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Real-weighted general factors on subcubic graphs

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Abstract

General factors generalize the concept of graph matchings and have been extensively studied in combinatorial optimization. Given a graph G where each vertex v is assigned a set $\pi(v)$ of feasible degrees (called a degree constraint), the general factor problem seeks a (spanning) subgraph F of G such that $\deg_F(v) \in \pi(v)$ for all v of G. When all degree constraints are symmetric Δ-matroids, the problem is solvable in polynomial-time. The weighted general factor problem further extends this by incorporating edge weights, and the goal is to find a general factor that maximizes the total weight in an edge-weighted graph. In this paper, we propose a strongly polynomial-time algorithm for the real-weighted general factor problem on subcubic graphs by establishing a refined structural result that ensures the optimality of weighted graph factors. As an application of our result, we obtain a strongly polynomial-time algorithm for the terminal backup problem, a variant of the Steiner tree problem. Furthermore, we provide a characterization theorem for matching-gadget realizable degree constraints.

Keywords Graph factor · Terminal backup problem · Matching-gadget · Strongly polynomial-time · Symmetric Δ-matroid

Mathematics Subject Classification 90C27 Combinatorial optimization · 05C85 Graph algorithms

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1 Introduction

A *matching* in an undirected graph is a subset of the edges such that no two share a common vertex. A *perfect matching* is a matching that covers all vertices of the graph. Graph matching is one of the most fundamental problems in both graph theory and combinatorial optimization, with elegant structural results and efficient algorithms as described in the monograph of Lovász and Plummer [32] and in standard textbooks [28, 37]. One of the most celebrated results on graph matchings is Edmonds' blossom algorithm [13, 14], which provides a polynomial-time solution to the *weighted (perfect) matching problem*. In this problem, each edge is assigned a weight and the goal is to find a (perfect) matching of maximum total weight. Since then, many improvements and efficient algorithms have been developed [6, 16, 18–21, 23–25, 29], as summarized in Table III of [10].

The *f-factor problem* generalizes the perfect matching problem by assigning to each vertex $v \in V$ of $G = (V, E)$ a non-negative integer $f(v)$. The goal is to find a (spanning) subgraph $F = (V_F, E_F)$ of G such that $\deg_F(v) = f(v)$ for every $v \in V$.[1] This includes the perfect matching problem as a special case when $f(v) = 1$ for every $v \in V$. Both the unweighted and weighted versions of the f-factor problem can be efficiently solved via gadget reductions to the perfect matching problem [15]. Tutte's theorem [41] provides a key characterization for the existence of an f-factor, generalizing his result for perfect matchings [40]. Since then, graph factors have been extensively studied, leading to various extensions including *b-matchings,* *$[a, b]$-factors,* *(g, f)-factors, parity (g, f)-factors*, and *anti-factors*, and several types of characterization theorems proved for the existence of such factors. We refer the readers to the book [1] and the survey [35] for a comprehensive treatment of the developments on the topic of graph factors.

In the early 1970 s, Lovász introduced a further generalization of the above factor problems [30, 31], known as the *general factor problem* (GFP). For any nonnegative integer n, let $[n]$ denote $\{0, 1, \dots, n\}$. A *degree constraint* D of arity n is a subset of $[n]$.[2] We say that a degree constraint D has a *gap of length* $k \geq 0$ if there exists $p \in D$ such that $p + 1, \dots, p + k \notin D$ and $p + k + 1 \in D$. An instance Ω of the GFP consists of a graph $G = (V, E)$ and a mapping π that assigns to each vertex $v \in V$ a degree constraint $\pi(v) \subseteq [\deg_G(v)]$, where the arity of $\pi(v)$ is $\deg_G(v)$. The task is to find a subgraph, if one exists, F of G such that $\deg_F(v) \in \pi(v)$ for every $v \in V$. We say F is a general graph of the instance Ω. The case $\pi(v) = \{0, 1\}$ for every $v \in V$ is the matching problem, and the case $\pi(v) = \{1\}$ for every $v \in V$ is the perfect matching problem. Cornuéjols showed that the GFP is solvable in polynomial-time if each degree constraint has gaps of length at most 1 [7]. The GFP becomes NP-complete when gaps of length at least 2 are allowed [7, 31], except in cases where the constraints are either all 0-*valid* or they are all 1-*valid*. A degree constraint D of arity k is 0-valid if $0 \in D$, and 1-valid if $k \in D$. When all constraints are 0-valid, the

[1] In graph theory, a graph factor is usually a spanning subgraph. Here, without causing ambiguity, we allow F to be an arbitrary subgraph including the empty graph and we adapt the convention that $\deg_F(v) = 0$ if $v \in V \backslash V_F$.

[2] Each degree constraint is always associated with an arity. Two degree constraints are different if they have different arities although they may be the same set of integers.

empty graph is a factor. When all constraints are 1-valid, the underlying graph is a factor of itself. In both cases, the GFP is trivially tractable.

In this paper, we consider the *weighted general factor problem* (WGFP) where each edge is assigned a real-valued weight and the task is to find an optimal general factor of the maximum total weight. We suppose that each degree constraint has gaps of length at most 1 for which the unweighted GFP is known to be polynomial-time solvable. In some cases, the WGFP can be reduced to the weighted matching or perfect matching problem via gadget constructions, making it solvable in polynomial-time. Such degree constraints are called *matching-realizable* (see Definition A.1). For instance, the degree constraint $D = [b]$ where $b > 0$ for the *b-matching problem* is matching realizable [42]. The *weighted b-matching problem* is interesting in its own right in combinatorial optimization and has been well studied with many elaborate algorithms developed [3, 17, 22, 33, 36]. Besides b-matchings, Cornuéjols showed that the *parity interval* constraint $D = \{g, g+2, \ldots, f\}$ where $f \geq g \geq 0$ and $f \equiv g \mod 2$ (i.e., all feasible degrees are odd or all are even), is matching realizable [7], and Szabó showed that the *interval* constraint $D = \{g, g+1, \ldots, f\}$ where $f \geq g \geq 0$, for (g, f)-factors is matching realizable [39]. Thus, when all degree constraints are intervals or parity intervals, the WGFP can be reduced to the weighted matching problem (with some vertices constrained to degree exactly 1), making it solvable in polynomial-time using Edmonds' algorithm. Szabó proposed an alternative approach for this problem [39]. In addition, by reducing the WGFP with interval and parity interval constraints to the weighted (g, f)-factor problem, a faster algorithm was obtained in [11] based on Gabow's algorithm [17].

In [39], Szabó conjectured that the WGFP is solvable in polynomial-time even without restricting each degree constraint to be an interval or a parity interval, as long as each degree constraint has gaps of length at most 1. To investigate the conjecture, a natural question arises: *Are there additional cases of the WGFP that can be solved in polynomial-time via gadget reductions to weighted matchings?* Equivalently, *are there other degree constraints that are matching realizable?* In this paper, we answer this question in the negative.

Theorem 1.1 *A degree constraint with gaps of length at most* 1 *is matching realizable if and only if it is an interval or a parity interval.*

1.1 Previous results

With the answer to the above question being negative, new algorithms need to be devised for the WGFP with degree constraints that are not intervals or parity intervals. Unlike the weighted matching problem and the weighted b-matching problem for which various types of algorithms have been developed, only a few algorithms have been presented for the more general and challenging WGFