RET(H).
Input: An irreflexive graph G and a collection of lists $\mathbf{S} = \{S_v \subseteq V(H) \mid v \in V(G)\}$ such that, for all $v \in V(G)$, $|S_v| \in \{1, |V(H)|\}$.
Output: $N((G, \mathbf{S}) \to H)$.

Name: #Hom(H).
Input: An irreflexive graph G.
Output: $N(G \to H)$.

The list homomorphisms counting problem, defined as follows, is a generalisation of #RET(H).
Name: #LHom(H).
Input: An irreflexive graph G and a collection of lists $\mathbf{S} = \{S_v \subseteq V(H) \mid v \in V(G)\}$.
Output: $N((G, \mathbf{S}) \to H)$.

If there is an approximation-preserving reduction [12] from a problem A to a problem B, we write $A \leq_{\mathrm{AP}} B$.

Observation 6 ([21, Observation 1.2]). *Let H be a graph. Then #Hom(H) $\leq_{\mathrm{AP}}$ #Ret(H) $\leq_{\mathrm{AP}}$ #LHom(H).*

For approximately counting list homomorphisms Galanis, Goldberg and Jerrum [25] give the following complete classification.

Theorem 7 ([25]). *Let H be a connected graph.*

  i) If H is an irreflexive complete bipartite graph or a reflexive complete graph, then #LHom(H) is in FP.
  ii) Otherwise, if H is an irreflexive bipartite permutation graph or a reflexive proper interval graph, then #LHom(H) is approximation-equivalent to #BIS.
  iii) Otherwise, #LHom(H) is approximation-equivalent to #SAT.

1.3 Paper Outline

Theorem 1 is proved in Section 2. Theorem 2 is proved in Section 4. The corresponding #BIS-easiness follows mainly from Theorem 1. The corresponding #SAT-hardness results are collected in Section 3. Proving #SAT-hardness is the bulk of this work because of the combinatorial complexity

of designing reductions which establish #SAT-hardness for all square-free graphs (apart from reflexive cliques, irreflexive caterpillars and those in $\mathcal{H}_{\mathrm{BIS}}$).

2 #BIS-EASINESS RESULTS

In this section we prove Theorem 1, which states that approximately counting retractions to any graph from the class $\mathcal{H}_{\mathrm{BIS}}$ (Definition 10) is #BIS-equivalent. The proof is built on a method for generating #BIS-easiness results from [21, Section 2.2.1] which uses the framework of constraint satisfaction problems. Intuitively, the method takes as input two CSP instances, say I_{v} and I_{e}, and produces a graph $H_{I_{\mathrm{v}},I_{\mathrm{e}}}$ for which $\#\mathrm{RET}(H_{I_{\mathrm{v}},I_{\mathrm{e}}}) \leq_{\mathrm{AP}} \#\mathrm{BIS}$. The challenge is to find the right instances I_{v} and I_{e} and to identify and generate corresponding general classes of #BIS-easy graphs. For the convenience of the reader we repeat some definitions introduced in [21]. Let $\mathcal{L}$ be a set of Boolean relations.

Name: #CSP($\mathcal{L}$).
Input: A set of variables X and a set of constraints C, where each constraint applies a relation from $\mathcal{L}$ to a list of variables from X.
Output: The number of assignments $\sigma\colon X \to \{0, 1\}$ that satisfy all constraints in C.

Imp $= \{(0, 0), (0, 1), (1, 1)\}$ is an arity-two Boolean relation. The constraint Imp(x, y) ensures that, in any satisfying assignment σ, we have $\sigma(x) \implies \sigma(y)$.

Definition 8 ([21, Definition 2.6]). Let $I_{\mathrm{v}} = (X, C_{\mathrm{v}})$ and $I_{\mathrm{e}} = (X, C_{\mathrm{e}})$ be instances of #CSP($\{$Imp$\}$). We define the undirected graph $H_{I_{\mathrm{v}},I_{\mathrm{e}}}$ as follows. The vertices of $H_{I_{\mathrm{v}},I_{\mathrm{e}}}$ are the satisfying assignments of I_{v}. Given any assignments σ and σ' in $V(H_{I_{\mathrm{v}},I_{\mathrm{e}}})$, there is an edge $\{\sigma, \sigma'\}$ in $H_{I_{\mathrm{v}},I_{\mathrm{e}}}$ if and only if the following holds: For every constraint Imp(x, y) in C_{e}, we have $\sigma(x) \implies \sigma'(y)$ and $\sigma'(x) \implies \sigma(y)$.

Lemma 9 ([21, Lemmas 2.5 and 2.8]). Let $I_{\mathrm{v}} = (X, C_{\mathrm{v}})$ and $I_{\mathrm{e}} = (X, C_{\mathrm{e}})$ be instances of #CSP($\{$Imp$\}$). Then $\#\mathrm{RET}(H_{I_{\mathrm{v}},I_{\mathrm{e}}}) \leq_{\mathrm{AP}} \#\mathrm{BIS}$.

Definition 10. A graph H is in $\mathcal{H}_{\mathrm{BIS}}$ if it can be defined as follows. For some positive integer Q, the vertex set $V(H)$ is of the form $V(H) = \bigcup_{i=0}^{Q} K_i \cup \bigcup_{i=1}^{Q} B_i$ where $K_0, \ldots, K_Q$ induce reflexive cliques in H, and $B_1, \ldots, B_Q$ are disjoint sets of unlooped degree-1 vertices (called *bristles*). There are $Q + 2$ vertices $p_0, \ldots, p_{Q+1}$ in $V(H)$ such that each clique K_i contains both p_i and p_{i+1}. The intersection of the cliques is given as follows.

- For $i \in [Q]$, $K_{i-1} \cap K_i = \{p_i\}$.
- For $i, j \in \{0, \ldots, Q\}$ with $|j - i| > 1$, $K_i \cap K_j = \emptyset$.

The size of each set B_i of bristles satisfies $0 \leq |B_i| \leq (|K_{i-1}| - 1) \cdot (|K_i| - 1)$. Finally, the edge set of H is given as follows.

$$E(H) = \bigcup_{i=0}^{Q}(K_i \times K_i) \cup \bigcup_{i=1}^{Q}(\{p_i\} \times B_i).$$

For an example graph from the class $\mathcal{H}_{\mathrm{BIS}}$ see Figure 5.

Let $H \in \mathcal{H}_{\mathrm{BIS}}$ be as defined in Definition 10. The high-level-structure of the proof of Theorem 1 is as follows. We first define two instances I_{v} and I_{e} of #CSP($\{$Imp$\}$), then we establish that H is isomorphic to $H_{I_{\mathrm{v}},I_{\mathrm{e}}}$ (Lemma 21), which then allows us to apply Lemma 9.

> To give more intuition we will use a running example where H is the graph depicted in Figure 5. To separate this example from the rest of the proof we use text boxes.

Let V^* be the set of looped vertices in H, i.e. $V^* = \bigcup_{i=0}^{Q} K_i$ and let $X = \{x_v \mid v \in V^* \setminus \{p_0\}\}$ be a set of Boolean variables. We fix an ordering "<" on the vertices of V^* with two properties: (1) In K_i,
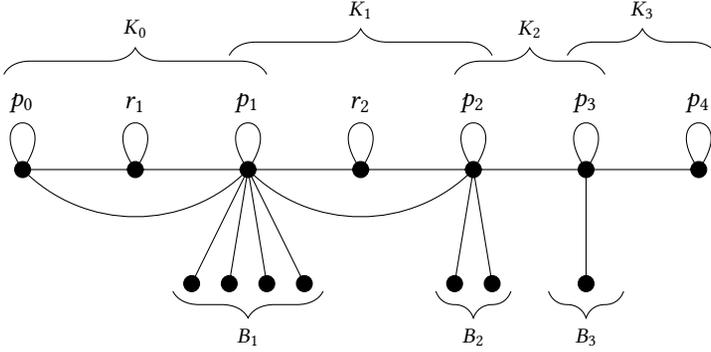
Fig. 5. Example graph from $\mathcal{H}_{\text{BIS}}$ for $Q = 3$. Note that $|B_1| = 4 = (|K_0| - 1) \cdot (|K_1| - 1)$, $|B_2| = 2 = (|K_1| - 1) \cdot (|K_2| - 1)$ and $|B_3| = 1 = (|K_2| - 1) \cdot (|K_3| - 1)$.

p_i is the smallest vertex and p_{i+1} is the largest; (2) The order of the K_i's is respected in the sense that, for any pair of distinct vertices $u, v \in V^*$, if there is an $i < j$ such that $u \in K_i$ and $v \in K_j$, then $u < v$. We define $U = \{\text{Imp}(x_u, x_v) \mid u > v\}$.

Consider $I_v^* = (X, U)$ and $I_e^* = (X, U)$. The graph $H_{I_v^*, I_e^*}$ is simply a reflexive path on $|V^*|$ vertices. We will construct I_v from I_v^* by choosing a subset C_v of U — this allows the creation of bristles. Similarly we will construct I_e from I_e^* by choosing a subset C_e of U — this creates reflexive cliques amongst the vertices of the reflexive path in $H_{I_v^*, I_e^*}$. In order to define C_v and C_e, for $i \in \{0, \ldots, Q\}$, we define the sets of constraints $D_e(i)$ (the constraints that we will delete from U to define C_e) as follows.

$$D_e(i) = \{\text{Imp}(x_u, x_v) \in U \mid u, v \in K_i \setminus \{p_i\}\}. \tag{1}$$

For $i \geq 1$ (i.e. for $i \in [Q]$), we define the sets of constraints $D_v(i)$ (the constraints we will delete from U to define C_v). The definition of $D_v(i)$ is a bit more involved and uses the following sets:

$$A(i) = \{\text{Imp}(x_u, x_v) \in U \mid u \in K_i \setminus \{p_i\}, \ v \in K_{i-1} \setminus \{p_{i-1}\}\}. \tag{2}$$

In order to model the set of bristles B_i we will "delete" exactly $|B_i|$ constraints that belong to $A(i)$ from C_v. As a means to specify which constraints will be deleted we define an order on the constraints in U (which uses the order $<$ on the vertices in V^* which we fixed previously). There are several orders which would work.

Definition 11. We define an order "$\preceq$" on U. Let $\text{Imp}(x_u, x_v), \text{Imp}(x_{u'}, x_{v'}) \in U$. Then we have $\text{Imp}(x_u, x_v) \preceq \text{Imp}(x_{u'}, x_{v'})$ if one of the following holds:

- $u < u'$.
- $u = u'$ and $v \geq v'$.

If $\text{Imp}(x_u, x_v) \preceq \text{Imp}(x_{u'}, x_{v'})$ and the ordered pair (u, v) is distinct from the ordered pair (u', v') then $\text{Imp}(x_u, x_v) \prec \text{Imp}(x_{u'}, x_{v'})$.

Now let $D_v(i)$ be the $|B_i|$ smallest elements of $A(i)$ with respect to $\preceq$ as given in Definition 11. Note that by the definition of $A(i)$ (Equation (2)) and the bound on $|B_i|$ (Definition 10) this is well-defined since

$$|B_i| \leq (|K_{i-1}| - 1) \cdot (|K_i| - 1) = |A(i)|.$$

Finally, we define C_v and C_e as follows.

$$C_v = U \setminus \left(\bigcup_{i=1}^{Q} D_v(i) \right) \quad \text{and} \quad C_e = U \setminus \left(\bigcup_{i=0}^{Q} D_e(i) \right). \tag{3}$$

In our running example, order the variables in V^* from left to right. We have $X = \{x_{r_1}, x_{p_1}, x_{r_2}, x_{p_2}, x_{p_3}, x_{p_4}\}$ as the set of variables of I_v and I_e. Since $|B_1| = 4$ the set $D_v(1)$ contains the 4 smallest elements of $A(1) = \{\text{Imp}(x_{r_2}, x_{p_1}), \text{Imp}(x_{r_2}, x_{r_1}), \text{Imp}(x_{p_2}, x_{p_1}), \text{Imp}(x_{p_2}, x_{r_1})\}$, which means $D_v(1) = A(1)$. Similarly, since $|B_2| = 2$, the set $D_v(2)$ contains the 2 smallest elements of $A(2) = \{\text{Imp}(x_{p_3}, x_{p_2}), \text{Imp}(x_{p_3}, x_{r_2})\}$, which means $D_v(2) = A(2)$. Finally, since $|B_3| = 1$, we have $D_v(3) = A(3) = \{\text{Imp}(x_{p_4}, x_{p_3})\}$. Thus,

$$C_v = \{\text{Imp}(x_{p_4}, x_{p_2}), \text{Imp}(x_{p_4}, x_{r_2}), \text{Imp}(x_{p_4}, x_{p_1}), \text{Imp}(x_{p_4}, x_{r_1}),$$
$$\text{Imp}(x_{p_3}, x_{p_1}), \text{Imp}(x_{p_3}, x_{r_1}), \text{Imp}(x_{p_2}, x_{r_2}), \text{Imp}(x_{p_1}, x_{r_1})\}. \tag{4}$$

Regarding the edge constraints we have $D_e(0) = \{\text{Imp}(x_{p_1}, x_{r_1})\}$, $D_e(1) = \{\text{Imp}(x_{p_2}, x_{r_2})\}$, $D_e(2) = \emptyset$ and $D_e(3) = \emptyset$ and hence

$$C_e = U \setminus \{\text{Imp}(x_{p_1}, x_{r_1}), \text{Imp}(x_{p_2}, x_{r_2})\}. \tag{5}$$

Recall that the satisfying assignments of I_v correspond to the vertices of H_{I_v, I_e}. For $v \in V^*$ let $\sigma_v \colon X \to \{0, 1\}$ be the assignment with

$$\sigma_v(x_u) = \begin{cases} 1, & \text{if } u \leq v \\ 0, & \text{otherwise,} \end{cases}$$

where $u \in V^* \setminus \{p_0\}$ (i.e. $x_u \in X$). Note that σ_{p_0} is the all-zero assignment since p_0 is the minimum vertex in V^*. The reason that we did not introduce a variable for p_0 in the definition of X is that its role is captured by the all-zero assignment to X. We will call assignments of the form σ_v *path assignments*. The path assignments inherit an order from the order on the set V^* that we fixed, i.e. $\sigma_u < \sigma_v$ if and only if $u < v$.

Lemma 12. *All path assignments satisfy $I_v = (X, C_v)$. If $\sigma_{p_0}, \ldots, \sigma_{p_{Q+1}}$ are the path assignments ordered by $<$, then they form a reflexive path in H_{I_v, C_v}. The reflexive path is not necessarily induced by its vertices.*

PROOF. This proof merely requires that $C_v \subseteq U$ and $C_e \subseteq U$ — it does not use the detailed definitions of C_v and C_e.

First, note that the assignments σ_v satisfy the #CSP($\{\text{Imp}\}$)-instance (X, U). Thus, they will still be satisfying assignments if we delete constraints from U.

We now investigate the edges between the vertices $\sigma_{p_0}, \ldots, \sigma_{p_{Q+1}}$ of H_{I_v, I_e}. From Definition 8 recall that, given any satisfying assignments σ and σ' of I_v, there is an edge $\{\sigma, \sigma'\}$ in H_{I_v, I_e} if and only if the following holds:

$$\text{For every constraint } \text{Imp}(x, y) \text{ in } C_e, \text{ we have } \sigma(x) \implies \sigma'(y) \text{ and } \sigma'(x) \implies \sigma(y). \tag{6}$$

Using (6) it can easily be checked that for $C_e = U$ (and therefore for all $C_e \subseteq U$), the vertices $\sigma_{p_0}, \ldots, \sigma_{p_{Q+1}}$ form a reflexive path, i.e. these vertices are looped and for $v \in V^* \setminus \{p_0\}$ and $v' = \max\{u \in V^* \mid u < v\}$ we have that $\{\sigma_{v'}, \sigma_v\}$ is an edge in H_{I_v, I_e}. □

Lemma 12 shows that even if we were to enforce all constraints in U both in the modelling of vertices ($I_v^* = (X, U)$) and of edges ($I_e^* = (X, U)$), the graph $H_{I_v^*, I_e^*}$ always contains a reflexive path on $|V^*|$ vertices. As noted earlier, $H_{I_v^*, I_e^*}$ is precisely this reflexive path. Our definition of C_e, given in (3), ensures that $C_e \subseteq U$ so it uses a subset of the constraints, which leads to the possibilitiy of more edges in H_{I_v, I_e}. The idea behind the construction is that a vertex $v \in V^*$ will be modelled by the satisfying assignment σ_v, which is a vertex in H_{I_v, I_e}. The previous lemma shows that these assignments form a reflexive path. We now show how the clique structure is modelled. Intuitively, the deleted constraints allow shortcuts that form cliques along the underlying path of path assignments. The following observation follows immediately from the definition of $D_e(i)$ in (1).

Observation 13. *For all $i \in [Q]$, $v \in V^*$ with $p_i < v$ and all $k \in \{0, \ldots, Q\}$, $\mathrm{Imp}(x_v, x_{p_i}) \notin D_e(k)$ and consequently $\mathrm{Imp}(x_v, x_{p_i}) \in C_e$ (since $\mathrm{Imp}(x_v, x_{p_i}) \in U$).*

Lemma 14. *Two vertices $u, v \in V^*$ are adjacent in H if and only if σ_u is adjacent to σ_v in H_{I_v, I_e}.*

Proof. First we consider the case where $u, v \in V^*$ are adjacent. Then there exists an index i such that $u, v \in K_i$. Consequently $p_i \le u \le p_{i+1}$ and $p_i \le v \le p_{i+1}$. Therefore, for $w \in V^*$, if $w \le p_i$ then $\sigma_u(x_w) = 1$ and $\sigma_v(x_w) = 1$, and if $w > p_{i+1}$ then $\sigma_u(x_w) = 0$ and $\sigma_v(x_w) = 0$. Let $\mathrm{Imp}(x_t, x_s)$ be some constraint in C_e. Then $\mathrm{Imp}(x_t, x_s) \notin D_e(i)$ and, by the Definition of $D_e(i)$ in (1), we have at least one of $s \le p_i$ or $t > p_{i+1}$. Using these facts, we can verify (6) for $\sigma = \sigma_u$ and $\sigma' = \sigma_v$ which shows that σ_u is adjacent to σ_v in H_{I_v, I_e}.

Now assume that u and v are not adjacent. Then there exist indices $i \ne j$ from $\{0, \ldots, Q\}$ such that $u \in K_i \setminus \{p_{i+1}\}$, $v \notin K_i$, $v \in K_j \setminus \{p_{j+1}\}$ and $u \notin K_j$. Without loss of generality assume $i < j$. Then $u < p_{i+1} < v$ so $\sigma_v(x_v) = 1$ and $\sigma_u(x_{p_{i+1}}) = 0$. However, by Observation 13, $\mathrm{Imp}(x_v, x_{p_{i+1}}) \in C_e$ and consequently σ_u and σ_v are not adjacent in H_{I_v, I_e}. □

In our example, the vertices $\sigma_{p_0}, \sigma_{r_1}, \sigma_{p_1}, \sigma_{r_2}, \sigma_{p_2}, \sigma_{p_3}, \sigma_{p_4}$ form a reflexive path (as guaranteed by Lemma 12). Furthermore, $\{\sigma_{p_0}, \sigma_{p_1}\}$ is an edge since $\mathrm{Imp}(x_{p_1}, x_{r_1}) \notin C_e$ by (5), and $\{\sigma_{p_1}, \sigma_{p_2}\}$ is an edge since $\mathrm{Imp}(x_{p_2}, x_{r_2}) \notin C_e$ by (5). The subgraph of H_{I_v, I_e} induced by the path assignments is as depicted in Figure 6 and is isomorphic to $H[V^*] = H[\{p_0, r_1, p_1, r_2, p_2, p_3, p_4\}]$ (as is guaranteed by Lemma 14).
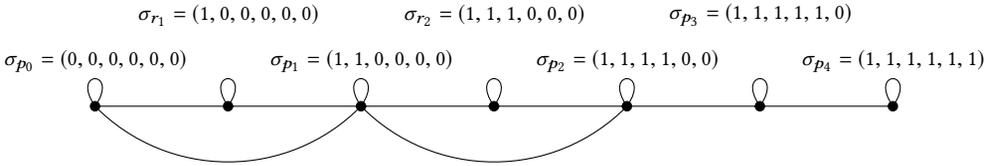
Fig. 6. The subgraph of H_{I_v, I_e} induced by the path assignments. If a vertex corresponds to a path assignment σ_v of I_v then its label is of the form $\sigma_v = (\sigma_v(x_{r_1}), \sigma_v(x_{p_1}), \sigma_v(x_{r_2}), \sigma_v(x_{p_2}), \sigma_v(x_{p_3}), \sigma_v(x_{p_4}))$.

With Lemma 14 we have established the clique structure of the looped vertices. Next we will model the bristles that are attached to the vertices $p_1, \ldots, p_Q$. To do this, we consider satisfying assignments of I_v that are not path assignments, i.e. that are not of the form σ_v. These assignments are called *bristle assignments*.

Definition 15. An assignment $\beta \colon X \to \{0, 1\}$ is a bristle assignment if and only if there exist $u, v \in V^* \setminus \{p_0\}$ with $u < v$ such that $\beta(x_u) = 0$ and $\beta(x_v) = 1$.

Definition 16. For $i \in [Q]$, $a \in K_{i-1} \setminus \{p_{i-1}\}$ and $b \in K_i \setminus \{p_i\}$ let $\beta_i[a,b] \colon X \to \{0,1\}$ be the assignment with

$$\beta_i[a,b](x_u) = \begin{cases} 1, & \text{if } u < a \\ 0, & \text{if } a \leq u \leq p_i \\ 1, & \text{if } p_i < u \leq b \\ 0, & \text{otherwise (if } b < u), \end{cases} \tag{7}$$

where $u \in V^* \setminus \{p_0\}$. Hence $\beta_i[a,b]$ is of the following form.

x_u		$\ldots$		x_a	$\ldots$	x_{p_i}		$\ldots$		x_b		$\ldots$		$x_{p_{Q+1}}$
$\beta_i[a,b](x_u)$	1	$\ldots$	1	0	$\ldots$	0	1	$\ldots$	1	0	$\ldots$	0

Note that a is the minimum index for which $\beta_i[a,b]$ takes value 0, and b is the maximum index for which $\beta_i[a,b]$ takes value 1. We say that a bristle assignment is *good* if it is of the form $\beta_i[a,b]$.

Observation 17. *Since $C_v \subseteq U$ the following are equivalent:*

(I) *$\beta_i[a,b]$ is a satisfying assignment of I_v.*
(II) *For all $s, t \in V^* \setminus \{p_0\}$ with $a \leq s \leq p_i$ and $p_i < t \leq b$ it holds that $\mathrm{Imp}(x_t, x_s) \notin C_v$.*

Lemma 18. *Every bristle assignment that satisfies $I_v = (X, C_v)$ is good.*

PROOF. The property of C_v that is used in this proof is that $U \setminus \bigcup_{i \in [Q]} A(i) \subseteq C_v$ and therefore if a constraint from U is not in C_v it has to be in one of the sets $A(i)$.

Let β be a bristle assignment that satisfies I_v. Let a be the minimum index with $\beta(x_a) = 0$ and let b be the maximum index with $\beta(x_b) = 1$. Since β is a bristle assignment, $b > a$. Then, since β is a satisfying assignment, $\mathrm{Imp}(x_b, x_a) \notin C_v$. Therefore $\mathrm{Imp}(x_b, x_a) \in A(i)$ for some $i \in [Q]$, and therefore (by the definition of $A(i)$), $b \in K_i \setminus \{p_i\}$ and $a \in K_{i-1} \setminus \{p_{i-1}\}$. This is consistent with Definition 16. We will show that $\beta = \beta_i[a,b]$. Let $u \in V^* \setminus \{p_0\}$. We investigate the value of $\beta(x_u)$ depending on u to show that β takes values as given in (7):

- If $u < a$, by the minimality of a we have $\beta(x_u) = 1$.
- If $u = a$, then $\beta(x_u) = \beta(x_a) = 0$ by the choice of a.
- If $a < u \leq p_i$, then $u \notin K_i \setminus \{p_i\}$ which implies $\mathrm{Imp}(x_u, x_a) \notin A(i)$ and consequently, since $\mathrm{Imp}(x_u, x_a)$ is in U, we have $\mathrm{Imp}(x_u, x_a) \in C_v$ (also using the fact that $\mathrm{Imp}(x_u, x_a)$ cannot be in any of the other sets $A(k)$ since $a \in K_{i-1} \setminus \{p_{i-1}\}$). Therefore it holds that $\beta(x_u) = 0$.
- If $p_i < u < b$, then $u \notin K_{i-1}$ which implies $\mathrm{Imp}(x_b, x_u) \notin A(i)$ and consequently, since $\mathrm{Imp}(x_b, x_u)$ is in U, we have $\mathrm{Imp}(x_b, x_u) \in C_v$ (also using the fact that $\mathrm{Imp}(x_b, x_u)$ cannot be in any of the other sets $A(k)$ since $b \in K_i \setminus \{p_i\}$). Therefore it holds that $\beta(x_u) = 1$.
- If $u = b$, then $\beta(x_u) = \beta(x_b) = 1$ by the choice of b.
- If $b < u$, by the maximality of b we have $\beta(x_u) = 0$.

$\square$

We can now check which of the (good) bristle assignments satisfy I_v in the running example. First, consider the set $D_v(1)$. We already showed, in the text box containing (4), that $D_v(1) = \{\mathrm{Imp}(x_{r_2}, x_{p_1}), \mathrm{Imp}(x_{r_2}, x_{r_1}), \mathrm{Imp}(x_{p_2}, x_{p_1}), \mathrm{Imp}(x_{p_2}, x_{r_1})\}$. From the definition of C_v (Equation (3)),

   (i) $\mathrm{Imp}(x_{r_2}, x_{p_1}) \notin C_v$.
   (ii) $\mathrm{Imp}(x_{r_2}, x_{r_1}) \notin C_v$.
  (iii) $\mathrm{Imp}(x_{p_2}, x_{p_1}) \notin C_v$.
  (iv) $\mathrm{Imp}(x_{p_2}, x_{r_1}) \notin C_v$.

The good bristle assignments of the form $\beta_1[a, b]$ have $a \in K_0 \setminus \{p_0\} = \{r_1, p_1\}$ and $b \in K_1 \setminus \{p_1\} = \{r_2, p_2\}$ so they are $\beta_1[p_1, r_2]$, $\beta_1[r_1, r_2]$, $\beta_1[p_1, p_2]$ and $\beta_1[r_1, p_2]$, which are as follows:

x_u	x_{r_1}	x_{p_1}	x_{r_2}	x_{p_2}	x_{p_3}	x_{p_4}
$\beta_1[p_1, r_2](x_u)$	1	0	1	0	0	0
$\beta_1[r_1, r_2](x_u)$	0	0	1	0	0	0
$\beta_1[p_1, p_2](x_u)$	1	0	1	1	0	0
$\beta_1[r_1, p_2](x_u)$	0	0	1	1	0	0

We now apply Observation 17. From (i) it follows that $\beta_1[p_1, r_2]$ satisfies I_v. Similarly, from (i) and (ii) it follows that $\beta_1[r_1, r_2]$ satisfies I_v. From (i) and (iii) it follows that $\beta_1[p_1, p_2]$ satisfies I_v. Finally, from (i), (ii), (iii) and (iv) it follows that $\beta_1[r_1, p_2]$ satisfies I_v. The four bristle assignments that we have checked correspond to the four bristles in B_1 in Figure 5. Similarly, one can check, from the definitions of $D_v(2)$ and $D_v(3)$, that $\beta_2[p_2, p_3], \beta_2[r_2, p_3]$ and $\beta_3[p_3, p_4]$ are the only other satisfying bristle assignments.

Lemma 19. *For every bristle assignment β that satisfies $I_v = (X, C_v)$ there exists $i \in [Q]$ such that σ_{p_i} is the only neighbour of β in H_{I_v, I_e}.*

    PROOF. Here we will use the fact that $U \setminus \bigcup_{i \in [Q]} A(i) \subseteq C_v$ and the fact that $C_e = U \setminus \left(\bigcup_{k=0}^{Q} D_e(k) \right)$.

Let β be a bristle assignment that satisfies I_v. By Lemma 18, β is good, i.e. there exist $i \in [Q]$, $a \in K_{i-1} \setminus \{p_{i-1}\}$ and $b \in K_i \setminus \{p_i\}$ such that $\beta = \beta_i[a, b]$. We will show that σ_{p_i} is the only neighbour of $\beta = \beta_i[a, b]$ in H_{I_v, I_e}.

Let ψ be some satisfying assignment of I_v which is adjacent to $\beta_i[a, b]$. Note that $\beta_i[a, b](x_{p_i}) = 0$. This fact together with Observation 13 (which states that for all $u \in V^*$ with $p_i < u$ we have $\mathrm{Imp}(x_u, x_{p_i}) \in C_e$) implies that, for all $u > p_i$, $\psi(x_u) = 0$.

Let v be the minimum vertex with $p_i < v$. Then $p_i < v \leq b$ and therefore $\beta_i[a, b](x_v) = 1$. The choice of v ensures that for all vertices $u \in V^*$, $u \leq p_i$ iff $u < v$. Therefore, if $u \leq p_i$ we have $\mathrm{Imp}(x_v, x_u) \in C_e$ (since $v \in K_i \setminus \{p_i\}$ but $u \notin K_i \setminus \{p_i\}$ and hence $\mathrm{Imp}(x_v, x_u) \notin D_e(i)$) and consequently $\psi(x_u) = 1$. Summarising, we obtain

$$\text{for all } u \leq p, \psi(x_u) = 1 \text{ and for all } u > p_i, \psi(x_u) = 0.$$

Thus, $\psi = \sigma_{p_i}$.

It remains to check that $\beta_i[a, b]$ and σ_{p_i} are in fact adjacent. To this end we verify (6):

Claim: If $\mathrm{Imp}(x_t, x_s) \in C_e$ then $\beta_i[a, b](x_t) \Rightarrow \sigma_{p_i}(x_s)$.

Proof of the claim: We check for possible violations. The only relevant s and t are those for which $s < t$, $\beta_i[a, b](x_t) = 1$ and $\sigma_{p_i}(x_s) = 0$, i.e., all s and t satisfying $p_i < s < t \leq b \leq p_{i+1}$. However, constraints of this form are in $D_e(i)$ and hence are not in C_e. *(End of the proof of the claim.)*

Claim: If $\mathrm{Imp}(x_t, x_s) \in C_e$ then $\sigma_{p_i}(x_t) \Rightarrow \beta_i[a, b](x_s)$.

Proof of the claim: Again we check for violations. The only relevant s and t are those for which $s < t$, $\sigma_{p_i}(x_t) = 1$ and $\beta_i[a, b](x_s) = 0$, i.e., all s and t satisfying $p_{i-1} < a \leq s < t \leq p_i$. However, constraints of this form are in $D_e(i - 1)$ and hence are not in C_e. *(End of the proof of the claim.)* □

Lemma 20. *For each $i \in [Q]$ there are exactly $|B_i|$ good bristle assignments that satisfy I_v and are adjacent to σ_{p_i} in H_{I_v, I_e}.*

PROOF. Every good bristle assignment is of the form $\beta_i[a, b]$ for some $i \in [Q]$, $a \in K_{i-1} \setminus \{p_{i-1}\}$ and $b \in K_i \setminus \{p_i\}$. In the proof of Lemma 19 we have shown that the only neighbour of $\beta_i[a, b]$ is σ_{p_i}. Then the following claim completes the proof of the lemma:

Claim: $\beta_i[a, b]$ satisfies I_v if and only if $\text{Imp}(x_b, x_a)$ is among the $|B_i|$ smallest elements of $A(i)$.

Proof of the claim: Since $a \in K_{i-1} \setminus \{p_{i-1}\}$ and $b \in K_i \setminus \{p_i\}$ we have $\text{Imp}(x_b, x_a) \in A(i)$.

First consider the case where $\text{Imp}(x_b, x_a)$ is one of the $|B_i|$ smallest elements of $A(i)$. From Observation 17 we know that $\beta_i[b, a]$ satisfies I_v if and only if for all $s, t \in V^* \setminus \{p_0\}$ with $a \leq s \leq p_i$ and $p_i < t \leq b$ it holds that $\text{Imp}(x_t, x_s) \notin C_v$. Note that each such $\text{Imp}(x_t, x_s)$ is in $A(i)$ and by Definition 11, $\text{Imp}(x_t, x_s) \leq \text{Imp}(x_b, x_a)$. Thus, $\text{Imp}(x_t, x_s) \in D_v(i)$ and we obtain $\text{Imp}(x_t, x_s) \notin C_v$ as required.

Now consider the remaining case where $\text{Imp}(x_b, x_a)$ is not among the the $|B_i|$ smallest elements of $A(i)$. Then, by our choice of $D_v(i)$, $\text{Imp}(x_b, x_a) \notin D_v(i)$. Moreover, for all $k \neq i$, $\text{Imp}(x_b, x_a) \notin D_v(k)$ since $\text{Imp}(x_b, x_a) \in A(i)$, $D_v(k) \subseteq A(k)$ and $A(k) \cap A(i) = \emptyset$. Hence $\text{Imp}(x_b, x_a) \in C_v$ and consequently $\beta_i[b, a]$ does not satisfy I_v by Observation 17. *(End of the proof of the claim.)* □

Lemma 21. *Let $I_v = (X, C_v)$ and $I_e = (X, C_e)$. Then H_{I_v, I_e} is isomorphic to H.*

PROOF. Here we collect the previous results. By Lemma 12 the path assignments satisfy I_v (and hence are vertices of H_{I_v, I_e}). Lemma 14 shows that the subgraph of H_{I_v, I_e} induced by the path assignments is isomorphic to $H[V^*]$. By Lemma 18 all other satisfying assignments are good bristle assignments and by Lemma 19 each of these good bristle assignments has a unique neighbour, which is among $\sigma_{p_1}, \ldots, \sigma_{p_Q}$. Then Lemma 20 shows that there are exactly $|B_i|$ good bristle assignments adjacent to σ_{p_i}. □

We can now prove Theorem 1, which we re-state at this point for the convenience of the reader.

Theorem 1. *Let H be a graph in $\mathcal{H}_{\text{BIS}}$. Then approximately counting retractions to H is #BIS-equivalent under approximation-preserving reductions.*

PROOF. Let $H \in \mathcal{H}_{\text{BIS}}$. The #BIS-easiness part of Theorem 1 follows directly from Lemmas 21 and 9. The #BIS-hardness part follows from $\#\text{HOM}(H) \leq_{\text{AP}} \#\text{RET}(H)$ (Observation 6) together with the fact that #BIS $\leq_{\text{AP}} \#\text{HOM}(H)$ for all connected graphs H other than reflexive cliques and irreflexive stars [24, Theorem 1]. □

In our running example we conclude by showing in Figure 7 how H is encoded as H_{I_v, I_e}. We have already demonstrated which assignments are satisfying and thus are vertices of H_{I_v, I_e}. Using (5) and (6) it is straightforward to verify (with some work) that each satisfying bristle assignment of the form $\beta_i[a, b]$ is in fact adjacent only to σ_{p_i}.
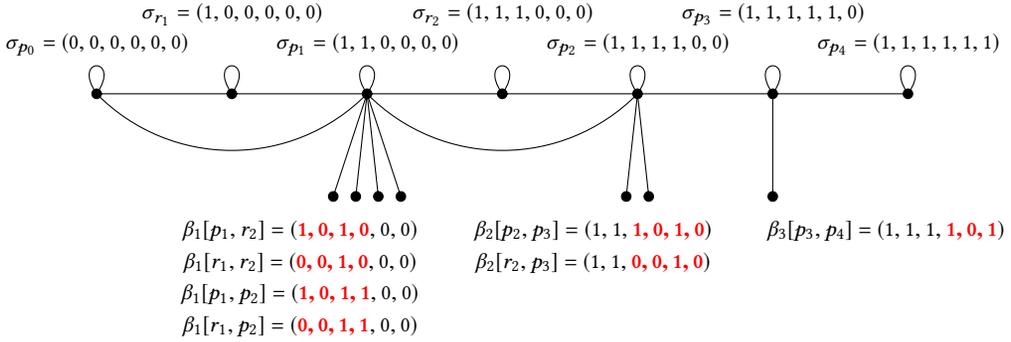
$$\sigma_{r_1} = (1, 0, 0, 0, 0, 0) \qquad \sigma_{r_2} = (1, 1, 1, 0, 0, 0) \qquad \sigma_{p_3} = (1, 1, 1, 1, 1, 0)$$
$$\sigma_{p_0} = (0, 0, 0, 0, 0, 0) \qquad \sigma_{p_1} = (1, 1, 0, 0, 0, 0) \qquad \sigma_{p_2} = (1, 1, 1, 1, 0, 0) \qquad \sigma_{p_4} = (1, 1, 1, 1, 1, 1)$$

$$\beta_1[p_1, r_2] = (\mathbf{1, 0, 1, 0}, 0, 0) \qquad \beta_2[p_2, p_3] = (1, 1, \mathbf{1, 0, 1, 0}) \qquad \beta_3[p_3, p_4] = (1, 1, 1, \mathbf{1, 0, 1})$$
$$\beta_1[r_1, r_2] = (\mathbf{0, 0, 1, 0}, 0, 0) \qquad \beta_2[r_2, p_3] = (1, 1, \mathbf{0, 0, 1, 0})$$
$$\beta_1[p_1, p_2] = (\mathbf{1, 0, 1, 1}, 0, 0)$$
$$\beta_1[r_1, p_2] = (\mathbf{0, 0, 1, 1}, 0, 0)$$

Fig. 7. The graph H_{I_v, I_e}. If a vertex corresponds to a satisfying assignment ρ of I_v then its label is of the form $\rho = (\rho(x_{r_1}), \rho(x_{p_1}), \rho(x_{r_2}), \rho(x_{p_2}), \rho(x_{p_3}), \rho(x_{p_4}))$.

3 #SAT-HARDNESS RESULTS

From previous results we already know the following:

- Theorem 5 classifies the complexity of approximately counting retractions to H for all graphs H that are both square-free and triangle-free (i.e. have girth at least 5).
- For irreflexive H, the proof of Theorem 5 does not use triangle-freeness — [21, Theorem 2.3] gives a trichotomy for approximately counting retractions to the class of irreflexive square-free graphs.

Therefore, we investigate square-free graphs that contain at least one triangle and at least one looped vertex. It turns out that we have to work through a number of technical cases to cover all #SAT-hard graphs with these properties (and hence all #SAT-hard square-free graphs).

For a positive integer q the graph WR_q is a looped star on $q + 1$ vertices (the underlying star has q degree-1 vertices). The (reflexive) *net* is a looped triangle where each vertex of the triangle has an additional looped neighbour, as shown in the following illustration:
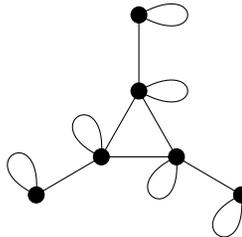
Here is an overview of the cases that we consider:

- In Section 3.2 we show that mixed triangles induce #SAT-hardness.
- In Section 3.3 we show in which cases the neighbourhood of a looped vertex induces #SAT-hardness.

- In Sections 3.4, 3.5 and 3.6 we show #SAT-hardness for square-free graphs with an induced WR$_3$, with an induced net and with an induced reflexive cycle of length at least 5, respectively. Essentially, these three sections deal with the graphs from the excluded subgraph characterisation of reflexive proper interval graphs (see [25, Section 1 and Appendix A] for the details about this characterisation).

3.1 Retractions and Neighbourhoods

Definition 22. For a graph H and a vertex $v \in V(H)$ we define the *(distance-1) neighbourhood* of v as $\Gamma_H(v) = \{u \in V(H) \mid \{u, v\} \in E(H)\}$. (In particular, this might include v itself.) Then $\deg_H(v) = |\Gamma_H(v)|$ is the *degree* of v. More generally, the *distance-k neighbourhood* of v is defined as $\Gamma_H^k(v) = \{u \in V(H) \mid$ There is a walk $W = u, w_1, \ldots, w_{k-1}, v$ (on k edges) in $H\}$. Let U be a subset of $V(H)$. Then $\Gamma_H(U) = \bigcap_{v \in U} \Gamma_H(v)$ is the *set of common neighbours* of the vertices in U.

The following well-known and simple observation shows that, for approximately counting retractions, hardness carries over from subgraphs that are induced by the neighbourhood of a vertex.

Observation 23. *Let H be a graph and let u be a vertex of H. Then $\#\mathrm{R}\textsc{et}(H[\Gamma_H(u)]) \leq_{\mathrm{AP}} \#\mathrm{R}\textsc{et}(H)$.*

Proof. Let $(G, \mathbf{S})$ be an input to $\#\mathrm{R}\textsc{et}(H[\Gamma_H(u)])$, let $v_1, \ldots, v_n$ be the vertices of G and $\mathbf{S} = \{S_v \mid v \in V(G)\}$. Let w be a vertex distinct from the vertices in G. Then we construct the graph G' with vertices $V(G') = V(G) \cup \{w\}$ and edges $E(G') = E(G) \cup \{\{w, v_i\} \mid i \in [n]\}$. We set $\mathbf{S}' = \{S'_v \mid v \in V(G')\}$, where

$$S'_v = \begin{cases} \{u\}, & \text{if } v = w \\ S_v, & \text{if } v \in V(G) \text{ and } |S_v| = 1 \\ V(H), & \text{otherwise.} \end{cases}$$

Then $N\big((G, \mathbf{S}) \to H[\Gamma_H(u)]\big) = N\big((G', \mathbf{S}') \to H\big)$. □

3.2 Square-Free Graphs with Mixed Triangles

Lemma 24. *Let H be a square-free graph which contains a triangle with exactly two looped and one unlooped vertex. Then $\#SAT \leq_{\mathrm{AP}} \#\mathrm{R}\textsc{et}(H)$.*

Proof. Let b_1, b_2, r be a triangle in H, where b_1 and b_2 are looped and r is unlooped. Consider the neighbourhood $\Gamma_H(b_1) \cap \Gamma_H(b_2) = \Gamma_H(\{b_1, b_2\})$. Since H is square-free, $H[\Gamma_H(b_1) \cap \Gamma_H(b_2)]$ is precisely the triangle b_1, b_2, r. The problem $\#\mathrm{H}\textsc{om}(H[\{b_1, b_2, r\}])$ corresponds to counting independent sets where vertices not in the independent set have a weight of 2 and vertices in the independent set have weight 1. It is well-known that approximately counting weighted independent sets is #SAT-hard, see for instance [30, Lemma 2]. This gives $\#SAT \leq_{\mathrm{AP}} \#\mathrm{H}\textsc{om}(H[\Gamma_H(b_1) \cap \Gamma_H(b_2)])$. From Observation 6 it follows immediately that $\#SAT \leq_{\mathrm{AP}} \#\mathrm{R}\textsc{et}(H[\Gamma_H(b_1) \cap \Gamma_H(b_2)])$.

Finally, one can easily observe that $\#\mathrm{R}\textsc{et}(H[\Gamma_H(b_1) \cap \Gamma_H(b_2)]) \leq_{\mathrm{AP}} \#\mathrm{R}\textsc{et}(H)$: Let $(G, \mathbf{S})$ be an input to $\#\mathrm{R}\textsc{et}(H[\Gamma_H(b_1) \cap \Gamma_H(b_2)])$ and let $\mathbf{S} = \{S_v \mid v \in V(G)\}$. Let w_1 and w_2 be vertices distinct from the vertices in G. Then we construct the graph G' with vertices $V(G') = V(G) \cup \{w_1, w_2\}$ and edges $E(G') = E(G) \cup \big(V(G) \times \{w_1, w_2\}\big)$. We set $\mathbf{S}' = \{S'_v \mid v \in V(G')\}$, where

$$S'_v = \begin{cases} \{b_1\}, & \text{if } v = w_1 \\ \{b_2\}, & \text{if } v = w_2 \\ S_v, & \text{if } v \in V(G) \text{ and } |S_v| = 1 \\ V(H), & \text{otherwise.} \end{cases}$$

Then $N\big((G, \mathbf{S}) \rightarrow H[\Gamma_H(b_1) \cap \Gamma_H(b_2)]\big) = N\big((G', \mathbf{S}') \rightarrow H\big)$. ☐

Lemma 25. *Let H be a square-free graph which contains a triangle with exactly two unlooped and one looped vertex. Then #SAT $\leq_{\mathrm{AP}}$ #RET(H).*

PROOF. Let H_T be a triangle in H with vertices b, r_1 and r_2, where b is looped and both r_1 and r_2 are unlooped. Let $H' = H[\Gamma_H(b)]$. By Observation 23 it holds that #RET(H') $\leq_{\mathrm{AP}}$ #RET(H). Suppose we can show that #RET($H'[\{b, r_1\}]$) $\leq_{\mathrm{AP}}$ #RET(H'). Then $H'[\{b, r_1\}]$ is a single edge with one looped (b) and one unlooped vertex (r_1) and it is well-known that counting homomorphisms to this graph corresponds to counting independent sets, which in turn is known to be #SAT-hard ([12, Theorem 3]). Summarising we have

$$\text{\#SAT} \leq_{\mathrm{AP}} \text{\#HOM}(H'[\{b, r_1\}]) \leq_{\mathrm{AP}} \text{\#RET}(H'[\{b, r_1\}]) \leq_{\mathrm{AP}} \text{\#RET}(H') \leq_{\mathrm{AP}} \text{\#RET}(H),$$

where the second reduction is from Observation 6. This proves the lemma. It remains to prove the following claim.

Claim: #RET($H'[\{b, r_1\}]$) $\leq_{\mathrm{AP}}$ #RET(H').
Proof of the claim: For $u \in V(H')$ let $w(u)$ be the number of common neighbours of r_1 and u in H'. Then $w(b) = 2$ since r_1 and b have two common neighbours: r_2 and b, and these are their only common neighbours as H' is square-free. Similarly, $w(r_1) = 2$ as the "common" neighbours in this case are simply the neighbours of r_1, which are only b and r_2 (since H' is square-free). Now let $u \in V(H') \setminus \{b, r_1\}$. The vertex b is a common neighbour of u and r_1 since every vertex in H' is a neighbour of b. It turns out that b is the only common neighbour of u and r_1: Suppose there exists a vertex $u' \neq b$ in H' which is a common neighbour of u and r_1. If $u' = u$ (see Figure 8 on the left) then u is adjacent to r_1. Additionally, u is then looped and hence $u \neq r_2$. Then u, r_1, r_2, b is a

Fig. 8. Contradictions to the square-freeness of the graph H'.

square. If otherwise $u' \neq u$ then u, u', r_1, b is a square (see Figure 8 on the right), both cases give a contradiction. So we have shown that, for $u \in V(H')$,

$$w(u) = 2 \text{ if } u \in \{b, r_1\}, \text{ and } w(u) = 1 \text{ otherwise.} \tag{8}$$

Intuitively, we will use this fact to "boost" the vertices b and r_1 and make them exponentially more likely to be used by a homomorphism to H'.

Let q be the number of vertices of H'. Now let $(G, \mathbf{S})$ be an n-vertex input to #RET($H'[\{b, r_1\}]$) and let ε be the desired precision. As usual, from $(G, \mathbf{S})$ we define an input $(G', \mathbf{S}')$ to #RET(H'). We introduce a vertex p distinct from the vertices of G that will serve as a pin to the vertex r_1 in H'. Then, for each $v \in V(G)$ we introduce an independent set on s vertices all of which are connected only to p and v. The parameter s will depend on the input size, specifically we set $s = n^2$. Intuitively it is clear that this gadget introduces a weight equal to $w(u)^s$ for each vertex $u \in V(H')$. For sufficiently large s, the image of v is likely to be b or r_1. This implies the statement of the lemma. The reader that trusts this intuition can skip reading the following calculations.

We give the full details for the sake of completeness: For each $v \in V(G)$, let I_v be an independent set of size s with vertices distinct from the remaining vertices of G'. Then G' is the graph with vertices $V(G') = V(G) \cup \{p\} \cup \bigcup_{v \in V(G)} I_v$ and edges $E(G') = E(G) \cup \bigcup_{v \in V(G)} (\{v, p\} \times I_v)$. We set $\mathbf{S}' = \{S'_v \mid v \in V(G')\}$, where

$$S'_v = \begin{cases} \{r_1\}, & \text{if } v = p \\ S_v, & \text{if } v \in V(G) \text{ and } |S_v| = 1 \\ V(H'), & \text{otherwise.} \end{cases}$$

We say that a homomorphism $h \in \mathcal{H}((G', \mathbf{S}'), H')$ is *full* if $h(V(G)) \subseteq \{b, r_1\}$. Let Z^* be the number of full homomorphisms from $(G', \mathbf{S}')$ to H'. Let Z_0 be the number of non-full homomorphisms from $(G', \mathbf{S}')$ to H'. Then

$$N\big((G', \mathbf{S}') \to H'\big) = Z^* + Z_0. \tag{9}$$

For $h \in \mathcal{H}((G, \mathbf{S}), H')$, let $Z(h)$ be the number of homomorphisms $h' \in \mathcal{H}((G', \mathbf{S}'), H')$ for which $h = h'|_{V(G)}$. By the construction of G', every vertex $v \in V(G)$ with $h(v) \in \{b, r_1\}$ contributes a factor of 2^s to $Z(h)$, whereas a vertex $v \in V(G)$ with $h(v) \notin \{b, r_1\}$ contributes a factor of 1 to $Z(h)$. It follows that

$$Z^* = \sum_{h \in \mathcal{H}((G, \mathbf{S}), H'),\ h \text{ full}} Z(h) = 2^{sn} \cdot N\big((G, \mathbf{S}) \to H'[\{b, r_1\}]\big), \tag{10}$$

and

$$Z_0 = \sum_{h \in \mathcal{H}((G, \mathbf{S}), H'),\ h \text{ non-full}} Z(h) \leq 2^{s(n-1)} \cdot N\big((G, \mathbf{S}) \to H'\big) \leq 2^{s(n-1)} \cdot q^n.$$

Therefore,

$$Z_0 / 2^{sn} \leq 2^{-s} \cdot q^n \leq 1/4, \tag{11}$$

where the last inequality holds for sufficiently large n by the choice $s = n^2$. Summarising, by (9) and (10), we have

$$N\big((G, \mathbf{S}) \to H'[\{b, r_1\}]\big) = \frac{Z^*}{2^{sn}} \leq \frac{N\big((G', \mathbf{S}') \to H'\big)}{2^{sn}}$$

and, using (9), (10) as well as (11), we obtain

$$\frac{N\big((G', \mathbf{S}') \to H'\big)}{2^{sn}} = \frac{Z^*}{2^{sn}} + \frac{Z_0}{2^{sn}} \leq N\big((G, \mathbf{S}) \to H'[\{b, r_1\}]\big) + 1/4.$$

Hence $N\big((G', \mathbf{S}') \to H'\big)/2^{sn} \in \big[N\big((G, \mathbf{S}) \to H'[\{b, r_1\}]\big), N\big((G, \mathbf{S}) \to H'[\{b, r_1\}]\big) + 1/4\big]$. Let Q be the solution returned by an oracle call to #RET(H') with input $((G', \mathbf{S}'), \varepsilon/21)$, i.e. an approximation of $N\big((G', \mathbf{S}') \to H'\big)$. Then the output $\lfloor Q/2^{sn} \rfloor$ approximates $N\big((G, \mathbf{S}) \to H'[\{b, r_1\}]\big)$ with the desired precision as was shown in [12, Proof of Theorem 3]. *(End of the proof of the claim.)* □

3.3 Square-Free Neighbourhoods of a Looped Vertex

Now we consider graphs of the form $X(k_1, k_2, k_3)$ (see Figure 9). Why are we interested in these graphs? Let H be a square-free graph with a looped vertex b and let H not contain any mixed triangle as a subgraph. Then consider $H[\Gamma_H(b)]$, the graph induced by the neighbourhood of b. Since H does not contain mixed triangles, the unlooped neighbours of b do not have any neighbours in $H[\Gamma_H(b)]$ apart from b. Since H is square-free, $H[\Gamma_H(b)]$ is square-free as well and therefore a looped neighbour $u \neq b$ of b can have at most one additional neighbour apart from b and u itself (within $H[\Gamma_H(b)]$). It follows that $H[\Gamma_H(b)]$ is of the form $X(k_1, k_2, k_3)$. Note that $X(k_1, k_2, 0)$ does not contain any cycles. Therefore the hardness results for graphs of this form come from the classification for graphs of girth at least 5 (Theorem 5). The remaining cases ($k_3 \geq 1$) are covered in this work. As an overview in advance, we will obtain #SAT $\leq_{AP}$ #RET($X(k_1, k_2, k_3)$) in the following

cases (where in most cases we actually show the stronger result #SAT $\leq_{AP}$ #Hom($X(k_1, k_2, k_3)$) — #SAT-hardness for #Ret($X(k_1, k_2, k_3)$) then follows from Observation 6.):

- $k_3 = 0$ and
  - $k_2 = 0$ and $k_1 \geq 1$ (Theorem 5)
  - $k_2 = 1$ and $k_1 \geq 1$ (Theorem 5)
  - $k_2 = 2$ and $k_1 \geq 2$ (Theorem 5)
- $k_3 = 1$ and
  - $k_2 = 0$ and $k_1 \geq 1$ (Lemma 36)
  - $k_2 = 1$ and $k_1 \geq 3$ (Lemma 37)
- $k_3 = 2$, $k_2 = 0$ and $k_1 \geq 5$ (Lemma 38)
- $k_2 + k_3 \geq 3$ (Lemma 40)

Following the classification for graphs of the form $X(k_1, k_2, k_3)$ we give a hardness result (Lemma 39) which uses properties of the distance-2 neighbourhood of a looped vertex b in H.
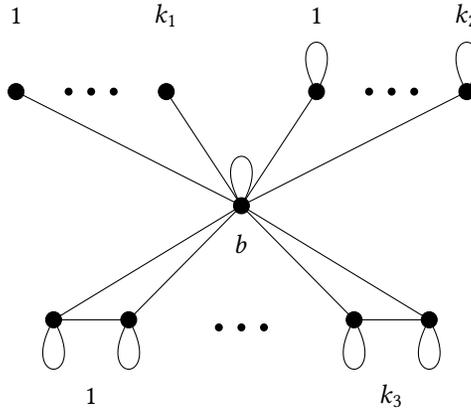
Fig. 9. The graph $X(k_1, k_2, k_3)$.

A useful and well-known tool for proving hardness results for approximate counting problems are gadgets based on complete bipartite graphs where two states dominate (see, e.g., [12, Lemma 25], [31, Section 5], [43, Lemma 5.1] and [21, Lemma 2.30]). Let $F(H) = \{u \in V(H) \mid \Gamma_H(u) = V(H)\}$. One can use the described tool to show that, under certain conditions, a homomorphism from a complete bipartite graph to H will typically map one side to $F(H)$ and the other to $V(H)$. In this case it is then easy to reduce from counting independent sets to obtain #SAT-hardness. Formally, we use the version stated by Kelk [43]:

Lemma 26 ([43, Lemma 5.1]). *Let H be a graph with $\emptyset \subsetneq F(H) \subsetneq V(H)$. Suppose that, for every pair (S, T) with $\emptyset \subseteq S, T \subseteq V(H)$ satisfying $S \subseteq \Gamma_H(T)$ and $T \subseteq \Gamma_H(S)$, at least one of the following holds:*

(1) $S = F(H)$.
(2) $T = F(H)$.
(3) $|S| \cdot |T| < |F(H)| \cdot |V(H)|$.

Then #SAT $\leq_{AP}$ #Hom(H).

In order to prove Lemmas 36 and 37 we will use Lemma 26 and, in addition, a reduction from the problem of counting large cuts, which is formally defined as follows: A *cut* of a graph G is a

partition of $V(G)$ into two subsets (the order of this pair is ignored) and the size of a cut is the number of edges that have exactly one endpoint in each of these two subsets.

Name: #LargeCut.

Input: An integer $K \geq 1$ and a connected graph G in which every cut has size at most K.

Output: The number of size-K cuts in G.

The full details of the proof involve analysing different types of homomorphisms. The most important part of the results leading up to the proof of Lemmas 36 and 37 are the Tables 1 and 2, respectively. These tables show which types of homomorphisms represent a significant share of the overall number of homomorphisms that we are interested in. The crucial question is whether we can ensure that the right types of homomorphisms dominate this number. We desire two properties. First, the number of homomorphisms should be dominated by homomorphisms of two distinct types. Second, these two types should interact in an "anti-ferromagnetic" way.

The following definitions and preliminary technical results resemble the ones introduced in [21, Section 2.2.2]. We are going to use the graph $J(p, q, t)$ (see Figure 10) as a vertex gadget. This gadget was originally introduced in [12]. In general it is a good candidate when looking for gadgets to prove reductions from #LargeCut. Here is the formal definition.
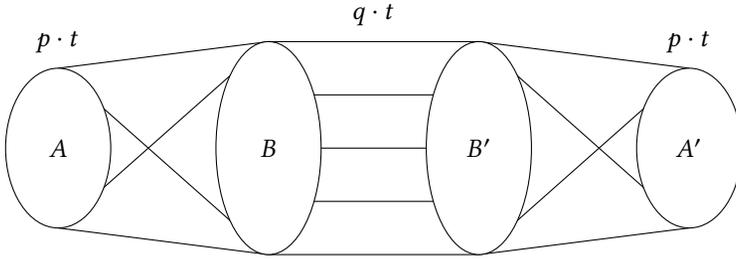

Fig. 10. The graph $J(p, q, t)$.

Let p, q and t be positive integers. Let A and A' be independent sets of size $p \cdot t$ and let B and B' be independent sets of size $q \cdot t$. The set of edges M between B and B' forms a perfect matching. Then $J(p, q, t)$ is the graph for which the vertex set is the union of A, B, B' and A'. The edges are $(A \times B) \cup M \cup (B' \times A')$.

Let H be a graph and let h be a homomorphism from $J(p, q, t)$ to H. Let $h(B, B') = \{(h(u), h(v)) \mid u \in B, v \in B', \{u, v\} \in E(H)\}$. We say that h has *type* $(h(A), h(B, B'), h(A'))$. In general, a tuple $T = (T_1, T_2, T_3)$ is an *H-type* if $T_1, T_3 \subseteq V(H)$ and $T_2 \subseteq \{(x, y) \mid \{x, y\} \in E(H)\}$. Let $A(T) = T_1$, $B(T) = \{x \mid \exists y \, (x, y) \in T_2\}$, $B'(T) = \{y \mid \exists x \, (x, y) \in T_2\}$ and $A'(T) = T_3$.

An H-type T is *non-empty* (with respect to $J(p, q, t)$) if there exists a homomorphism from $J(p, q, t)$ to H that has type T. Otherwise, T is called an *empty H-type*. From the definition of $J(p, q, t)$ we observe the following.

Observation 27. *Let H be a graph. An H-type T is non-empty if and only if*

(1) T_1, T_2 and T_3 are non-empty,

(2) $T_1 \times B(T) \subseteq E(H)$,

(3) $B'(T) \times T_3 \subseteq E(H)$.

Let T and T' be H-types. We write $T \subseteq T'$ if, for $i \in [3]$, we have $T_i \subseteq T_i'$. An H-type T is *maximal* if it is non-empty and every H-type T' with $T' \neq T$, $T \subseteq T'$ is empty. The following analysis (Lemma 28) is contained in the proof of [21, Lemma 2.25].

Lemma 28. *Let H be a graph and let $T = (T_1, T_2, T_3)$ be a maximal H-type. Then T is completely defined by $B(T)$ and $B'(T)$ since*

(1) $T_1 = \Gamma_H(B(T))$, $T_2 = E(B(T), B'(T))$ and $T_3 = \Gamma_H(B'(T))$.
(2) $B(T) = \Gamma_H(\Gamma_H(B(T)))$ and $B'(T) = \Gamma_H(\Gamma_H(B'(T)))$.

Given an H-type $T = (T_1, T_2, T_3)$ we define $N(T)$ to be the number of homomorphisms in $\mathcal{H}(J(p, q, t), H)$ that have type T. We also set $\widehat{N}(T) = |T_1|^{pt}|T_2|^{qt}|T_3|^{pt}$. For non-empty T, $\widehat{N}(T)$ is a close approximation to $N(T)$:

Lemma 29 ([21, Lemma 2.20]). *Let H be a graph. Let p and q be positive integers. There exists a positive integer t_0 such that for all $t \geq t_0$ and all H-types T that are non-empty with respect to $J(p, q, t)$, it holds that*

$$\frac{\widehat{N}(T)}{2} \leq N(T) \leq \widehat{N}(T).$$

Lemma 30 ([21, Lemma 2.23]). *Let H be a connected graph with at least 2 vertices. Let T be a non-empty H-type that is not maximal. Then there exists a non-empty H-type T^* such that $\widehat{N}(T) \leq \left(\frac{2|E(H)|-1}{2|E(H)|}\right)^t \widehat{N}(T^*)$.*
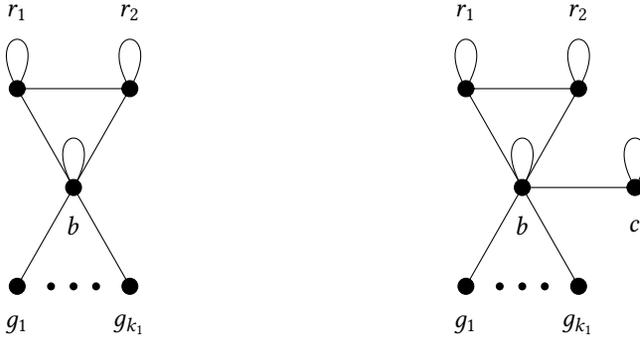

Fig. 11. The graphs $X(k_1, 0, 1)$ (on the left) and $X(k_1, 1, 1)$ (on the right).

Table 1. Maximal types of the homomorphisms from $J(p, q, t)$ to $X(k_1, 0, 1)$, where the vertices of $X(k_1, 0, 1)$ are labelled as in Figure 11 (on the left). Each line i corresponds to a type $T_i = \left(A(T_i), E(B(T_i), B'(T_i)), A'(T_i)\right)$. To shorten the notation we set $\mathcal{G} = \{g_j \mid j \in [k_1]\}$.

	$A(T)$	$B(T)$	$B'(T)$	$A'(T)$	$\widehat{N}(T)$
T_1	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{b\}$	$\{b\}$	$\{r_1, r_2, b\} \cup \mathcal{G}$	$(3 + k_1)^{pt} \cdot 1^{qt} \cdot (3 + k_1)^{pt}$
T_2	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$(3 + k_1)^{pt} \cdot 3^{qt} \cdot 3^{pt}$
T_3	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{b\}$	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{b\}$	$(3 + k_1)^{pt} \cdot (3 + k_1)^{qt} \cdot 1^{pt}$
T_4	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$3^{pt} \cdot 9^{qt} \cdot 3^{pt}$
T_5	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{b\}$	$3^{pt} \cdot (9 + k_1)^{qt} \cdot 1^{pt}$
T_6	$\{b\}$	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{r_1, r_2, b\} \cup \mathcal{G}$	$\{b\}$	$1^{pt} \cdot (9 + 2k_1)^{qt} \cdot 1^{pt}$

Let $T = \left(A(T), E(B(T), B'(T)), A'(T)\right)$ be an H-type. Then we call T *symmetric* to the H-type $T' = \left(A'(T), E(B'(T), B(T)), A(T)\right)$. Clearly, $\widehat{N}(T) = \widehat{N}(T')$.

Lemma 31. *Let $H = X(k_1, 0, 1)$. Then all maximal H-types are listed in Table 1 (apart from those that are symmetric to a listed H-type). For each listed H-type T the last column of the table gives the corresponding value $\widehat{N}(T)$.*

PROOF. Let $H = X(k_1, 0, 1)$ and let T be a maximal H-type. We claim that

$$B(T), B'(T) \in \{\{b\}, \{r_1, r_2, b\}, \{r_1, r_2, b, g_1, \ldots, g_{k_1}\}\}$$

for the following reasons (we give the arguments for $B(T)$, they are identical for $B'(T)$):

- Since b is a neighbour of every vertex in H, from item (2) of Lemma 28 we obtain $b \in B(T)$.
- If, for some $i \in [k_1]$, we have $g_i \in B(T)$ then $\Gamma_H(B(T)) = \{b\}$ as b is the only neighbour of g_i. By item (2) of Lemma 28 it follows that $B(T) = \{r_1, r_2, b, g_1, \ldots, g_{k_1}\}$.
- If $B(T) = \{r_1, b\}$, then $\Gamma_H(B(T)) = \{r_1, r_2, b\}$. By item (2) of Lemma 28 this gives $B(T) = \{r_1, r_2, b\}$, a contradiction.
- $B(T) = \{r_2, b\}$ gives a contradiction in the same way.

Table 1 then lists all possible combinations of sets $B(T)$ and $B'(T)$. From Lemma 28 it follows that these sets determine T completely (and $A(T), A'(T)$ are given accordingly). By definition, $T_1 = A(T)$ and $T_3 = A'(T)$. From item (1) of Lemma 28 it has to hold that $T_2 = E(B(T), B'(T))$. Then $\widehat{N}(T) = |T_1|^{pt}|T_2|^{qt}|T_3|^{pt}$ can be computed from the given sets in each row. □

Table 2. Maximal types of the homomorphisms from $J(p, q, t)$ to $X(k_1, 1, 1)$, where the vertices of $X(k_1, 1, 1)$ are labelled as in Figure 11 (on the right). Each line i corresponds to a type $T_i = \left(A(T_i), E\big(B(T_i), B'(T_i)\big), A'(T_i)\right)$. To shorten the notation we set $\mathcal{G} = \{g_j \mid j \in [k_1]\}$.

	$A(T)$	$B(T)$	$B'(T)$	$A'(T)$	$\widehat{N}(T)$
T_1	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$\{b\}$	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$(4 + k_1)^{pt} \cdot 1^{qt} \cdot (4 + k_1)^{pt}$
T_2	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$\{b, c\}$	$\{b, c\}$	$(4 + k_1)^{pt} \cdot 2^{qt} \cdot 2^{pt}$
T_3	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$(4 + k_1)^{pt} \cdot 3^{qt} \cdot 3^{pt}$
T_4	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$(4 + k_1)^{pt} \cdot (4 + k_1)^{qt} \cdot 1^{pt}$
T_5	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$2^{pt} \cdot 4^{qt} \cdot 2^{pt}$
T_6	$\{b, c\}$	$\{b, c\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$2^{pt} \cdot 4^{qt} \cdot 3^{pt}$
T_7	$\{b, c\}$	$\{b, c\}$	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$2^{pt} \cdot (6 + k_1)^{qt} \cdot 1^{pt}$
T_8	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$3^{pt} \cdot 9^{qt} \cdot 3^{pt}$
T_9	$\{r_1, r_2, b\}$	$\{r_1, r_2, b\}$	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$3^{pt} \cdot (10 + k_1)^{qt} \cdot 1^{pt}$
T_{10}	$\{b\}$	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{r_1, r_2, b, c\} \cup \mathcal{G}$	$\{b\}$	$1^{pt} \cdot (12 + 2k_1)^{qt} \cdot 1^{pt}$

Lemma 32. *Let $H = X(k_1, 1, 1)$. Then all maximal H-types are listed in Table 2 (apart from those that are symmetric to a listed H-type). For each listed H-type T the last column of the table gives the corresponding value $\widehat{N}(T)$.*

PROOF. Let $H = X(k_1, 1, 1)$ and let T be a maximal H-type. We claim that

$$B(T), B'(T) \in \{\{b\}, \{b, c\}, \{r_1, r_2, b\}, \{r_1, r_2, b, c, g_1, \ldots, g_{k_1}\}\}.$$

The remainder of the proof is analogous to that of Lemma 31 with only one additional argument:

- If $r_1 \in B(T)$ and $c \in B(T)$, then $\Gamma_H(B(T)) = \{b\}$. By item (2) of Lemma 28 it follows that $B(T) = \{r_1, r_2, b, c, g_1, \ldots, g_{k_1}\}$. The same is true if both $r_2 \in B(T)$ and $c \in B(T)$.

□

Lemma 33. *Let $k_1 \in [7]$. Consider the types T_i, $i \in [6]$ given by Table 1. Then there is a $\gamma \in (0, 1)$ and positive integers p and q such that, for all $i \in [6]$, $i \neq 5$ and all positive integers t, we have $\widehat{N}(T_i) \leq \gamma^t \widehat{N}(T_5)$.*

Proof. Let

$$\mathcal{L} = \left\{ \frac{\log\left(\frac{(3+k_1)^2}{3}\right)}{\log(9+k_1)}, \frac{\log(3+k_1)}{\log\left(\frac{9+k_1}{3}\right)}, \frac{\log\left(\frac{3+k_1}{3}\right)}{\log\left(\frac{9+k_1}{3+k_1}\right)}, \frac{\log 3}{\log\left(\frac{9+k_1}{9}\right)} \right\}$$

and

$$R = \frac{\log 3}{\log\left(\frac{9+2k_1}{9+k_1}\right)}.$$

For each of the seven possible values of k_1 we can check (for example by computer) that every member of $\mathcal{L}$ is less than R. Thus, we can choose p and q so that

$$\forall L \in \mathcal{L}, \ L < \frac{q}{p} < R. \tag{12}$$

We check the sought-for bound for each $i \in [6]$, $i \neq 5$:

T_1: $\frac{\widehat{N}(T_1)}{\widehat{N}(T_5)} = ((3+k_1)^2/3)^{pt}(1/(9+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $((3+k_1)^2/3)^p < (9+k_1)^q$ which is equivalent to $\log((3+k_1)^2/3)/\log(9+k_1) < q/p$. This is true by (12).

T_2: $\frac{\widehat{N}(T_2)}{\widehat{N}(T_5)} = (3+k_1)^{pt}(3/(9+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $(3+k_1)^p < ((9+k_1)/3)^q$. This is true by (12).

T_3: $\frac{\widehat{N}(T_3)}{\widehat{N}(T_5)} = ((3+k_1)/3)^{pt}((3+k_1)/(9+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $((3+k_1)/3)^p < ((9+k_1)/(3+k_1))^q$. This is true by (12).

T_4: $\frac{\widehat{N}(T_4)}{\widehat{N}(T_5)} = 3^{pt}(9/(9+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $3^p < ((9+k_1)/9)^q$. This is true by (12).

T_6: $\frac{\widehat{N}(T_6)}{\widehat{N}(T_5)} = (1/3)^{pt}((9+2k_1)/(9+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $((9+2k_1)/(9+k_1))^q < 3^p$. This is true by (12).

□

Lemma 34. *Let $k_1 \in \{3, 4, 5, 6\}$. Consider the types T_i, $i \in [10]$ given by Table 2. Then there is a $\gamma \in (0, 1)$ and positive integers p and q such that, for all $i \in [10]$, $i \neq 9$ and all positive integers t, we have $\widehat{N}(T_i) \leq \gamma^t \widehat{N}(T_9)$.*

Proof. Let

$$\mathcal{L} = \left\{ \frac{\log\left(\frac{(4+k_1)^2}{3}\right)}{\log(10+k_1)}, \frac{\log(4+k_1)}{\log\left(\frac{10+k_1}{3}\right)}, \frac{\log\left(\frac{4+k_1}{3}\right)}{\log\left(\frac{10+k_1}{4+k_1}\right)}, \frac{\log 3}{\log\left(\frac{10+k_1}{9}\right)} \right\}$$

and

$$R = \frac{\log 3}{\log\left(\frac{12+2k_1}{10+k_1}\right)}.$$

For each of the four possible values of k_1 we can check (for example by computer) that every member of $\mathcal{L}$ is less than R. Thus, we can choose p and q so that

$$\forall L \in \mathcal{L}, \ L < \frac{q}{p} < R. \tag{13}$$

Suppose that T and T' are types listed in Table 2 which are distinct from T_9 and have the property that $\widehat{N}(T') \leq \widehat{N}(T)$ for all $k_1 \in \{3, 4, 5, 6\}$. Then the sought-for bound automatically holds for T' if it holds for T.

We check the sought-for bound for each $i \in [10]$, $i \neq 9$:

T_1: $\dfrac{\widehat{N}(T_1)}{\widehat{N}(T_9)} = ((4+k_1)^2/3)^{pt}(1/(10+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $((4+k_1)^2/3)^p < (10+k_1)^q$. This is true by (13).

T_2: $\widehat{N}(T_2) \leq \widehat{N}(T_3)$.

T_3: $\dfrac{\widehat{N}(T_3)}{\widehat{N}(T_9)} = (4+k_1)^{pt}(3/(10+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $(4+k_1)^p < ((10+k_1)/3)^q$. This is true by (13).

T_4: $\dfrac{\widehat{N}(T_4)}{\widehat{N}(T_9)} = ((4+k_1)/3)^{pt}((4+k_1)/(10+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $((4+k_1)/3)^p < ((10+k_1)/(4+k_1))^q$. This is true by (13).

T_5: $\widehat{N}(T_5) \leq \widehat{N}(T_8)$.

T_6: $\widehat{N}(T_6) \leq \widehat{N}(T_8)$, for all $k \in \{3, 4, 5, 6\}$.

T_7: $\dfrac{\widehat{N}(T_7)}{\widehat{N}(T_9)} = (2/3)^{pt}((6+k_1)/(10+k_1))^{qt} < \gamma^t$ is fulfilled for $2/3 \cdot (6+k_1)/(10+k_1) < \gamma < 1$.

T_8: $\dfrac{\widehat{N}(T_8)}{\widehat{N}(T_9)} = 3^{pt}(9/(10+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $3^p < ((10+k_1)/9)^q$. This is true by (13).

T_{10}: $\dfrac{\widehat{N}(T_{10})}{\widehat{N}(T_9)} = (1/3)^{pt}((12+2k_1)/(10+k_1))^{qt} < \gamma^t$ is fulfilled for some sufficiently large $\gamma < 1$ if and only if $((12+2k_1)/(10+k_1))^q < 3^p$. This is true by (13).

$\square$

Remark 35. We point out that the proofs of Lemmas 33 and 34 break for larger k_1. (For larger k_1 there exists some lower bound on p/q which exceeds some upper bound on that ratio.) Lemma 34 also breaks for $k_1 = 1$ and $k_1 = 2$. This matches the results from Section 2 which show that approximately counting retractions to $X(k_1, 1, 1)$ is actually #BIS-easy for these values of k_1.

Lemma 36. If $k_1 \geq 1$, then $\#SAT \leq_{AP} \#H\textsc{om}(X(k_1, 0, 1))$.

PROOF. We make a case distinction depending on k_1. The first case is the main work of the proof and we use the dominance of the type T_5 from Table 1 for $k_1 \leq 6$ as shown in Lemma 33. The second case ($k_1 \geq 7$) then follows from Lemma 26.

Case 1: $k_1 \in [6]$. We use a reduction from #L\textsc{argeCut}, which is known to be #SAT-hard (see [12]). Let G and K be an input to #L\textsc{argeCut}, n be the number of vertices of G and $\varepsilon \in (0, 1)$ be the parameter of the desired precision. To shorten notation let $H = X(k_1, 0, 1)$. From G we construct an input G' to $\#H\textsc{om}(H)$ by introducing vertex and edge gadgets. We assume that the vertices of H are labelled as in Figure 11 (on the left).

Let p, q be positive integers that fulfil (12). Note that p and q only depend on k_1 which is a parameter of H and therefore does not depend on the input G. We define the parameter t of the gadget graph $J(p, q, t)$ to be $t = n^4$. We also define a new parameter $s = n + 2$.

For each vertex $v \in V(G)$ we introduce a vertex gadget G'_v which is a graph $J(p, q, t)$ as given in Figure 10. We denote the corresponding sets A, B, B', A' by A_v, B_v, B'_v and A'_v, respectively. We say that two gadgets G'_u and G'_v are adjacent if u and v are adjacent in G.

For every edge $e = \{u, v\} \in E(G)$ we introduce an edge gadget as follows. We introduce two size-s independent sets, denoted by S_e and S'_e. As shown in Figure 12 we construct the set of edges

$$E'_e = (B_u \times S_e) \cup (B'_u \times S'_e) \cup (B_v \times S'_e) \cup (B'_v \times S_e).$$
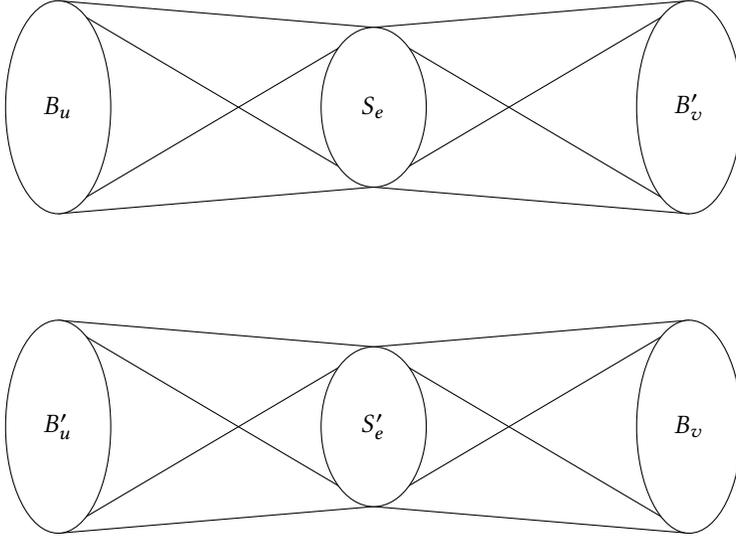
Fig. 12. The edge gadget for the edge $e = \{u, v\}$.

Putting the pieces together, G' is the graph with

$$V(G') = \bigcup_{v \in V(G)} V(G'_v) \cup \bigcup_{e \in E(G)} \left(S_e \cup S'_e\right) \qquad \text{and} \qquad E(G') = \bigcup_{v \in V(G)} E(G'_v) \cup \bigcup_{e \in E(G)} E'_e.$$

Let h be a homomorphism from G to H, v be some vertex of G and G'_v be the corresponding vertex gadget. Then $h|_{V(G'_v)}$ corresponds to a homomorphism from $J(p, q, t)$ to H and therefore has an H-type.

We say that a homomorphism from G' to H is *full* if its restriction to each vertex gadget is either of type T_5 (from Table 1) or of its symmetric type (let us call it T'_5). The cut corresponding to a full homomorphism h partitions $V(G)$ into those vertices v for which $h|_{G'_v}$ has type T_5 and those for which $h|_{G'_v}$ has type T'_5. We say that a full homomorphism is K-*large* if the size of the corresponding cut is equal to K, otherwise we say that the homomorphism is K-*small*. Consider a full homomorphism h from G' to H.

- For an edge $e = \{u, v\}$ of G suppose $h|_{G'_u}$ has type T_5 and $h|_{G'_v}$ has type T'_5. Note that by the definition of the edge gadget, we have $h(S_e) \subseteq \Gamma_H(h(B_u)) \cap \Gamma_H(h(B'_v))$. Then the vertices in S_e can be mapped to any of $\{r_1, r_2, b\}$, whereas all vertices in S'_e have to be mapped to b (the sole common neighbour of r_1, r_2, b and the vertices in $\mathcal{G}$).
- Suppose instead that $h|_{G'_u}$ and $h|_{G'_v}$ have the same type T_5 or T'_5. Then the homomorphism h has to map the vertices in both S_e and S'_e to b.

Thus, every pair of adjacent gadgets of different types contributes a factor of 3^s to the number of full homomorphisms, whereas every pair of adjacent gadgets of the same type only contributes a factor of 1. Recall the definition of $N(T)$ as the number of homomorphisms from $J(p, q, t)$ to H that have type T. Then for $\ell \geq 1$ every size-ℓ cut of G arises in $2 \cdot N(T_5)^n \cdot 3^{s\ell}$ ways as a full homomorphism from G' to H.

Let L be the number of solutions to #LargeCut with input G and K (our goal is to approximate this number). We partition the homomorphisms from G' to H into three different sets. Z^* is the number of K-large (full) homomorphisms, Z_1 is the number of homomorphisms that are full but K-small and Z_2 is the number of non-full homomorphisms. Then we have $L = Z^*/(2N(T_5)^n 3^{sK})$

and $N(G' \to H) = Z^* + Z_1 + Z_2$. Thus it remains to show that $(Z_1 + Z_2)/(2N(T_5)^n 3^{sK}) \le 1/4$ for our choice of p, q, t and s. Under this assumption we then have $N(G' \to H)/(2N(T_5)^n 3^{sK}) \in [L, L + 1/4]$ and a single oracle call to determine $N(G' \to H)$ with precision $\delta = \varepsilon/21$ suffices to determine L with the sought-for precision as demonstrated in [12].

Now we prove $(Z_1 + Z_2)/(2N(T_5)^n 3^{sK}) \le 1/4$. As there are at most 2^n ways to assign a type T_5 or T_5' to the n vertex gadgets in G' we have $Z_1 \le 2^n \cdot N(T_5)^n \cdot 3^{s(K-1)}$. Then we obtain the following bound since $s = n + 2$:

$$\frac{Z_1}{2N(T_5)^n 3^{sK}} \le \frac{2^n N(T_5)^n 3^{s(K-1)}}{2N(T_5)^n 3^{sK}} = \frac{2^n}{2 \cdot 3^s} \le \frac{1}{8}.$$

We obtain a similar bound for Z_2: From Lemmas 30, 31 and 33 we know that for our choice of p and q there exists $\gamma \in (0, 1)$ such that for every H-type T that is not T_5 or T_5' we have $\widehat{N}(T) \le \gamma^t \widehat{N}(T_5)$. Using Lemma 29 this gives $N(T) \le 2\gamma^t N(T_5)$ for sufficiently large t with respect to p, q and k_1 (which do not depend on the input G). Since $t = n^4$ we can assume that t is sufficiently large with respect to p and q as otherwise the input size is bounded by a constant (in which case we can solve #LargeCut in constant time).

For each H-type $T = (T_1, T_2, T_3)$, the cardinality of each set T_i is bounded from above by $\max\{|V(H)|, 2|E(H)|\} = 12 + 2k_1$ and hence there are at most $\left(2^{12+2k_1}\right)^3$ different types. Furthermore, as H has $3 + k_1$ vertices, there are at most $(3 + k_1)^{2sn^2}$ possible functions from the at most $2sn^2$ vertices in $\bigcup_{e \in E(G)}(S_e \cup S_e')$ to vertices in H. Since $t = n^4$ and $s = n + 2$ we obtain

$$\frac{Z_2}{2N(T_5)^n 3^{sK}} \le \frac{\left(2^{12+2k_1}\right)^{3n} \cdot N(T_5)^{n-1} \cdot 2\gamma^t N(T_5) \cdot (3 + k_1)^{2sn^2}}{2N(T_5)^n 3^{sK}}$$

$$= \gamma^t \cdot \frac{\left(2^{12+2k_1}\right)^{3n}(3 + k_1)^{2sn^2}}{3^{sK}} \le \frac{1}{8}.$$

The last inequality holds for sufficiently large n as

$$\frac{\left(2^{12+2k_1}\right)^{3n}(3 + k_1)^{2sn^2}}{3^{sK}} \le C^{n^3}$$

for some constant C that only depends on H, but not on the input G, whereas $t = n^4$. **(End of Case 1)**

Case 2: $k_1 \ge 7$. We will show that in this case we can apply Lemma 26 to obtain #SAT $\le_{AP}$ #Hom(H). We have $F(H) = \{b\}$. Therefore, $|F(H)| \cdot |V(H)| = |V(H)| = 3 + k_1 \ge 10$. Let (S, T) be a pair with $\emptyset \subseteq S, T \subseteq V(H)$ satisfying $S \subseteq \Gamma_H(T)$, $T \subseteq \Gamma_H(S)$ and both $S \ne \{b\}$ and $T \ne \{b\}$ to meet the requirements of Lemma 26. We have to show that $|S| \cdot |T| < |F(H)| \cdot |V(H)| = 3 + k_1$. Note that for every vertex $u \ne b$ of $H = X(k_1, 0, 1)$ it holds that $|\Gamma_H(u)| \le 3$. Therefore, $|S| \le |\Gamma_H(T)| \le 3$ and analogously $|T| \le |\Gamma_H(S)| \le 3$. Hence

$$|S| \cdot |T| \le 9 < 10 \le 3 + k_1.$$

(End of Case 2) □

Lemma 37. *If $k_1 \ge 3$, then #SAT $\le_{AP}$ #Hom($X(k_1, 1, 1)$).*

Proof. This proof is very similar to the proof of Lemma 36. We give the details for the sake of completeness. As before, we make a case distinction depending on k_1. The first case is about the dominance of the type T_9 from Table 2 for $k_1 \in \{3, 4, 5, 6\}$ as shown in Lemma 34. Otherwise, we use Lemma 26.

Case 1: $k_1 \in \{3, 4, 5, 6\}$. We use a reduction from #LargeCut (#SAT-hard by [12]). Let G and K be an input to #LargeCut, n be the number of vertices of G and $\varepsilon \in (0, 1)$ be the parameter of the desired precision. To shorten notation let $H = X(k_1, 1, 1)$. We assume that the vertices of H are labelled as in Figure 11 (on the right).

Let p, q be positive integers that fulfil (13). Note that p and q only depend on k_1 which is a parameter of H and therefore does not depend on the input G. We will define the parameter t of the gadget graph J to be $t = n^4$. We also define a new parameter $s = n + 2$.

For each vertex $v \in V(G)$ we introduce a vertex gadget G'_v which is a graph $J(p, q, t)$ as given in Figure 10. We denote the corresponding sets A, B, B', A' by A_v, B_v, B'_v and A'_v, respectively. We say that two gadgets G'_u and G'_v are adjacent if u and v are adjacent in G. For every edge $e = \{u, v\} \in E(G)$ we use exactly the same edge gadget as in the proof of Lemma 36 (see Figure 12). G' is the graph with

$$V(G') = \bigcup_{v \in V(G)} V(G'_v) \cup \bigcup_{e \in E(G)} (S_e \cup S'_e) \qquad \text{and} \qquad E(G') = \bigcup_{v \in V(G)} E(G'_v) \cup \bigcup_{e \in E(G)} E'_e.$$

Let h be a homomorphism from G to H, v be some vertex of G and G'_v be the corresponding vertex gadget. Then $h|_{V(G'_v)}$ corresponds to a homomorphism from $J(p, q, t)$ to H and therefore has an H-type.

We say that a homomorphism from G' to H is *full* if its restriction to each vertex gadget is either of type T_9 (from Table 2) or of its symmetric type (let us call it T'_9). The cut corresponding to a full homomorphism h partitions $V(G)$ into those vertices v for which $h|_{G'_v}$ has type T_9 and those for which $h|_{G'_v}$ has type T'_9. We say that a full homomorphism is *K-large* if the size of the corresponding cut is equal to K, otherwise we say that the homomorphism is *K-small*. Consider a full homomorphism h from G' to H.

- For an edge $e = \{u, v\}$ of G suppose $h|_{G'_u}$ has type T_9 and $h|_{G'_v}$ has type T'_9. Note that by the definition of the edge gadget, we have $h(S_e) \subseteq \Gamma_H(h(B_u)) \cap \Gamma_H(h(B'_v))$. Then the vertices in S_e can be mapped to any of $\{r_1, r_2, b\}$, whereas all vertices in S'_e have to be mapped to b (the sole common neighbour of r_1, r_2, b, c and the vertices in $\mathcal{G}$).
- Suppose instead that $h|_{G'_u}$ and $h|_{G'_v}$ have the same type T_9 or T'_9. Then the homomorphism h has to map the vertices in both S_e and S'_e to b.

Thus, every pair of adjacent gadgets of different types contributes a factor of 3^s to the number of full homomorphisms, whereas every pair of adjacent gadgets of the same type only contributes a factor of 1. Recall the definition of $N(T)$ as the number of homomorphisms from $J(p, q, t)$ to H that have type T. Then for $\ell \geq 1$ every size-ℓ cut of G arises in $2 \cdot N(T_9)^n \cdot 3^{s\ell}$ ways as a full homomorphism from G' to H.

Let L be the number of solutions to #LargeCut with input G and K (our goal is to approximate this number). We partition the homomorphisms from G' to H into three different sets. Z^* is the number of K-large (full) homomorphisms, Z_1 is the number of homomorphisms that are full but K-small and Z_2 is the number of non-full homomorphisms. Then we have $L = Z^*/(2N(T_9)^n 3^{sK})$ and $N(G' \to H) = Z^* + Z_1 + Z_2$. Thus it remains to show that $(Z_1 + Z_2)/(2N(T_9)^n 3^{sK}) \leq 1/4$ for our choice of p, q, t and s. Under this assumption we then have $N(G' \to H)/(2N(T_9)^n 3^{sK}) \in [L, L + 1/4]$ and a single oracle call to determine $N(G' \to H)$ with precision $\delta = \varepsilon/21$ suffices to determine L with the sought-for precision as demonstrated in [12].

Now we prove $(Z_1 + Z_2)/(2N(T_9)^n 3^{sK}) \leq 1/4$. As there are at most 2^n ways to assign a type T_9 or T'_9 to the n vertex gadgets in G' we have $Z_1 \leq 2^n \cdot N(T_9)^n \cdot 3^{s(K-1)}$. Then we obtain the following

bound since $s = n + 2$:

$$\frac{Z_1}{2N(T_9)^n 3^{sK}} \leq \frac{2^n N(T_9)^n 3^{s(K-1)}}{2N(T_9)^n 3^{sK}} = \frac{2^n}{2 \cdot 3^s} \leq \frac{1}{8}.$$

We obtain a similar bound for Z_2: From Lemmas 30, 32 and 34 we know that for our choice of p and q there exists $\gamma \in (0, 1)$ such that for every H-type T that is not T_9 or T_9' we have $\widehat{N}(T) \leq \gamma^t \widehat{N}(T_9)$. Using Lemma 29 this gives $N(T) \leq 2\gamma^t N(T_9)$ for sufficiently large t with respect to p, q and k_1. Since $t = n^4$ we can assume that t is sufficiently large with respect to p and q as otherwise the input size is bounded by a constant (in which case we can solve #LARGECUT in constant time).

For each H-type $T = (T_1, T_2, T_3)$, the cardinality of each set T_i is bounded from above by $\max\{|V(H)|, 2|E(H)|\} = 16 + 2k_1$ and hence there are at most $\left(2^{16+2k_1}\right)^3$ different types. Furthermore, as H has $4 + k_1$ vertices, there are at most $(4 + k_1)^{2sn^2}$ possible functions from the at most $2sn^2$ vertices in $\bigcup_{e \in E(G)}(S_e \cup S_e')$ to vertices in H. Since $t = n^4$ and $s = n + 2$ we obtain

$$\frac{Z_2}{2N(T_9)^n 3^{sK}} \leq \frac{\left(2^{16+2k_1}\right)^{3n} \cdot N(T_9)^{n-1} \cdot 2\gamma^t N(T_9) \cdot (4 + k_1)^{2sn^2}}{2N(T_9)^n 3^{sK}}$$

$$= \gamma^t \cdot \frac{\left(2^{16+2k_1}\right)^{3n}(4 + k_1)^{2sn^2}}{3^{sK}} \leq \frac{1}{8}.$$

The last inequality holds for sufficiently large n as

$$\frac{\left(2^{16+2k_1}\right)^{3n}(4 + k_1)^{2sn^2}}{3^{sK}} \leq C^{n^3}$$

for some constant C that only depends on H, but not on the input G, whereas $t = n^4$. **(End of Case 1)**

Case 2: $k_1 \geq 6$. We will show that in this case we can apply Lemma 26 to obtain #SAT $\leq_{AP}$ #HOM(H). We have $F(H) = \{b\}$. Therefore, $|F(H)| \cdot |V(H)| = |V(H)| = 4 + k_1 \geq 10$. Let (S, T) be a pair with $\emptyset \subseteq S, T \subseteq V(H)$ satisfying $S \subseteq \Gamma_H(T)$, $T \subseteq \Gamma_H(S)$ and both $S \neq \{b\}$ and $T \neq \{b\}$ to meet the requirements of Lemma 26. We have to show that $|S| \cdot |T| < |F(H)| \cdot |V(H)| = 4 + k_1$. Note that for every vertex $u \neq b$ of $H = X(k_1, 1, 1)$ it holds that $|\Gamma_H(u)| \leq 3$. Therefore, $|S| \leq |\Gamma_H(T)| \leq 3$ and analogously $|T| \leq |\Gamma_H(S)| \leq 3$. Hence

$$|S| \cdot |T| \leq 9 < 10 \leq 4 + k_1.$$

(End of Case 2) □

Lemma 38. *If $k_1 \geq 5$, then #SAT $\leq_{AP}$ #HOM($X(k_1, 0, 2)$).*

PROOF. Let $k_1 \geq 5$. To shorten notation let $H = X(k_1, 0, 2)$. Again we use Lemma 26 to obtain #SAT $\leq_{AP}$ #HOM(H). We have $F(H) = \{b\}$ and $|F(H)| \cdot |V(H)| = |V(H)| = 5 + k_1 \geq 10$. The remainder of the proof is identical to Case 2 in the proof of Lemma 37. □

Lemma 39. *Let H be a graph and $b \in V(H)$ be a looped vertex with an unlooped neighbour $g \in V(H)$. If $|\Gamma_H(g)| \geq 2$ (g has at least 2 neighbours in H) and for all $u \in \Gamma_H(b) \setminus \{g\}$ we have $|\Gamma_H(u) \cap \Gamma_H(g)| = 1$ (b is the only common neighbour of g and u), then #SAT $\leq_{AP}$ #RET(H).*

PROOF. Let H' be the graph obtained by replacing the vertex g in $H[\Gamma_H(b)]$ by an independent set I of size $|\Gamma_H(g)|^s$, where $s = 2\left\lceil \log_{|\Gamma_H(g)|}(|\Gamma_H(b)|) \right\rceil$. This is well-defined as $|\Gamma_H(g)| > 1$. The choice of s will become clear in a moment. Within the graph $H[\Gamma_H(b)]$, g is adjacent only to b by the assumption that $|\Gamma_H(b) \cap \Gamma_H(g)| = 1$. Therefore, each vertex in I shares an edge only with b. The transformation is depicted in Figure 13.
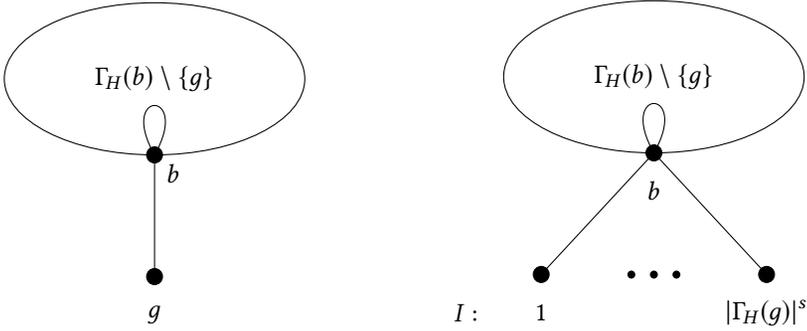
Fig. 13. $H[\Gamma_H(b)]$ on the left and H' on the right.

First, we will show #SAT-hardness for #Ret(H') and we will apply Lemma 26 to achieve this. Let us check that the requirements are met. First note that $F(H') = \{b\}$. Now we have to show that, for every pair (S, T) with $\emptyset \subsetneq S, T \subseteq V(H')$ satisfying $S \subseteq \Gamma_{H'}(T)$, $T \subseteq \Gamma_{H'}(S)$ and both $S \neq \{b\}$ and $T \neq \{b\}$ it holds that $|S| \cdot |T| < |F(H')| \cdot |V(H')| = |V(H')|$. First note that if there exists a vertex $u \in I \cap S$ then $T = \{b\}$ as b is the only neighbour of u in H'. Hence $I \cap S = \emptyset$. By the same reasoning it holds that $I \cap T = \emptyset$. Then $|S|, |T| \leq |\Gamma_{H'}(b) \setminus I| \leq |\Gamma_H(b)|$ and, by our choice $s = 2\left\lceil \log_{|\Gamma_H(g)|}(|\Gamma_H(b)|) \right\rceil$, we can conclude that

$$|S| \cdot |T| \leq |\Gamma_H(b)|^2 \leq |\Gamma_H(g)|^s < |V(H')| = |F(H')| \cdot |V(H')|.$$

This proves #SAT $\leq_{AP}$ #Ret(H').

To complete the proof of the lemma we show the following claim.

Claim: #Ret(H') $\leq_{AP}$ #Ret(H).

Proof of the claim:

Let $(G, \mathbf{S})$ be an input to #Ret(H'). Let w be a weight function on the vertices in $\Gamma_H(b)$ with $w(g) = |\Gamma_H(g)|^s$ and $w(u) = 1$ for all $u \in \Gamma_H(b) \setminus \{g\}$. By the construction of H' it is standard that

$$N\big((G, \mathbf{S}) \to H'\big) = \sum_{h \in \mathcal{H}((G,\mathbf{S}),H[\Gamma_H(b)])} \prod_{v \in V(G)} w(h(v)). \tag{14}$$

Now consider the vertices in $\Gamma_H(b)$. For $u \in \Gamma_H(b)$, let $w'(u)$ be the number of common neighbours of u and g in H. (It is essential that we regard all neighbours in H, not just the neighbours in $\Gamma_H(b)$.) By definition $w'(u) = |\Gamma_H(g)|$ if $u = g$ and, by the assumptions of this lemma, $w(u) = 1$ if $u \in \Gamma_H(b) \setminus \{g\}$. Hence, for all $u \in \Gamma_H(b)$ we have

$$w(u) = w'(u)^s. \tag{15}$$

Intuitively, we will use the fact that g has larger "weight" w' compared to the other vertices in $\Gamma_H(b)$ to "boost" the vertex g and make it more likely (by a factor of $|\Gamma_H(g)|^s$) to be used in a homomorphism to $H[\Gamma_H(b)]$.

Here are the details: Let $(G, \mathbf{S})$ be an input to #Ret(H'). We construct a graph G' from G in the following way. Let β and γ be vertices that are distinct from the vertices in G. Intuitively, β and γ will serve as "pins" to b and g, respectively. In addition, for each $v \in V(G)$, we introduce a independent set I_v of size s, see Figure 14. Then G' is the graph with vertices $V(G') = V(G) \cup \{\beta, \gamma\} \cup \bigcup_{v \in V(G)} I_v$ and edges $E(G') = E(G) \cup \big(V(G) \times \{\beta\}\big) \cup \bigcup_{v \in V(G)}\big(I_v \times \{v, \gamma\}\big)$.

Consider a homomorphism h from $(G', \mathbf{S}')$ to H. Then, since every vertex in $V(G)$ is a neighbour of β and $h(\beta) = b$, $h|_{V(G)}$ is a homomorphism from $(G, \mathbf{S})$ to $H[\Gamma_H(b)]$. Furthermore, by the construction
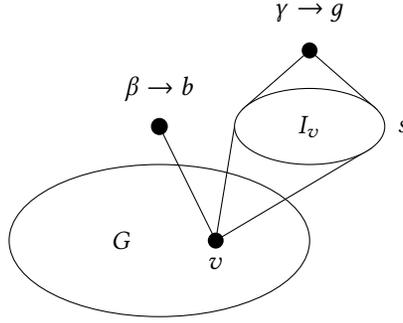
Fig. 14. The vertex gadget used in the construction of G'.

of G', for each $v \in V(G)$ and each $u \in I_v$, it holds that $h(u)$ is a common neighbour of $h(v)$ and g in H. Recall that there are $w'(h(v))$ such common neighbours. Thus, using (14) and (15), we conclude

$$
\begin{aligned}
N\big((G', \mathbf{S}') \to H\big) &= \sum_{h \in \mathcal{H}((G,\mathbf{S}), H[\Gamma_H(b)])} \prod_{v \in V(G)} w'(h(v))^s \\
&= \sum_{h \in \mathcal{H}(G, H[\Gamma_H(b)])} \prod_{v \in V(G)} w(h(v)) \\
&= N\big((G, \mathbf{S}) \to H'\big).
\end{aligned}
$$

□

3.4 Square-Free Graphs with an Induced WR$_3$

This section can be seen as an extension of Section 3.3 as it essentially shows #SAT-hardness for graphs of the form $X(k_1, k_2, k_3)$ where $k_2 + k_3 \geq 3$. Consider a square-free graph H with an induced WR$_3$. Suppose that there is an induced WR$_3$ such that the neighbourhood of its center b does not contain any triangles. Then $H[\Gamma_H(b)]$ is subject to Theorem 5 which shows #SAT $\leq_{\mathrm{AP}}$ #RET($H[\Gamma_H(b)]$). Then, #SAT $\leq_{\mathrm{AP}}$ #RET(H) by Observation 23.

However, when considering square-free graphs as opposed to graphs of girth at least 5, $H[\Gamma_H(b)]$ might contain triangles. The smallest open case is displayed in Figure 15.
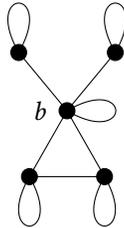
Fig. 15. Smallest square-free graph with induced WR$_3$ for which it remains to prove hardness.

The goal of this section is to prove Lemma 40.

Lemma 40. *Let H be a square-free graph. If H contains a* WR$_3$ *as an induced subgraph then* #SAT $\leq_{\mathrm{AP}}$ #RET(H).

First we will introduce some preliminary results.

Definition 41. Let $\left\{{a \atop b}\right\}$ be the *Stirling number of the second kind*, i.e. the number of surjective functions from a set of a elements to a set of b elements.

Lemma 42 ([12, Lemma 18]). *If a and b are positive integers and $a \geq 2b \ln b$, then*

$$b^a \left(1 - \exp\left(-\frac{a}{2b}\right)\right) \leq \left\{{a \atop b}\right\} \leq b^a.$$

Corollary 43. *If a and b are positive integers with $a \geq 2b \ln 2b$ and $b \geq 1$, then*

$$\frac{b^a}{2} \leq \left\{{a \atop b}\right\} \leq b^a.$$

The proof of Lemma 40 uses the same general idea as the proof of [28, Theorem 2] — namely that approximating the partition function of the q-state ferromagnetic Potts model is #SAT-equivalent if $q \geq 3$ and, in addition, we are allowed to specify that certain vertices have to have a specific spin. Crucially, #SAT-hardness is known only if this single-vertex "pinning" is allowed. In general, the complexity of approximating the partition function of the Potts model is still unresolved and an important open problem. The approach of simulating ferromagnetic Potts with "pinning" to obtain hardness results has been used before, for instance in the proofs of [29, Lemma 3.6] and [21, Lemma 2.2]. The gadgets we use here to accomplish the reduction are tailored to the specific problem and different from the gadgets used in similar reductions.

As in the proof of [28, Theorem 2] we use a reduction from the problem of counting so-called multiterminal cuts. We introduce the corresponding definitions from [28]. A *multiterminal cut* of a graph G with distinguished vertices $\tau_1, \ldots, \tau_q$ (called *terminals*) is a set of edges $E' \subseteq E(G)$ that disconnects the terminals (i.e. ensures that there is no path in $(V(G), E(G) \setminus E')$ that connects any two distinct terminals). The *size* of a multiterminal cut is its cardinality. We consider the following computational problem.

Name: #MULTITERMINALCUT(q).
Input: A connected irreflexive graph G with q distinct terminals $\tau_1, \ldots, \tau_q \in V(G)$ and a positive integer K. The input has the property that every multiterminal cut has size at least K.
Output: The number of size-K multiterminal cuts of G with terminals $\tau_1, \ldots, \tau_q$.

Lemma 44 ([28, Section 4]). *Let $q \geq 3$. Then #MULTITERMINALCUT(q) $\equiv_{AP}$ #SAT.*

Definition 45. Let $I = (G, \tau_1, \ldots, \tau_q, K)$ be an instance of #MULTITERMINALCUT(q). $\Phi(I) = \{\phi\colon V(G) \to [q] \mid \phi(\tau_i) = i, \ i \in [q]\}$ is the set of *separating functions* from $V(G)$ to $[q]$. For $\phi \in \Phi(I)$ let $\text{Cut}(\phi) = \{\{u, v\} \in E(G) \mid \phi(u) \neq \phi(v)\}$ and, for $i \in [q]$, let $\text{Mon}_i(\phi) = \{\{u, v\} \in E(G) \mid \phi(u) = \phi(v) = i\}$. Finally, let $\Phi^*(I) = \{\phi \in \Phi(I) \mid |\text{Cut}(\phi)| = K\}$.

Observation 46. *Let $I = (G, \tau_1, \ldots, \tau_q, K)$ be an instance of #MULTITERMINALCUT(q). For each $\phi \in \Phi(I)$, $\text{Cut}(\phi)$ is a multiterminal cut of I. On the other hand, each size-K multiterminal cut splits the graph G into exactly q connected components (as otherwise there would exist a multiterminal cut of size less than K). Hence each size-K multiterminal cut corresponds exactly to the function $\phi \in \Phi(I)$ for which $\phi(v) = i$ if v is in the same connected component as τ_i. Thus, $\Phi^*(I)$ is the subset of functions in $\Phi(I)$ that correspond to size-K multiterminal cuts. Let $T(I)$ be the number of size-K multiterminal cuts of the instance I. Then $T(I) = |\Phi^*(I)|$.*

Now we have all the tools at hand to prove the main lemma of this section.

Lemma 40. Let H be a square-free graph. If H contains a WR_3 as an induced subgraph then #SAT $\leq_{AP}$ #RET(H).

Proof. Suppose that H contains a mixed triangle as an induced subgraph, then the statement of this lemma follows from Lemmas 24 and 25. Hence, for the remainder of this proof let H be a square-free graph with an induced WR_3 without any induced mixed triangle subgraphs. We choose a vertex b such that b is the center of an induced WR_3. We consider the graph $H[\Gamma_H(b)]$ which is the subgraph of H that is induced by the neighbourhood of b. For ease of notation we set $H_b = H[\Gamma_H(b)]$. Let U be the set of unlooped neighbours of b. Since H does not contain any mixed triangles, for each $u \in U$, b is the only neighbour of u in H_b. Since H is square-free, so is H_b. Therefore every looped neighbour $w \neq b$ of b has degree $\deg_{H_b}(w) \in \{2, 3\}$. By the choice of b, b has at least 4 neighbours including itself, i.e. we have $\deg_{H_b}(b) = \deg_H(b) \geq 4$. Let $x_1, \ldots, x_k$ be the looped neighbours of degree 2 and $x_{k+1}, y_{k+1}, \ldots, x_q, y_q$ be the looped neighbours of degree 3, where for each $i \in \{k + 1, \ldots, q\}$ we have $\{x_i, y_i\} \in E(H_b)$ (where we use that looped vertices can only have looped neighbours since b is the sole neighbour of vertices in U). The graph H_b is depicted in Figure 16.
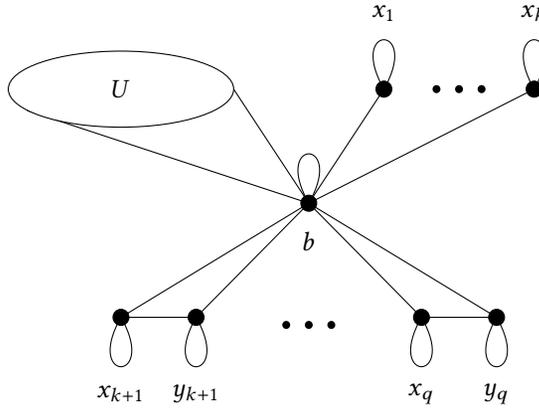
Fig. 16. The graph H_b.

We will give a reduction from #MULTITERMINALCUT(q) to #RET(H_b). By the choice of b we have $q \geq 3$. This gives the desired reduction since

$$\text{#SAT} \leq_{AP} \text{#MULTITERMINALCUT(q)} \leq_{AP} \text{#RET}(H_b) \leq_{AP} \text{#RET}(H),$$

where the first reduction is from Lemma 44 and the last reduction is from Observation 23.

Let $I = (G, \tau_1, \ldots, \tau_q, K)$ be an instance of #MULTITERMINALCUT(q) and let $\varepsilon \in (0, 1)$ be the desired precision bound. Let $n = |V(G)|$ and $m = |E(G)|$. From the instance I we construct an instance $(J, \mathbf{S})$ of #RET(H_b). We will need some parameters whose relevance will become clear later in the proof. Let $s = n^5$ and $t = n^2$.

The intuition behind the gadgets that will be used in this proof is the following. For every vertex v in G we introduce a huge clique C_v. The image of such a clique under a homomorphism to H_b tends to be a reflexive clique, i.e. tends to be of the form $\Gamma_{H_b}(x_i)$. There are q such neighbourhoods. These will correspond to the q different states that a vertex $v \in V(G)$ can be in. We will have to add some attachments to the clique C_v to balance out the fact that $\Gamma_{H_b}(x_i)$ is a clique on 2 vertices if $i \leq k$, whereas it is a clique on 3 vertices if $i > k$. For each edge $\{u, v\} \in E(G)$ we introduce a

gadget that favours the case where u and v have identical states (i.e. the corresponding cliques have the same image under homomorphisms to H_b).

Here are the details. First we define the graph J. We introduce q distinct vertices $p_1, \ldots, p_q$ which will serve as "pins" to the vertices $x_1, \ldots, x_q$. In the first part of the construction we will only use the vertices $p_1, \ldots, p_k$. For every vertex $v \in V(G)$ we introduce a graph J_v (the "vertex gadget") as follows. Let C_v be a clique on s vertices. For each vertex w in this clique we introduce k distinct vertices $\{w_1, \ldots, w_k\}$. Then J_v is the graph with vertices

$$V(J_v) = \{p_1, \ldots, p_k\} \cup V(C_v) \cup \bigcup_{w \in V(C_v)} \{w_1, \ldots, w_k\}$$

and edges

$$E(J_v) = E(C_v) \cup \bigcup_{w \in V(C_v)} \bigcup_{i \in [k]} \{\{w, w_i\}, \{w_i, p_i\}\}.$$

The graph J_v is depicted in Figure 17. Note that the vertices $p_1, \ldots p_k$ are identical over all vertex gadgets whereas the remaining vertices are distinct for each v.
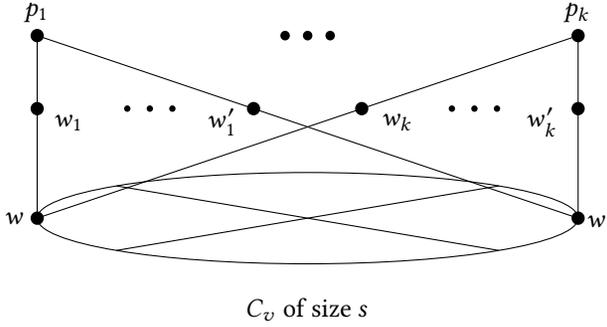
Fig. 17. The graph J_v for a vertex v.

For every edge $e = \{u, v\} \in E(G)$ we introduce a graph J_e together with a set of edges E_e (the "edge gadget"). The graph J_e is defined in precisely the same way as J_v but uses the parameter t instead of s. We denote the corresponding clique by C_e. Further, we set $E_e = (V(C_u) \cup V(C_v)) \times V(C_e)$. The edge gadget is depicted in Figure 18.
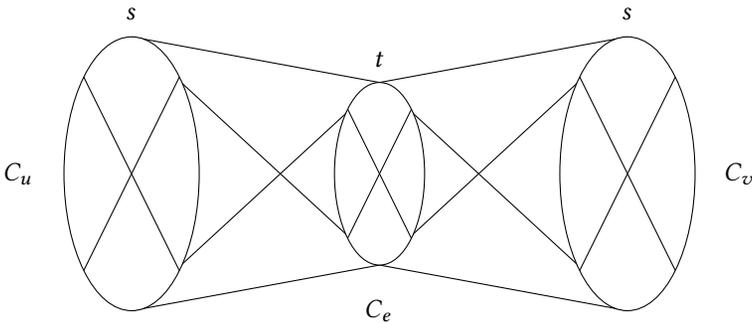

Fig. 18. The edge gadget for an edge $e = \{u, v\}$. The edges to the vertices $\{w_1, \ldots, w_k \mid w \in C_u \cup C_e \cup C_v\}$ and $\{p_1, \ldots, p_k\}$ are omitted.

Finally, J is the graph with vertices

$$V(J) = \{p_{k+1}, \ldots, p_q\} \cup \bigcup_{v \in V(G)} V(J_v) \cup \bigcup_{e \in E(G)} V(J_e)$$

and edges

$$E(J) = \bigcup_{i \in [q]} \left(V(C_{\tau_i}) \times \{p_i\}\right) \cup \bigcup_{v \in V(G)} E(J_v) \cup \bigcup_{e \in E(G)} \left(E(J_e) \cup E_e\right).$$

Note that the first set in the union is a set of edges for each terminal τ_i. The purpose of these edges will be to ensure that the corresponding graph J_{τ_i} is in the right "state" (the one corresponding to i). Here we now use all of the p_i, not just the first k as we did in the construction of J_v and J_e.

Next we define the lists $\mathbf{S} = \{S_v \subseteq V(H_b) \mid v \in V(J)\}$. We set

$$S_v = \begin{cases} \{x_i\}, & \text{if } v = p_i, \ i \in [q] \\ V(H_b), & \text{otherwise.} \end{cases}$$

For a homomorphism $h \in \mathcal{H}((J, \mathbf{S}), H_b)$ and a vertex $v \in V(G)$ we say that the image $h(V(C_v))$ is the *state* of v (under h).

A *pinned configuration* is a tuple $(z, z_1, \ldots, z_k)$ of vertices of H_b such that, for each $i \in [k]$, $\{z, z_i\}$ and $\{z_i, x_i\}$ are edges of H_b. Note that the vertices $(w, w_1, \ldots, w_k)$ of C_v (see Figure 17) have to map to a pinned configuration under a homomorphism from $(J, \mathbf{S})$ to H_b. For $z \in V(H_b)$ let $f(z)$ be the number of pinned configurations $(z, z_1, \ldots, z_k)$. We have

$$f(b) = 2^k \qquad \text{(All } z_j \text{ can be either } x_j \text{ or } b.) \tag{16}$$

$$f(x_i) = 2 \ (\forall i \in [k]) \qquad (z_i \text{ can be either } x_i \text{ or } b, \text{ all other } z_j \text{ have to be } b.) \tag{17}$$

$$f(x_i) = f(y_i) = 1 \ (\forall i \in \{k+1, \ldots, q\}) \quad \text{(All } z_j \text{ have to be } b.) \tag{18}$$

$$f(u) = 1 \ (\forall u \in U) \qquad \text{(All } z_j \text{ have to be } b.) \tag{19}$$

We say that a vertex $v \in V(G)$ is *full* (under h) if the following conditions are met:

- There exists $i \in [q]$ such that $h(V(C_v)) = \Gamma_{H_b}(x_i)$.
- For every element $z \in h(V(C_v))$ and every pinned configuration $(z, z_1, \ldots, z_k)$ there exists a vertex w in the clique C_v such that $h(w, w_1, \ldots, w_k) = (z, z_1, \ldots, z_k)$ (elementwise).

We call an edge $e = \{u, v\} \in E(G)$ *monochromatic* (under h) if u and v have the same state. Otherwise, we say that e is *dichromatic*. We say that the homomorphism h is *full* if every vertex $v \in V(G)$ is full under h. We say that a full homomorphism h is *K-small* if there are at most K dichromatic edges under h, otherwise we say that it is *K-large*.

Let Z^* be the number of full homomorphisms that are K-small. Further, let Z_1 be the number of full homomorphisms that are K-large and let Z_2 be the number of non-full homomorphisms. Then

$$N\left((J, \mathbf{S}) \to H_b\right) = Z^* + Z_1 + Z_2.$$

Let T be the sought-for number of size-K multiterminal cuts of the instance $I = (G, \tau_1, \ldots, \tau_q, K)$. We will now investigate the way in which the number Z^* relates to T. Recall the definitions about separating functions from Definition 45. In particular, we will use the sets $\Phi(I)$, $\Phi^*(I)$ and $\text{Cut}(\phi)$. To shorten notation within the scope of this proof, we write Φ when we mean $\Phi(I)$ and Φ^* when we mean $\Phi^*(I)$. From Observation 46 we know that $T = |\Phi^*|$.

For a function $\phi \in \Phi$ we say that a homomorphism $h \in \mathcal{H}((J, \mathbf{S}), H_b)$ *agrees* with ϕ if, for each vertex v of G, the state of v under h is $h(V(C_v)) = \Gamma_{H_b}(x_{\phi(v)})$. Note that, by the construction of J, under a full homomorphism a terminal τ_i has state $\Gamma_{H_b}(x_i)$. Therefore, each full homomorphism h agrees with exactly one $\phi \in \Phi$.

If h agrees with ϕ, then it follows that $\mathrm{Cut}(\phi)$ is exactly the set of dichromatic edges under h. Hence, each K-small full homomorphism agrees with exactly one function $\phi \in \Phi^*$ and each K-large full homomorphism agrees with exactly one function $\phi \in \Phi \setminus \Phi^*$. Let Z_ϕ be the number of full homomorphisms that agree with $\phi \in \Phi$. Then

$$Z^* = \sum_{\phi \in \Phi^*} Z_\phi \quad \text{and} \quad Z_1 = \sum_{\phi \in \Phi \setminus \Phi^*} Z_\phi. \tag{20}$$

Let $\phi \in \Phi$. We are interested in the number Z_ϕ. What are the possible full homomorphisms h that agree with ϕ?

Observation A Let $v \in V(G)$. We consider possible images of the vertices of J_v. For h to agree with ϕ, the state of v is fixed to be $\Gamma_{H_b}(x_{\phi(v)})$, where this set can be either of the form $\{b, x_i\}$ or of the form $\{b, x_i, y_i\}$. From Equations (16) and (17) it follows that there are a total of $2^k + 2$ pinned configurations $(z, z_1, \ldots, z_k)$ with $z \in \{b, x_i\}$. Similarly, from Equations (16) and (18) it follows that there are a total of $2^k + 1 + 1$ pinned configurations with $z \in \{b, x_i, y_i\}$. As h is full, each possible pinned configuration has to be used at least once by the s vertices in C_v. As a consequence each vertex $v \in V(G)$ contributes a factor of $\left\{{s \atop 2^k+2}\right\}$ to Z_ϕ.

Observation B Let $e = \{u, v\}$. What are the possible images of the vertices of J_e? We make a case distinction depending on e.

- Let $e = \{u, v\} \in \mathrm{Cut}(\phi)$. Then, as h is full, $h(V(C_u))$ and $h(V(C_v))$ are different states from the set $\{\Gamma_{H_b}(x_i) \mid i \in [q]\}$. By the definition of J we have that $h(V(C_e)) \subseteq h(V(C_u)) \cap h(V(C_v))$. It follows that $h(V(C_e)) = \{b\}$. There are 2^k pinned configurations with $z = b$. Each of the t vertices of C_e can have any of these 2^k pinned configurations. Therefore, e contributes a factor of 2^{kt} to Z_ϕ.
- Let $e = \{u, v\} \notin \mathrm{Cut}(\phi)$. Then, as h is full, $h(V(C_u)) = h(V(C_v)) = \Gamma_{H_b}(x_i)$ for some $i \in [q]$. Then $h(C_e) \subseteq \Gamma_{H_b}(x_i)$, where $\Gamma_{H_b}(x_i)$ is of the form $\{b, x_i\}$ or of the form $\{b, x_i, y_i\}$. As before there are $2^k + 2$ corresponding pinned configurations. Therefore, e contributes a factor of $(2^k + 2)^t$ to Z_ϕ.

Summarising, we obtain

$$Z_\phi = \left\{{s \atop 2^k + 2}\right\}^n (2^k)^{t|\mathrm{Cut}(\phi)|}(2^k + 2)^{t(m - |\mathrm{Cut}(\phi)|)}. \tag{21}$$

For each $\phi \in \Phi^*$ we have $|\mathrm{Cut}(\phi)| = K$. Then, using the fact that $|\Phi^*| = T$, we obtain

$$Z^* = \sum_{\phi \in \Phi^*} Z_\phi$$

$$= \sum_{\phi \in \Phi^*} \left\{{s \atop 2^k + 2}\right\}^n (2^k)^{tK}(2^k + 2)^{t(m-K)}$$

$$= \left\{{s \atop 2^k + 2}\right\}^n (2^k)^{tK}(2^k + 2)^{t(m-K)} \cdot T.$$

To shorten the notation let

$$L = \left\{{s \atop 2^k + 2}\right\}^n (2^k)^{tK}(2^k + 2)^{t(m-K)}.$$

We want to approximately compute the value T, where we have shown that $T = Z^*/L$. Assume for now that we have

$$N\big((J, \mathbf{S}) \to H_b\big)/L \in [Z^*/L, Z^*/L + 1/4] = [T, T + 1/4].$$

Then consider the algorithm which makes a #RET(H_b) oracle call with input $((J, S), \varepsilon/21)$ to obtain a value Q and returns $\lfloor Q/L \rfloor$. This algorithm approximates T with the desired error bound ε as is shown in [12, Proof of Theorem 3]. It remains to show the following claim.

Claim: $N\big((J, S) \to H_b\big)/L \in [Z^*/L, Z^*/L + 1/4]$.

Proof of the claim: Recall that $N\big((J, S) \to H_b\big) = Z^* + Z_1 + Z_2$. Clearly, we have $N\big((J, S) \to H_b\big)/L \geq Z^*/L$. We will show $Z_1/L \leq 1/8$ and $Z_2/L \leq 1/8$ to prove $N\big((J, S) \to H_b\big)/L \leq Z^*/L + 1/4$.

Recall that Z_1 is the number of K-large full homomorphisms. Using (20) and (21) and the fact that for each $\phi \in \Phi \setminus \Phi^*$ we have $|\mathrm{Cut}(\phi)| \geq K + 1$ we obtain

$$Z_1 \leq \sum_{\phi \in \Phi \setminus \Phi^*} \left\{ {s \atop 2^k + 2} \right\}^n (2^k)^{t(K+1)} (2^k + 2)^{t(m-K-1)}$$

$$\leq q^n \left\{ {s \atop 2^k + 2} \right\}^n (2^k)^{t(K+1)} (2^k + 2)^{t(m-K-1)}$$

where the last inequality follows from the fact that there are q^n functions in Φ. Then

$$\frac{Z_1}{L} \leq \left(\frac{2^k}{(2^k + 2)} \right)^t q^n \leq 1/8,$$

where the last inequality holds for sufficiently large n by our choice of $t = n^2$ and the fact that $2^k/(2^k + 2) < 1$.

Recall that Z_2 is the number of homomorphisms that are not full. How many non-full homomorphisms h are there? In general, there are at most $2^{|V(H_b)|}$ possible states $h(V(C_v))$ for any vertex $v \in V(G)$. By the same arguments as given in Observation A, each full vertex under h contributes a factor of $\left\{ {s \atop 2^k+2} \right\}$ to Z_2. Since $s = n^5$ the requirements of Corollary 43 are met for sufficiently large n and we obtain

$$(2^k + 2)^s / 2 \leq \left\{ {s \atop 2^k + 2} \right\}. \tag{22}$$

We will use this bound shortly. If h is not full, there has to exist at least one vertex $v \in V(G)$ which is not full under h, and consequently there are at most $n - 1$ full vertices under h.

Now assume that $v \in V(G)$ is not full under h. Then either v has state $h(V(C_v)) = \Gamma_{H_b}(x_i)$ for some $i \in [q]$ but not all of the $2^k + 2$ possible pinned configurations are used, or $h(V(C_v)) \notin \{\Gamma_{H_b}(x_i) \mid i \in [q]\}$ (which means that either $h(V(C_v)) \subsetneq \Gamma_{H_b}(x_i)$ or $h(V(C_v)) = \{b, u\}$ for some $u \in U$, by the fact that C_v is a large clique). In both cases the s vertices in C_v can each use at most $2^k + 1$ different pinned configurations. Hence, each non-full vertex contributes a factor of at most $(2^k + 1)^s$ to Z_2. In particular this factor is smaller (for sufficiently large n) than the factor contributed by full vertices (see (22)). Finally, for each edge e there are at most $|V(H_b)|^{(k+1)t}$ mappings from the $(k + 1) \cdot t$ vertices in $V(J_e) \setminus \{p_1, \dots, p_k\}$ to $V(H_b)$. Therefore,

$$Z_2 \leq 2^{|V(H_b)|n} \cdot \left\{ {s \atop 2^k + 2} \right\}^{n-1} \cdot (2^k + 1)^s \cdot |V(H_b)|^{(k+1)tm}.$$

Recall that $L = \left\{ {s \atop 2^k+2} \right\}^n (2^k)^{tK} (2^k + 2)^{t(m-K)} \geq \left\{ {s \atop 2^k+2} \right\}^n$. Then

$$\frac{Z_2}{L} \leq \frac{2^{|V(H_b)|n} \cdot (2^k + 1)^s \cdot |V(H_b)|^{(k+1)tm}}{\left\{ {s \atop 2^k+2} \right\}} \leq \left(\frac{2^k + 1}{2^k + 2} \right)^s \cdot 2 \cdot 2^{|V(H_b)|n} \cdot |V(H_b)|^{(k+1)n^4} \leq 1/8,$$

where the second inequality follows from (22) and the last inequality holds for sufficiently large n by our choice of $s = n^5$. This proves the claim and completes the proof. *(End of the proof of the claim.)* $\qquad \square$

3.5 Square-Free Graphs with an Induced Net

The goal of this section is to prove #SAT-hardness for square-free graphs with an induced net (see Figure 19). Note that the subgraphs of the net that are induced by a distance-1 neighbourhood of some vertex u of the net are of two forms. Either the corresponding subgraph is a looped edge (if $u \notin \{w_i \mid i \in [3]\}$) or it is isomorphic to a looped triangle where one vertex in the triangle has a single additional looped neighbour (if $u \in \{w_i \mid i \in [3]\}$). Approximately counting retractions to either of these two graphs is #BIS-easy (see Theorem 1). Therefore we cannot use these subgraphs in our hardness proof, so we need to work harder.
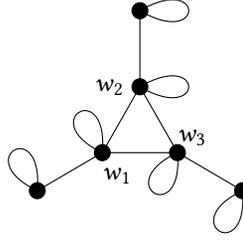
Fig. 19. The *net*.

In our proof (Lemma 48) we use the same general approach that we used to prove Lemma 40. To make the approach work we have to find new gadgets tailored to the net. In one part of the reduction we will need to approximate real values by integers. To achieve this we use Dirichlet's approximation lemma, which has been used frequently in this line of research (see for instance [24]).

Lemma 47 (see e.g. [49, p. 34]). *Let $\lambda_1, \ldots, \lambda_d > 0$ be real numbers and N be a natural number. Then there exist positive integers $t_1, \ldots, t_d, r$ with $r \leq N$ such that $|r\lambda_i - t_i| \leq 1/N^{1/d}$ for every $i \in [d]$.*

Lemma 48. *Let H be a square-free graph that has the net (as displayed in Figure 19) as an induced subgraph. Then #SAT $\leq_{\text{AP}}$ #Ret(H).*

PROOF. Let H be a square-free graph with an induced net that is labelled as in Figure 19. Note that each of the vertices w_1, w_2 and w_3 might have additional neighbours in H. However, they cannot have further common neighbours as H is square-free. We use a reduction from #MultiTerminalCut(3) which is #SAT-hard under AP-reductions by Lemma 44. Let $I = (G, \tau_1, \tau_2, \tau_3, K)$ be an instance of #MultiTerminalCut(3) and $\varepsilon \in (0, 1)$ the desired precision bound. Let $n = |V(G)|$ and $m = |E(G)|$. We will construct an instance (J, S) of #Ret(H).

To construct this instance we will use a number of parameters which we introduce at this point. Let $s = n^2$. For $i \in [3]$ let $s_i = s \cdot \log_{|\Gamma_H(w_i)|/3} 2$. For $i \in [3]$, the value s_i is chosen such that

$$\left(\frac{|\Gamma_H(w_i)|}{3}\right)^{s_i} = 2^s. \tag{23}$$

It will become clear later in the proof why this is useful. (The important part is that the values $\left(\frac{|\Gamma_H(w_i)|}{3}\right)^{s_i}$, for $i \in [3]$, are identical and that the base of the exponent on the right-hand side is greater than 1.) We will now determine integers that approximate s_1, s_2 and s_3 using Dirichlet's approximation lemma. Let $\delta = \varepsilon/2$ and $\delta' = \log_{|V(H)|} e^\delta$ (the choices will become clear later in the proof). By Lemma 47 we obtain integers r, t_1, t_2 and t_3 of value at most $(m/\delta')^3 \in \text{poly}(n, \varepsilon^{-1})$ such that $|rs_i - t_i| \leq \delta'/m$ for $i \in [3]$.

We go on to define the graph J. Intuitively, for $i \in [3]$, the terminal τ_i will serve as a "pin" to the vertex w_i. For each vertex $v \in V(G)$ we introduce a graph J_v which is simply a star with center v and leaves $\{\tau_1, \tau_2, \tau_3\}$. Formally, the vertices of J_v are $V(J_v) = \{v, \tau_1, \tau_2, \tau_3\}$. and the edges are $E(J_v) = \{v\} \times \{\tau_1, \tau_2, \tau_3\}$. For each edge $e = \{u, v\} \in E(G)$ we introduce a graph J_e which is defined as follows. For $i \in [3]$, let I_e^i be disjoint independent sets of size t_i. Let $t = \sum_{i \in [3]} t_i$ and let $I_e = \bigcup_{i \in [3]} I_e^i$ (I_e is an independent set of size t). Then J_e is depicted in Figure 20 and formally defined as the graph with vertices

$$V(J_e) = I_e \cup \{u, v, \tau_1, \tau_2, \tau_3\}$$

and edges

$$E(J_e) = (\{u, v\} \times I_e) \cup \bigcup_{i=1}^{3} (I_e^i \times \{\tau_i\}).$$
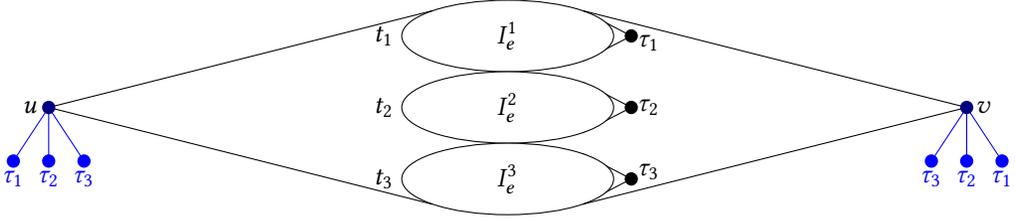
Fig. 20. The graph J_e for an edge $e = \{u, v\}$ is depicted in black. J_u and J_v are depicted in blue. Note that τ_1, τ_2 and τ_3 are global vertices. That is, τ_i is the same vertex in all of the different gadgets.

Then J is the graph with vertices

$$V(J) = \bigcup_{v \in V(G)} V(J_v) \cup \bigcup_{e \in E(G)} V(J_e)$$

and edges

$$E(J) = \bigcup_{v \in V(G)} E(J_v) \cup \bigcup_{e \in E(G)} E(J_e).$$

The corresponding set of lists $\mathbf{S} = \{S_v \subseteq V(H) \mid v \in V(J)\}$ is defined by

$$S_v = \begin{cases} \{w_i\}, & \text{if } v = \tau_i, \ i \in [3] \\ V(H), & \text{otherwise.} \end{cases}$$

Let h be a homomorphism from $(J, \mathbf{S})$ to H. By the definition of J_v every vertex $v \in V(G)$ is also a vertex of J. An edge $e = \{u, v\} \in E(G)$ is called *monochromatic* under h if $h(u) = h(v)$. Otherwise, it is called *dichromatic* under h. We say that h is *K-small* if the number of dichromatic edges under h is at most K. Otherwise, h is called *K-large*. Let Z^* be the number of K-small homomorphisms from $(J, \mathbf{S})$ to H and let Z_1 be the number of K-large homomorphisms. Clearly,

$$N\big((J, \mathbf{S}) \to H\big) = Z^* + Z_1.$$

Recall the definitions of separating functions from Definition 45 and, to shorten notation, define the sets $\Phi = \Phi(I)$ and $\Phi^* = \Phi^*(I)$. T is the number of size-K multiterminal cuts of the instance $I = (G, \tau_1, \tau_2, \tau_3, K)$. Our goal is to approximate T. From Observation 46 we know that $T = |\Phi^*|$.

We say that a homomorphism $h \in \mathcal{H}((J, \mathbf{S}), H)$ *agrees* with $\phi \in \Phi$ if, for each $v \in V(G)$, we have $h(v) = w_{\phi(v)}$. By definition of the lists $\mathbf{S}$, for each $i \in [3]$, a homomorphism h from $(J, \mathbf{S})$ to H has

to map τ_i to w_i. Furthermore, as v is adjacent to all three terminals and H is square-free we have $h(v) \in \{w_1, w_2, w_3\}$.

At this point we have introduced the gadget J_e and the graph J and have defined what it means for a homomorphism from J to H to agree with a function in Φ (which in turn corresponds to a multiterminal cut). All these definitions are heavily tailored to the graph H. The following steps, however, are very similar to those in the proof of Lemma 40. What complicates this proof in comparison to that of Lemma 40 is the fact that we need to use Dirichlet's approximation lemma to balance out the edge interactions. Here are the details.

Every homomorphism $h \in \mathcal{H}((J, S), H)$ agrees with exactly one function $\phi \in \Phi$. In particular, if h agrees with ϕ, then the dichromatic edges of h are exactly the multiterminal cut $\mathrm{Cut}(\phi)$. It follows that every K-small homomorphism agrees with exactly one function $\phi \in \Phi^*$ and every K-large homomorphism agrees with exactly one function $\phi \in \Phi \setminus \Phi^*$. For $\phi \in \Phi$ let Z_ϕ be the number of homomorphisms from (J, S) to H that agree with ϕ. Then

$$Z^* = \sum_{\phi \in \Phi^*} Z_\phi \quad \text{and} \quad Z_1 = \sum_{\phi \in \Phi \setminus \Phi^*} Z_\phi. \tag{24}$$

Let $\phi \in \Phi$. We are interested in the number Z_ϕ and investigate which homomorphisms h agree with ϕ. For each $v \in V(G)$ the image of J_v under h is fixed by the lists in S and the fact that $h(v) = \phi(v)$. Therefore, we only need to consider possible images of the graphs J_e. We make a case distinction depending on e.

- Let $e = \{u, v\} \in \mathrm{Cut}(\phi)$. (This means that e is dichromatic under h.) By the definition of J_e it follows that the image $h(I_e)$ is a subset of $\Gamma_H(h(u)) \cap \Gamma_H(h(v))$. The vertices $h(u)$ and $h(v)$ are distinct and are from $\{w_1, w_2, w_3\}$. As H is square-free it follows that $\Gamma_H(h(u)) \cap \Gamma_H(h(v)) = \{w_1, w_2, w_3\}$. In addition, each vertex of I_e is adjacent to one of the terminals. Since the images of these vertices are also in $\{w_1, w_2, w_3\}$ this does not bring any additional constraints. Summarising, since I_e has size t the edge e contributes a factor of 3^t to Z_ϕ.
- Let $e = \{u, v\} \in \mathrm{Mon}_i(\phi)$ for some $i \in [3]$. (This means that e is a monochromatic edge under h with $h(u) = h(v) = w_i$.) Then, for $j \in [3]$ with $j \neq i$, by the same arguments as before we have $h(I_e^j) \subseteq \{w_1, w_2, w_3\}$. However the vertices in I_e^i can be mapped to any neighbour of w_i. Therefore, each edge in $\mathrm{Mon}_i(\phi)$ contributes a factor of $|\Gamma_H(w_i)|^{t_i} \cdot 3^{t-t_i}$ to Z_ϕ.

Using this knowledge, for $\phi \in \Phi$, we have

$$Z_\phi = 3^{t|\mathrm{Cut}(\phi)|} \cdot \prod_{i \in [3]} \left(|\Gamma_H(w_i)|^{t_i} \cdot 3^{t-t_i} \right)^{|\mathrm{Mon}_i(\phi)|}.$$

Since $m = |\mathrm{Cut}(\phi)| + \sum_{i \in [3]} |\mathrm{Mon}_i(\phi)|$ we can simplify this expression to

$$Z_\phi = 3^{tm} \prod_{i \in [3]} (|\Gamma_H(w_i)|/3)^{t_i |\mathrm{Mon}_i(\phi)|}.$$

For $i \in [3]$, recall the fact that $|rs_i - t_i| \leq \delta'/m$, where $s_i = s \cdot \log_{|\Gamma_H(w_i)|/3} 2$. For an upper bound on Z_ϕ we use the fact that $t_i \leq rs_i + \delta'/m$. Also note that $|\Gamma_H(w_i)|$ is bounded above by $|V(H)|$ (which we use in the second inequality of the following expression). Furthermore, $\sum_{i \in [3]} |\mathrm{Mon}_i(\phi)| \leq m$ and $\delta' = \log_{|V(H)|} e^\delta$ (which we use in the third inequality of the following expression). Then

$$Z_\phi \leq 3^{tm} \prod_{i \in [3]} (|\Gamma_H(w_i)|/3)^{(rs_i + \delta'/m) \cdot |\mathrm{Mon}_i(\phi)|}$$

$$\leq 3^{tm} \cdot 2^{(\sum_{i \in [3]} |\mathrm{Mon}_i(\phi)|) \cdot rs} \cdot |V(H)|^{(\sum_{i \in [3]} |\mathrm{Mon}_i(\phi)|) \cdot \delta'/m}$$

$$\leq 3^{tm} \cdot 2^{(\sum_{i \in [3]} |\mathrm{Mon}_i(\phi)|) \cdot rs} \cdot e^\delta.$$

Analogously, for a lower bound on Z_ϕ we use the fact that $t_i \geq rs_i - \delta'/m$. We obtain $3^{tm} \cdot 2^{(\sum_{i \in [3]} |\text{Mon}_i(\phi)|) \cdot rs} \cdot e^{-\delta} \leq Z_\phi$. Summarising, we have

$$3^{tm} \cdot 2^{(\sum_{i \in [3]} |\text{Mon}_i(\phi)|) \cdot rs} \cdot e^{-\delta} \leq Z_\phi \leq 3^{tm} \cdot 2^{(\sum_{i \in [3]} |\text{Mon}_i(\phi)|) \cdot rs} \cdot e^{\delta}. \tag{25}$$

Putting these bounds on Z_ϕ into the expression for Z^* in (24) gives

$$\sum_{\phi \in \Phi^*} 3^{tm} \cdot 2^{(\sum_{i \in [3]} |\text{Mon}_i(\phi)|) \cdot rs} \cdot e^{-\delta} \leq Z^* \leq \sum_{\phi \in \Phi^*} 3^{tm} \cdot 2^{(\sum_{i \in [3]} |\text{Mon}_i(\phi)|) \cdot rs} \cdot e^{\delta}.$$

Since $\sum_{i \in [3]} |\text{Mon}_i(\phi)| = m - K$ for each $\phi \in \Phi^*$ and $|\Phi^*| = T$ we obtain

$$T \cdot 3^{tm} \cdot 2^{(m-K) \cdot rs} \cdot e^{-\delta} \leq Z^* \leq T \cdot 3^{tm} \cdot 2^{(m-K) \cdot rs} \cdot e^{\delta}.$$

We set $L = 3^{tm} 2^{(m-K)rs}$ to obtain

$$T \cdot e^{-\delta} \leq Z^*/L \leq T \cdot e^{\delta}. \tag{26}$$

Assume for now that $N\big((J, S) \to H\big)/L \in [Z^*/L, Z^*/L + 1/4]$. Then the algorithm that makes a #Ret(H) oracle call with input $((J, S), \varepsilon/42)$ returns a solution Q such that $Z^*/L \cdot e^{-\varepsilon/2} \leq \lfloor Q/L \rfloor \leq Z^*/L \cdot e^{\varepsilon/2}$ as was shown in [12, Proof of Theorem 3]. Using (26) and our choice of $\delta = \varepsilon/2$ this gives

$$T \cdot e^{-\varepsilon} \leq \lfloor Q/L \rfloor \leq T \cdot e^{\varepsilon}.$$

Therefore the output $\lfloor Q/L \rfloor$ approximates T with the desired precision. It remains to show the following claim.

Claim: $N\big((J, S) \to H\big)/L \in [Z^*/L, Z^*/L + 1/4]$.

Proof of the claim: Recall that $N\big((J, S) \to H\big) = Z^* + Z_1$. It is immediate that $N\big((J, S) \to H\big)/L \geq Z^*/L$. It remains to show that $Z_1/L \leq 1/4$.

To obtain the following expression we first use (24) and (25). The third inequality then uses the fact that, for every $\phi \in \Phi \setminus \Phi^*$, we have $|\text{Cut}(\phi)| \geq K + 1$ and hence $\sum_{i \in [3]} |\text{Mon}_i(\phi)| \leq m - (K + 1)$. Finally, in the fourth inequality we use the fact that $|\Phi \setminus \Phi^*| \leq |V(H)|^n$

$$\begin{aligned}
Z_1 &= \sum_{\phi \in \Phi \setminus \Phi^*} Z_\phi \\
&\leq \sum_{\phi \in \Phi \setminus \Phi^*} 3^{tm} \cdot 2^{(\sum_{i \in [3]} |\text{Mon}_i(\phi)|) \cdot rs} \cdot e^{\delta} \\
&\leq \sum_{\phi \in \Phi \setminus \Phi^*} 3^{tm} \cdot 2^{(m-K-1) \cdot rs} \cdot e^{\delta} \\
&\leq |V(H)|^n \cdot 3^{tm} \cdot 2^{(m-K-1) \cdot rs} \cdot e^{\delta}
\end{aligned}$$

Recall the definition $L = 3^{tm} \cdot 2^{(m-K) \cdot rs}$. It follows that

$$Z_1/L \leq \frac{|V(H)|^n \cdot e^{\delta}}{2^{rs}} \leq 1/4,$$

where the last inequality holds for sufficiently large n by our choice of $s = n^2$ and the fact that $r \geq 1$. This proves the claim and completes the proof. *(End of the proof of the claim.)* □

3.6 Square-Free Graphs with an Induced Reflexive Cycle of Length at least 5

The goal of this section is to show the following lemma. We build up to its proof, which is given at the end of this section.

Lemma 49. *Let H be a square-free graph. If H contains a reflexive cycle of length at least 5 as an induced subgraph then #SAT $\leq_{AP}$ #RET(H).*

Let H be a connected square-free graph with an induced reflexive cycle of length at least 5. If all cycles in H have length at least 5, then H has girth at least 5 and the complexity of #RET(H) is classified by Theorem 5. In the special case where H is reflexive this classification is straightforward to see: Either there is just a single cycle in H, then H is a pseudotree and #SAT-hardness follows from NP-hardness for the decision problem [18, Corollary 4.2, Theorem 5.1] together with [12, Theorem 1] — or there are multiple cycles (all of which have length at least 5), then there exists an induced WR_3 (as H is connected) and hardness follows from Lemma 40.

Thus, it remains to show hardness if H contains both a cycle of length at least 5 as well as a cycle of length at most 5, i.e. (since H is square-free) it contains a triangle. The hardness proof we give in this section will handle the case where H includes triangles but will not rely on this fact (i.e. it will also cover the before-mentioned case where all cycles have length at least 5 without relying on hardness results for the decision problem).

As mentioned before, it is known that approximately counting list homomorphisms to reflexive graphs with an induced cycle of length at least 4 is #SAT-hard [25, Lemma 3.4]. That proof makes use of a certain set of two-vertex lists. In the proof of Lemma 49 we will use single-vertex lists to simulate these two-vertex lists.

As we have already shown #SAT-hardness results for square-free graphs with an induced WR_3 or an induced net in Sections 3.4 and 3.5 we now focus on graphs that do not contain such subgraphs. When considering reflexive graphs it turns out that this leaves a class of graphs which we call reflexive triangle-extended cycles. We will also make use of these reflexive triangle-extended cycles when considering square-free graphs H that are not necessarily reflexive. When we do this we will restrict to the (reflexive) subgraph induced by the looped vertices of H.

Definition 50. A *reflexive triangle-extended cycle* of length q consists of a reflexive cycle $c_0, \ldots, c_{q-1}$ together with a set $\mathcal{I} \subseteq \{0, \ldots, q-1\}$, and a reflexive triangle $d_i, c_i, c_{i+1 \bmod q}$ for each $i \in \mathcal{I}$. An example of a reflexive triangle-extended cycle is depicted in Figure 21.
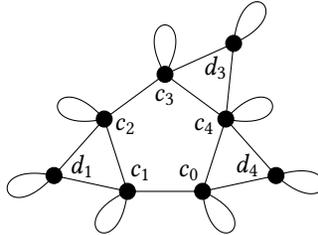

Fig. 21. Reflexive triangle-extended cycle with $q = 5$ and $\mathcal{I} = \{1, 3, 4\}$.

Definition 51. Analogously to Definition 50, a *reflexive triangle-extended path* is a reflexive path $c_0, \ldots, c_{q-1}$ together with a set $\mathcal{I} \subseteq \{0, \ldots, q-2\}$, and a reflexive triangle d_i, c_i, c_{i+1} for each $i \in \mathcal{I}$.

Lemma 52. *Let H be a connected reflexive square-free graph that does not contain an induced WR_3 and also does not contain an induced net. If H contains an induced cycle of length at least 5 then H is a reflexive triangle-extended cycle of length at least 5. Otherwise it is a reflexive triangle-extended path.*

PROOF. **Case 1: H contains an induced cycle $C = c_0, \ldots, c_{q-1}$ with $q \geq 5$.** If H is just the cycle C, then the statement of the lemma is true ($\mathcal{I} = \emptyset$). Otherwise, consider any $d \in V(H) \setminus V(C)$ with a neighbour $c \in V(C)$ (has to exist since H is connected).

Since H does not contain an induced WR_3 the vertex d is adjacent to a neighbour $c' \in V(C)$ of c. Let c^0 and c'' be the other neighbours of c and c' in C, respectively, i.e. $\Gamma_C(c) = \{c^0, c'\}$ and $\Gamma_C(c') = \{c, c''\}$. The vertices $\{c^0, c, c', c''\}$ are all distinct as C has length at least 5. As H is square-free we observe

$$\{d, c^0\} \notin E(H) \text{ and } \{d, c''\} \notin E(H). \tag{27}$$

The proof of the following claim directly proves that H is a reflexive triangle-extended cycle.

Claim: $\Gamma_H(d) = \{c, c'\}$.

Proof of the claim: Assume there exists a neighbour $d' \notin \{c, c'\}$ of d in H. By (27) we have $d' \notin \{c^0, c''\}$. Furthermore, since H is square-free, we obtain the following.

$$\text{There is no } u \neq d \text{ with } u \in \Gamma_H(c) \cap \Gamma_H(d') \text{ or } u \in \Gamma_H(c') \cap \Gamma_H(d'). \tag{28}$$

Let H' be the subgraph of H induced by the vertices $\{c^0, c, c', c'', d, d'\}$, see Figure 22. Then H' is a net (cf. Figure 19 where $c = w_1$, $d = w_2$ and $c' = w_3$).
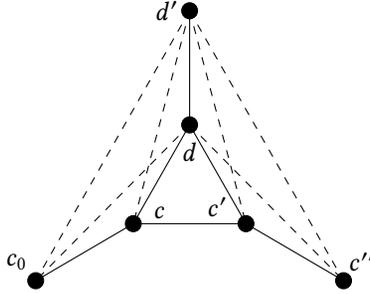

Fig. 22. The graph H' induced by $\{c^0, c, c', c'', d, d'\}$. Loops are omitted. Dashed lines show edges that cannot exist by the fact that H is square-free.

Because of (28) we have $\Gamma_{H'}(d') = \{d\}$. (The dashed edges incident to d' in Figure 22 cannot exist.) Because of (27) we have $\Gamma_{H'}(d) = \{d', c, c'\}$. (The dashed edges incident to d in Figure 22 cannot exist.) Finally, since C is an induced cycle, there are no edges between the vertices $\{c^0, c, c', c''\}$ outside of C. Therefore, H' is an *induced* net in H, which gives a contradiction. This proves the claim in Case 1. **(End of Case 1)**

Case 2: All induced cycles in H are triangles. This case is handled very similarly to the previous one: Let $P = c_0, \ldots, c_{q-1}$ be a maximal induced path in H. If H is just the path P, then the statement of the lemma is true ($\mathcal{I} = \emptyset$). Otherwise, let $d \in V(H) \setminus V(P)$ be a neighbour of $c \in V(P)$. We show that d is adjacent to a neighbour $c' \in V(P)$ of c:

- If c is an inner vertex of P then, since H does not contain an induced WR_3, d is also adjacent to a neighbour of c in P.

- If c is an endpoint of P, then d has to be adjacent to a vertex $c' \in V(P)$ with $c' \neq c$ as P is maximal induced. Without loss of generality assume that c' is the neighbour of d which is closest to c in P. Then c', c, d has to be a triangle (c' has to be a neighbour of c) as P is induced and all induced cycles in H are triangles.

Then the proof of the following claim shows that H is a reflexive triangle-extended path.

Claim: $\Gamma_H(d) = \{c, c'\}$.

Proof of the claim: Assume there exists a neighbour $d' \notin \{c, c'\}$ of d in H. We observe the following properties:

- The vertex d' does not have a neighbour in P: Assume the opposite and let $u \in P$ be a neighbour of d'. Without loss of generality let u be closer to c' than c in P. Furthermore, let u be the neighbour of d' in P which is closest to c'. Then the edge $\{d', u\}$ and the path d', d, c' close an induced cycle with P. If $u \neq c'$ this cycle has length greater than 3, a contradiction. If $u = c'$ we obtain a contradiction to the fact that H is square-free (see Figure 22).
- Both c and c' are inner points of P: Suppose, for contradiction, that c' is an end point of P. Since d' does not have a neighbour in P, replacing c' by d, d' in P gives an induced path P'. Moreover, P' is longer than P which is a contradiction to the maximality of P. This shows that c' cannot be an end point of P. Analogously c cannot be an end point of P.

Then let c^0 and c'' be the other neighbours of c and c' in P (they have to exist since c and c' are inner points of P). The remainder of the argument is analogous to the proof of the claim in Case 1.
(End of Case 2)   □

The goal of the remainder of this section is to prove Lemma 49. In order to prove Lemma 49 we work with the following parameterised version of the list homomorphism counting problem. Let H be a graph and $\mathcal{L}$ be a set of subsets of $V(H)$.

Name: #Hom$(H, \mathcal{L})$.
Input: An irreflexive graph G and a collection of lists $\mathsf{S} = \{S_v \in \mathcal{L} \mid v \in V(G)\}$.
Output: $N\big((G, \mathsf{S}) \to H\big)$.

We also use the following lemma.

Lemma 53 ([25, Proof of Lemma 3.4]). *Let H be a graph that contains an induced reflexive cycle $C = c_0, \ldots, c_{q-1}$ on $q \geq 4$ vertices. Let $\mathcal{L} = \{\{c_0, c_1\}, \{c_0, c_2\}, \ldots, \{c_0, c_{q-1}\}\}$. Then #Hom$(H, \mathcal{L}) \equiv_{AP}$ #SAT.*

The key to proving Lemma 49 is the following result. It states that for certain graphs that contain a reflexive triangle-extended cycle we can simulate each size-2 list of vertices in the corresponding cycle C by gadgets using only single-vertex lists.

Lemma 54. *Let H be a square-free graph that does not contain any mixed triangle as an induced subgraph and let H^* be the graph induced by the looped vertices of H. Suppose that H^* contains a connected component H^{**} that is a reflexive triangle-extended cycle, where $C = c_0, \ldots, c_{q-1}$ is the corresponding reflexive cycle as given by Definition 50 and the length of C is $q \geq 5$. Let $\mathcal{L}$ and $\mathcal{L}'$ be sets with*

$$\mathcal{L}' \subsetneq \mathcal{L} \subseteq \{\{c_0, c_1\}, \{c_0, c_2\}, \ldots, \{c_0, c_{q-1}\}\} \quad \text{such that} \quad |\mathcal{L}'| = |\mathcal{L}| - 1.$$

Let $\mathcal{L}'' = \mathcal{L}' \cup \big\{S \subseteq V(H) \mid |S| \in \{1, |V(H)|\}\big\}$. Then

$$\text{#Hom}(H, \mathcal{L}) \leq_{AP} \text{#Hom}(H, \mathcal{L}'').$$

PROOF. For the reflexive triangle-extended cycle H^{**} we use the notation (C, $\mathcal{I}$ and d_i for $i \in \mathcal{I}$) as given by Definition 50. Let $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$ be as given in the statement of the lemma. We have $\mathcal{L}' = \mathcal{L} \setminus \{\{c_0, c_\ell\}\}$ for some $\ell \in [q-1]$. Let $(G, \mathbf{S}^G)$ be an input to #HOM($H, \mathcal{L}$). Let $U = \{u \in V(G) \mid S_u^G = \{c_0, c_\ell\}\}$. Since $\{c_0, c_\ell\}$ is not part of $\mathcal{L}''$ the goal is to simulate $\{c_0, c_\ell\}$ using gadgetry and lists from $\mathcal{L}''$.

From $(G, \mathbf{S}^G)$ we define an instance $(J, \mathbf{S}^J)$ of #HOM($H, \mathcal{L}''$). To this end we will define, for each $u \in U$, a vertex gadget J_u and a corresponding set of lists $\mathbf{S}^u = \{S_v^u \in \mathcal{L}'' \mid v \in V(J_u)\}$. There are two distinct paths in C that connect c_0 and c_ℓ: $P_1 = c_0, c_1, \ldots, c_\ell$ and $P_2 = c_\ell, \ldots, c_{q-1}, c_0$. The graph J_u has two parts: a graph J_{P_1} and a graph J_{P_2}, which depend on the paths P_1 and P_2. We first define J_{P_1} and J_{P_2} (and the corresponding sets of lists $\mathbf{S}^1 = \{S_v^1 \in \mathcal{L}'' \mid v \in V(J_{P_1})\}$ and $\mathbf{S}^2 = \{S_v^2 \in \mathcal{L}'' \mid v \in V(J_{P_2})\}$) and then we describe the way in which they are connected to form J_u. The definition of J_{P_1} depends on ℓ, the number of edges of P_1:

- If ℓ is even, think of a path on $\ell/2$ edges. Let v^* be one of the end points of this path. We pin v^* to $c_{\ell/2}$ (the vertex in the "middle" of P_1). This graph is depicted in Figure 23 on the left. The graph J_{P_1} is then a modification of this graph where each vertex of the path is replaced by a clique of size 2 (apart from v^* which will be pinned to $c_{\ell/2}$ anyway). This modification will ensure that only looped vertices can be in the image of J_{P_1}. The graph J_{P_1} is depicted in Figure 23 (on the right) and is formally defined as follows: $V(J_{P_1}) = \{v_i, v_i' \mid i \in \{0, \ldots, \ell/2 - 1\}\} \cup \{v^*\}$, where all these vertices are distinct from the vertices of G (and distinct from the vertices used in other gadgets). $E(J_{P_1}) = \{\{v_i, v_i'\} \mid i \in \{0, \ldots, \ell/2 - 1\}\} \cup \{\{v_{i-1}, v_{i-1}'\} \times \{v_i, v_i'\} \mid i \in [\ell/2 - 1]\} \cup \{\{v_{\ell/2-1}, v^*\}, \{v_{\ell/2-1}', v^*\}\}$. We set $S_{v^*}^1 = \{c_{\ell/2}\}$ and $S_v^1 = V(H)$ for all $v \in V(J_{P_1}) \setminus \{v^*\}$
- If ℓ is odd, then J_{P_1} is defined very similarly to the previous case, see Figure 24. Formally, $V(J_{P_1}) = \{v_i, v_i' \mid i \in \{0, \ldots, \lfloor \ell/2 \rfloor\}\} \cup \{v_1^*, v_2^*\}$, where all these vertices are distinct from the vertices of G (and distinct from the vertices used in other gadgets). $E(J_{P_1}) = \{\{v_i, v_i'\} \mid i \in \{0, \ldots, \lfloor \ell/2 \rfloor\}\} \cup \{\{v_{i-1}, v_{i-1}'\} \times \{v_i, v_i'\} \mid i \in [\lfloor \ell/2 \rfloor]\} \cup \{\{v_{\lfloor \ell/2 \rfloor}, v_{\lfloor \ell/2 \rfloor}'\} \times \{v_1^*, v_2^*\}\}$. We set $S_{v_1^*}^1 = \{c_{\lceil \ell/2 \rceil}\}$, $S_{v_2^*}^1 = \{c_{\lfloor \ell/2 \rfloor}\}$ and $S_v^1 = V(H)$ for all $v \in V(J_{P_1}) \setminus \{v_1^*, v_2^*\}$.
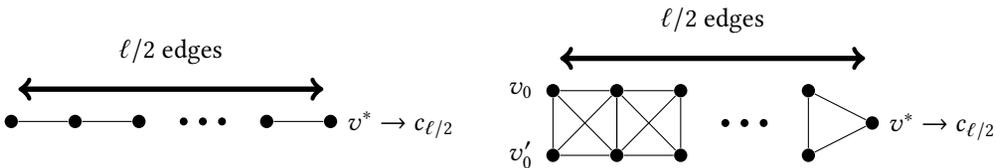

Fig. 23. Construction of the graph J_{P_1} for even ℓ. The label of the form $v^* \to c_{\ell/2}$ means that the vertex $v^* \in V(J_{P_1})$ is "pinned" to $c_{\ell/2} \in V(H)$ since $S_{v^*}^1 = \{c_{\ell/2}\}$.

This completes the definition of J_{P_1}. J_{P_2} is defined analogously. However, the length of P_2 is $q - \ell$ instead of ℓ. Furthermore, if $q - \ell$ is even, note that the vertex in the "middle" of $P_2 = c_\ell, \ldots, c_{q-1}, c_0$ is $c_{(q+\ell)/2}$ rather than $c_{\ell/2}$. (Accordingly, if $q - \ell$ is odd, we use $c_{\lceil (q+\ell)/2 \rceil}$ and $c_{\lfloor (q+\ell)/2 \rfloor}$ to "pin" to.) Formally, J_{P_2} is defined as follows:

- If $q - \ell$ is even, we have $V(J_{P_2}) = \{w_i, w_i' \mid i \in \{0, \ldots, (q-\ell)/2 - 1\}\} \cup \{w^*\}$ and $E(J_{P_2}) = \{\{w_i, w_i'\} \mid i \in \{0, \ldots, (q-\ell)/2 - 1\}\} \cup \{\{w_{i-1}, w_{i-1}'\} \times \{w_i, w_i'\} \mid i \in [(q-\ell)/2 - 1]\} \cup \{\{w_{(q-\ell)/2-1}, w^*\}, \{w_{(q-\ell)/2-1}', w^*\}\}$. We set $S_{w^*}^2 = \{c_{(q+\ell)/2}\}$ and $S_w^2 = V(H)$ for all $w \in V(J_{P_2}) \setminus \{w^*\}$
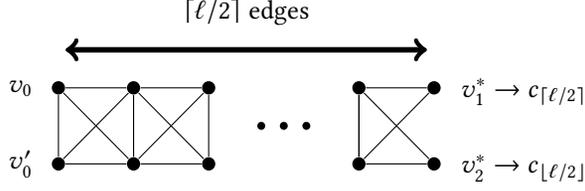
Fig. 24. The graph J_{P_1} for odd ℓ. A label of the form $a \to b$ means that the vertex $a \in V(J_{P_1})$ is "pinned" to $b \in V(H)$ since $S_a^1 = \{b\}$.

- If $q - \ell$ is odd, we have $V(J_{P_2}) = \{w_i, w_i' \mid i \in \{0, \ldots, \lfloor (q-\ell)/2 \rfloor\}\} \cup \{w_1^*, w_2^*\}$ and $E(J_{P_2}) = \{\{w_i, w_i'\} \mid i \in \{0, \ldots, \lfloor (q-\ell)/2 \rfloor\}\} \cup \{\{w_{i-1}, w_{i-1}'\} \times \{w_i, w_i'\} \mid i \in [\lfloor (q-\ell)/2 \rfloor]\} \cup \{\{w_{\lfloor (q-\ell)/2 \rfloor}, w_{\lfloor (q-\ell)/2 \rfloor}'\} \times \{w_1^*, w_2^*\}\}$. We set $S_{w_1^*}^2 = \{c_{\lceil (q+\ell)/2 \rceil}\}$, $S_{w_2^*}^2 = \{c_{\lfloor (q+\ell)/2 \rfloor}\}$ and $S_w^2 = V(H)$ for all $w \in V(J_{P_2}) \setminus \{w_1^*, w_2^*\}$.

We can now define the graph J_u (for $u \in U$): J_u is the graph obtained from J_{P_1} and J_{P_2} by identifying v_0 with w_0 and v_0' with w_0'. As an example, if P_1 has even length and P_2 has odd length, the graph J_u is depicted in Figure 25. Let $\mathcal{A}(u)$ denote the set that contains the two vertices $v_0 = w_0$ and $v_0' = w_0'$. The lists S^u of the vertices in J_u are the union of S^1 and S^2. (Note that this is well-defined as $S_{v_0}^1 = S_{v_0'}^1 = S_{w_0}^2 = S_{w_0'}^2 = V(H)$.) This completes the definition of J_u and S^u.

We can finally define the instance (J, S^J): J is the graph with vertices $V(J_u) = V(G) \setminus U \cup \bigcup_{u \in U} V(J_u)$ and edges

$$
\begin{aligned}
E(J) = & \big\{ \{v, v'\} \mid \{v, v'\} \in E(G) \text{ and } v, v' \in V(G) \setminus U \big\} \\
& \cup \big\{ \{v\} \times \mathcal{A}(v') \mid \{v, v'\} \in E(G) \text{ and } v \in V(G) \setminus U, v' \in U \big\} \\
& \cup \big\{ \mathcal{A}(v) \times \mathcal{A}(v') \mid \{v, v'\} \in E(G) \text{ and } v, v' \in U \big\} \\
& \cup \bigcup_{u \in U} E(J_u).
\end{aligned}
$$

Finally, $S^J = \{S_v^J \subseteq V(H) \mid v \in V(J)\}$ with

$$
S_v^J = \begin{cases} S_v^G, & \text{if } v \in V(G) \setminus U \\ S_v^u & \text{if otherwise } v \in V(J_u) \text{ for some } u \in U. \end{cases}
$$

Note that for each $v \in V(J)$ we have $S_v^J \in \mathcal{L}''$ and therefore (J, S^J) is a valid input to #Hom$(H, \mathcal{L}'')$.

To show how homomorphisms from (J, S^J) to H relate to homomorphisms from (G, S^G) to H we determine some properties of the gadget J_u. Consider the case where ℓ, the number of edges of P_1, is even, and $q - \ell$, the number of edges of P_2, is odd. (The other cases of ℓ and $q - \ell$ even or odd will be analogous.) Then J_u is the gadget depicted in Figure 25. Let the vertices of J_u be labelled accordingly. Now let h be a homomorphism from (J, S^J) to H.

Claim 1: For all $i \in \{0, \ldots, \ell/2 - 1\}$, the image $h(\{v_i, v_i'\})$ contains at least one looped vertex. For all $j \in \{0, \ldots, \lfloor (q-\ell)/2 \rfloor\}$, the image $h(\{w_j, w_j'\})$ contains at least one looped vertex.

Proof of Claim 1: First consider $h(\{v_{\ell/2-1}, v_{\ell/2-1}'\})$. Assume that both $h(v_{\ell/2-1})$ and $h(v_{\ell/2-1}')$ are unlooped vertices of H. Since $\{v_{\ell/2-1}, v_{\ell/2-1}'\}$ is an edge in J, $h(v_{\ell/2-1})$ and $h(v_{\ell/2-1}')$ have to be connected by an edge in H and therefore have to be different unlooped vertices. However, the vertices $v_{\ell/2-1}$ and $v_{\ell/2-1}'$ are also neighbours of v^* which is pinned to the looped vertex $c_{\ell/2}$. It
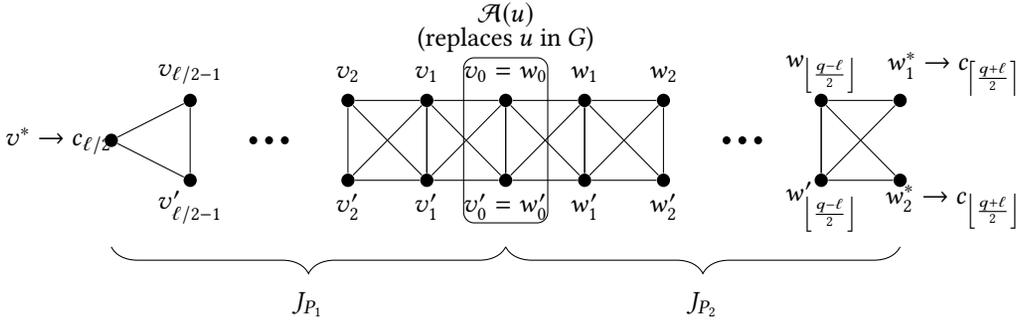
Fig. 25. The graph J_u if ℓ (the number of edges of P_1) is even and $q - \ell$ (the number of edges of P_2) is odd.

follows that $h(v_{\ell/2-1}), h(v'_{\ell/2-1}), c_{\ell/2}$ form a mixed triangle in H, a contradiction. This argument can be repeated iteratively for each $i = \ell/2 - 2, \ldots, 0$ (using the fact that $h(\{v_{i+1}, v'_{i+1}\})$ contains a looped vertex). The argument for $h(\{w_j, w'_j\})$ ($j \in \{0, \ldots, \lfloor (q - \ell)/2 \rfloor\}$) is analogous.

Claim 2: There exists a vertex $v \in \mathcal{A}(u)$ with $h(v) \in \{c_0, c_\ell\}$.

Proof of Claim 2: Let $\mathcal{A}(u) = \{v, v'\}$ where $h(v)$ is looped (this can be assumed without loss of generality by Claim 1). We will show that $h(v) \in \{c_0, c_\ell\}$: By Claim 1 and the construction of J_{P_1} there exists a walk on $\ell/2$ edges in H which uses looped vertices only and goes from $c_{\ell/2}$ to $h(v)$, which by assumption is looped itself. As this walk is looped it is in H^*, and as it contains $c_{\ell/2}$ it is in H^{**}. Hence,

$$h(v) \in \Gamma_{H^{**}}^{\ell/2}(c_{\ell/2}). \tag{29}$$

Similarly, by the construction of J_{P_2} we obtain

$$h(v) \in \Gamma_{H^{**}}^{\lceil (q-\ell)/2 \rceil}(c_{\lfloor (q+\ell)/2 \rfloor}) \cap \Gamma_{H^{**}}^{\lceil (q-\ell)/2 \rceil}(c_{\lceil (q+\ell)/2 \rceil}). \tag{30}$$

Since H^{**} is a reflexive triangle-extended cycle we have

$$\Gamma_{H^{**}}^{\ell/2}(c_{\ell/2}) = \{c_0, \ldots, c_\ell\} \cup \{d_i \mid i \in \mathcal{I} \cap \{0, \ldots, \ell - 1\}\}$$

and

$$\Gamma_{H^{**}}^{\lceil (q-\ell)/2 \rceil}(c_{\lfloor (q+\ell)/2 \rfloor}) \cap \Gamma_{H^{**}}^{\lceil (q-\ell)/2 \rceil}(c_{\lceil (q+\ell)/2 \rceil}) = \{c_\ell, \ldots, c_{q-1}, c_0\} \cup \{d_i \mid i \in \mathcal{I} \cap \{\ell, \ldots, q - 1\}\}.$$

Therefore, from Equations (29) and (30) it follows that $h(v) \in \{c_0, c_\ell\}$.

Claim 3: Let $v \in \mathcal{A}(u)$ and $h(v) \in \{c_0, c_\ell\}$. Then the image of the remaining vertices of J_u under h is determined completely. In particular, $h(\mathcal{A}(u)) = \{h(v)\}$.

Proof of Claim 3: Consider the case where $h(v) = c_0$ (the case $h(v) = c_\ell$ can be treated analogously). Since H^{**} is a reflexive triangle-extended cycle and a connected component of H^*, the walk $c_{\ell/2}, c_{\ell/2-1}, \ldots, c_0$ is the only $\ell/2$-edge walk on looped vertices in H that leads from $c_{\ell/2}$ to c_0 (since there are no reflexive shortcuts in C). Thus, by Claim 1 and the construction of J_{P_1}, we have $\forall i \in \{0, \ldots, \ell/2 - 1\}$, $c_i \in h(\{v_i, v'_i\})$. We assume without loss of generality (by renaming) that for each $i \in \{0, \ldots, \ell/2 - 1\}$ we have $h(v_i) = c_i$.

Now consider the image $h(v'_i)$ for some $i \in [\ell/2 - 1]$. The vertex v'_i is a neighbour of v_{i-1}, v_i and v_{i+1} (or alternatively v^* if $i = \ell/2 - 1$) in J. Therefore, by the fact that for each $i \in \{0, \ldots, \ell/2 - 1\}$ we have $h(v_i) = c_i$ and by the pinning that ensures $h(v^*) = c_{\ell/2}$, we know that $h(v'_i)$ has to be a neighbour of c_{i-1}, c_i and c_{i+1} in H. Since there is no edge between c_{i-1} and c_{i+1} we have

$h(v'_i) \notin \{c_{i-1}, c_{i+1}\}$. Then $h(v'_i) = c_i$ as otherwise $c_{i-1}, c_i, c_{i+1}, h(v'_i)$ would form a square in H which is a contradiction to the fact that H is square-free. So we have established that for each $i \in [\ell/2 - 1]$ it holds that $h(v_i) = h(v'_i) = c_i$.

Similarly one establishes that for each $i \in [\lfloor (q - \ell)/2 \rfloor]$ it holds that $h(\{w_i, w'_i\}) = c_{q-i}$: Note that by the construction of J_{P_2} the homomorphism h has to map $w_{\lfloor (q-\ell)/2 \rfloor}$ and $w'_{\lfloor (q-\ell)/2 \rfloor}$ to common neighbours of $c_{\lfloor (q+\ell)/2 \rfloor}$ and $c_{\lceil (q+\ell)/2 \rceil}$. Since H^{**} is a reflexive triangle-extended cycle and a connected component of H^*, the walk $c_{\lceil (q+\ell)/2 \rceil} \ldots, c_{q-1}, c_0$ is the only $\lfloor (q - \ell)/2 \rfloor$-edge walk on looped vertices in H that leads from a common neighbour of $c_{\lfloor (q+\ell)/2 \rfloor}$ and $c_{\lceil (q+\ell)/2 \rceil}$ to c_0. Therefore, we have $c_0 \in h(\{w_0, w'_0\})$ and $\forall i \in [\lfloor (q - \ell)/2 \rfloor], \; c_{q-i} \in h(\{w_i, w'_i\})$. Then by the same arguments as before we establish that for each $i \in [\lfloor (q - \ell)/2 \rfloor]$ it holds that $h(w_i) = h(w'_i) = c_{q-i}$.

Finally, v'_0 is a neighbour of $w_1, v_0(= w_0)$ and v_1. Hence, $h(v'_0)$ is a neighbour of $h(w_1) = c_{q-1}$, $h(v_0) = c_0$ and $h(v_1) = c_1$. Since there is no edge between c_{q-1} and c_1 we have $h(v'_0) \notin \{c_{q-1}, c_1\}$. Then $h(v'_0) = c_0$ as otherwise $c_{q-1}, c_0, c_1, h(v'_0)$ would form a square in H which is a contradiction to the fact that H is square-free. We obtain $h(v'_0) = h(v_0) = c_0$. This proves Claim 3.

From the construction of J together with Claim 2 and Claim 3 we obtain that $N\big((G, \mathbf{S}^G) \to H\big) = N\big((J, \mathbf{S}^J) \to H\big)$, which gives the sought-for reduction in the case where ℓ is even and $q - \ell$ is odd. All other cases of ℓ even or odd and $q - \ell$ even or odd can be treated analogously. □

Now we can prove Lemma 49.

Lemma 49. Let H be a square-free graph. If H contains a reflexive cycle of length at least 5 as an induced subgraph then #SAT $\leq_{AP}$ #RET(H).

PROOF. Suppose that H contains a mixed triangle as an induced subgraph, then the statement of this lemma follows from Lemmas 24 and 25. We can now assume that H does not contain any mixed triangle as an induced subgraph.

Let $C = c_0, \ldots, c_{q-1}$ be the reflexive cycle of length $q \geq 5$ in H. Let H^* be the graph induced by the looped vertices in H. If H (and hence H^*) contains an induced WR$_3$ or an induced net then #SAT $\leq_{AP}$ #RET(H) by Lemmas 40 and 48, respectively. Otherwise, the connected component of H^* that contains the cycle C has to be a reflexive triangle-extended cycle by Lemma 52 and therefore H fulfills the requirements of Lemma 54.

Let $\mathcal{L} = \{\{c_0, c_1\}, \{c_0, c_2\}, \ldots, \{c_0, c_{q-1}\}\}$. Then #HOM$(H, \mathcal{L}) \equiv_{AP}$ #SAT by Lemma 53. We can use Lemma 54 iteratively to obtain #HOM$(H, \mathcal{L}) \leq_{AP}$ #HOM$\big(H, \{S \subseteq V(H) \mid |S| \in \{1, |V(H)|\}\}\big)$. Note that by the problem definitions we have #HOM$\big(H, \{S \subseteq V(H) \mid |S| \in \{1, |V(H)|\}\}\big)$ = #RET(H). Summarising,

$$\text{\#SAT} \equiv_{AP} \text{\#HOM}(H, \mathcal{L}) \leq_{AP} \text{\#HOM}\big(H, \{S \subseteq V(H) \mid |S| \in \{1, |V(H)|\}\}\big) = \text{\#RET}(H).$$

□

4 PUTTING THE PIECES TOGETHER

This section contains the proof of Theorem 2 (we restate it here for convenience). We will use the following theorem, which is a consequence of the classification in [21] for approximately counting retractions to graphs of girth at least 5, since that proof does not use the fact that H is triangle-free for irreflexive H.

Theorem 55 ([21, Theorem 2.3]). *Let H be an irreflexive square-free graph.*

  i) *If every connected component of H is a star, then #RET(H) is in FP.*
  ii) *Otherwise, if every connected component of H is a caterpillar, then #RET(H) is approximation-equivalent to #BIS.*

iii) Otherwise, #RET(H) is approximation-equivalent to #SAT.

In the following lemma we collect the #SAT-hardness results which we use to prove Theorem 2.

Lemma 56. *Let H be a connected square-free graph other than a reflexive clique, a member of $\mathcal{H}_{\mathrm{BIS}}$, or an irreflexive caterpillar. Then #RET(H) is approximation-equivalent to #SAT.*

PROOF. If H is irreflexive then by assumption it is not a caterpillar. Thus #RET(H) is approximation-equivalent to #SAT by Theorem 55.

If H is not irreflexive, i.e. if H has at least one loop, then we collect different #SAT-hardness results proved throughout this work to show hardness. If H contains a mixed triangle as induced subgraph, then #SAT $\leq_{\mathrm{AP}}$ #RET(H) by Lemmas 24 and 25. If H does not contain a mixed triangle as an induced subgraph but contains a WR$_3$, a net or a reflexive cycle of length at least 5 as an induced subgraph, then #SAT $\leq_{\mathrm{AP}}$ #RET(H) by Lemmas 40, 48 and 49, respectively. It remains to show #SAT $\leq_{\mathrm{AP}}$ #RET(H) if H is a graph with the following properties:

- H is connected and square-free.
- H has at least one looped vertex.
- H is not a reflexive clique.
- $H \notin \mathcal{H}_{\mathrm{BIS}}$.
- H does not contain any of the following as an induced subgraph: a mixed triangle, a WR$_3$, a net, a reflexive cycle of length at least 5.

Let H^* be a connected component in the graph induced by the looped vertices in H. (It will turn out that H^* is actually the only connected component in this graph.) Then by the properties of H and Lemma 52 we know that H^* is a reflexive triangle-extended path. We recall the definition of a reflexive triangle-extended path from Definition 51: H^* is a reflexive path $c_0, \ldots, c_{q-1}$ together with a set $\mathcal{I} \subseteq \{0, \ldots, q-2\}$, and a reflexive triangle d_i, c_i, c_{i+1} for each $i \in \mathcal{I}$. Since H^* is not a reflexive clique it holds that $q - 1 \geq 2$.

Note that $H^* \in \mathcal{H}_{\mathrm{BIS}}$ (where the set of bristles is empty and c_i corresponds to p_i). For all $i \in \mathcal{I}$ the clique K_i has size 3, for $i \notin \mathcal{I}$ it has size 2. Since $H \notin \mathcal{H}_{\mathrm{BIS}}$ and H is connected, there exists a vertex u outside of H^* with a neighbour v in H^*. The vertex u has to be unlooped as otherwise it would be part of the reflexive connected component H^*. We consider four disjoint cases.

- If there exists a vertex $u \notin V(H^*)$ (u is unlooped) which is adjacent to a vertex $v \in V(H^*)$ and $\deg_H(u) \geq 2$, then consider two different cases:
  - If u is adjacent to a vertex $w \in \Gamma_H(v)$ with $w \neq v$, then $w \neq u$ since u is unlooped and u, v, w is a mixed triangle, a contradiction.
  - If v is the only neighbour of u in $\Gamma_H(v)$, then the requirements of Lemma 39 are met (with $b = v$ and $g = u$) and hence #SAT $\leq_{\mathrm{AP}}$ #RET(H).
- If there exists a vertex $u \notin V(H^*)$ which is adjacent to a vertex $v \in V(H^*)$, $\deg_H(u) = 1$ and $v \in \{d_i \mid i \in \mathcal{I}\}$, then $H[\Gamma_H(v)]$ is a graph of the form $X(k_1, 0, 1)$ where $k_1 \geq 1$ (cf. Figure 9) and therefore

  $$\text{\#SAT} \leq_{\mathrm{AP}} \text{\#HOM}(X(k_1, 0, 1)) \leq_{\mathrm{AP}} \text{\#RET}(X(k_1, 0, 1)) = \text{\#RET}(H[\Gamma_H(v)]) \leq_{\mathrm{AP}} \text{\#RET}(H),$$

  by Lemma 36, Observation 6 and Observation 23 (in the order of the reductions used).
- Suppose there exists a vertex $u \notin V(H^*)$ with $\deg_H(u) = 1$ that is adjacent to a vertex $v \in \{c_0, c_{q-1}\}$. Without loss of generality (by renaming the vertices of H^*) let $v = c_0$. If $0 \in \mathcal{I}$ then c_0 is part of a reflexive triangle d_0, c_0, c_1 in H^* and $H[\Gamma_H(v)]$ is a graph of the form $X(k_1, 0, 1)$ where $k_1 \geq 1$. Then we have #SAT $\leq_{\mathrm{AP}}$ #RET(H) by the same arguments used in the previous case. If otherwise $0 \notin \mathcal{I}$ then c_1 is the only neighbour of c_0 in H^* and $H[\Gamma_H(v)]$ is a graph of the form $X(k_1, 1, 0)$ where $k_1 \geq 1$. Then we use Theorem 5 to infer

that #SAT $\leq_{AP}$ #RET($X(k_1, 1, 0)$) (since $X(k_1, 1, 0)$ has girth at least 5 and therefore is subject to Theorem 5). It follows that #SAT $\leq_{AP}$ #RET(H) by the same arguments as in the previous case.

- If for every pair u, v of adjacent vertices with $u \notin V(H^*)$ and $v \in V(H^*)$ we have $\deg_H(u) = 1$ (u is a so-called bristle) and $v \in \{c_1, \ldots, c_{q-2}\}$, then H is the triangle-extended path H^* together with a number of bristles (all of which are attached to a vertex in $\{c_1, \ldots, c_{q-2}\}$). To match the notation of $\mathcal{H}_{BIS}$ in Definition 10 we set $Q = q - 2$ and, for all $i \in \{0, \ldots, Q + 1\}$, we set $p_i = c_i$. Further, if $i \in \mathcal{I}$ then $K_i = \{p_i, d_i, p_{i+1}\}$ and otherwise $K_i = \{p_i, p_{i+1}\}$. Note that $|K_i| \in \{2, 3\}$ which we will use in a moment. For each $i \in [Q]$ let B_i be the set of unlooped neighbours (bristles) of p_i. By the fact that all unlooped vertices of H have degree 1 and a neighbour in $\{c_1, \ldots, c_{q-2}\} = \{p_1, \ldots, p_Q\}$ we have $V(H) = \bigcup_{i=0}^{Q} K_i \cup \bigcup_{i=1}^{Q} B_i$ and $E(H) = \bigcup_{i=0}^{Q} (K_i \times K_i) \cup \bigcup_{i=1}^{Q} (\{p_i\} \times B_i)$. Since H^* is a triangle-extended path we can also verify the properties $K_{i-1} \cap K_i = \{p_i\}$ (for $i \in [Q]$) and $K_i \cap K_j = \emptyset$ (for $i, j \in \{0, \ldots, Q\}$ with $|j - i| > 1$).

  Therefore, since $H \notin \mathcal{H}_{BIS}$, there exists $i \in [Q]$ such that at least one of the following holds:
  (i) $|K_{i-1}| = |K_i| = 2$ and $|B_i| \geq 2$.
  (ii) $|K_{i-1}| = 2$ and $|K_i| = 3$ (or $|K_{i-1}| = 3$ and $|K_i| = 2$) and $|B_i| \geq 3$.
  (iii) $|K_{i-1}| = |K_i| = 3$ and $|B_i| \geq 5$.

  In all three cases we will show that the neighbourhood of p_i induces a #SAT-hard subgraph, i.e. that #SAT $\leq_{AP}$ #RET($H[\Gamma_H(p_i)]$). Then, by Observation 23, we obtain #SAT $\leq_{AP}$ #RET(H) which completes the proof of this case and with it the proof of the theorem.

  - If item (i) holds, then $H[\Gamma_H(p_i)]$ is of the form $X(k_1, 2, 0)$ where $k_1 \geq 2$. The graph $X(k_1, 2, 0)$ has girth at least 5 and therefore is subject to Theorem 5. Since $X(k_1, 2, 0)$ with $k_1 \geq 2$ is a mixed graph but not a partially bristled reflexive path we obtain #SAT $\leq_{AP}$ #RET($H[\Gamma_H(p_i)]$) by Theorem 5.
  - If item (ii) holds, then $H[\Gamma_H(p_i)]$ is of the form $X(k_1, 1, 1)$ where $k_1 \geq 3$. Then #SAT $\leq_{AP}$ #RET($H[\Gamma_H(p_i)]$) by Lemma 37.
  - If item (iii) holds, then $H[\Gamma_H(p_i)]$ is of the form $X(k_1, 0, 2)$ where $k_1 \geq 5$. Then #SAT $\leq_{AP}$ #RET($H[\Gamma_H(p_i)]$) by Lemma 38.

  $\square$

We will use the following remark to deal with graphs that have multiple connected components.

Remark 57 ([21, Remark 1.15]). Let H be a graph with connected components $H_1, \ldots, H_k$. On the one hand it holds that $\forall j \in [k]$, #RET(H_j) $\leq_{AP}$ #RET(H). On the other hand, given an oracle for each #RET(H_j), we can construct a polynomial-time algorithm for #RET(H).

Theorem 2. *Let H be a square-free graph.*

  i) *If every connected component of H is trivial then approximately counting retractions to H is in FP.*
  ii) *Otherwise, if every connected component of H is*
   - *trivial,*
   - *in the class $\mathcal{H}_{BIS}$, or*
   - *is an irreflexive caterpillar*
   then approximately counting retractions to H is #BIS-equivalent.
  iii) *Otherwise, approximately counting retractions to H is #SAT-equivalent.*

PROOF. If H is a trivial graph then #LHOM(H) $\in$ FP by the result of Dyer and Greenhill [14] (see Theorem 7). From #RET(H) $\leq_{AP}$ #LHOM(H) (Observation 6) it also follows that #RET(H) $\in$ FP. Then item *i)* follows from Remark 57.

If H is a graph for which item *i)* does not hold, then H has a connected component H' that is not a trivial graph. By Remark 57 we have #RET(H') $\leq_{AP}$ #RET(H). Then #BIS-hardness in item *ii)* follows from the reduction #HOM(H') $\leq_{AP}$ #RET(H') (Observation 6) together with the fact that #BIS $\leq_{AP}$ #HOM(H') since H' is a non-trivial connected graph [24, Theorem 1].

We will now prove #BIS-easiness in item *ii)*. If H' is a trivial graph we have already pointed out that #RET(H') $\in$ FP and hence #RET(H') is trivially #BIS-easy. If $H' \in \mathcal{H}_{BIS}$ then #RET(H') $\leq_{AP}$ #BIS by Theorem 1. If H' is an irreflexive caterpillar then #RET(H') $\leq_{AP}$ #BIS by Theorem 55. Hence, #BIS-easiness in item *ii)* follows from Remark 57.

If H is a graph for which item *ii)* does not hold, then H has a connected component H' that is not trivial, not a member of $\mathcal{H}_{BIS}$ and not an irreflexive caterpillar. Then #RET(H') is approximation-equivalent to #SAT by Lemma 56 and #RET(H') $\leq_{AP}$ #RET(H) by Remark 57. This proves item *iii)*. □

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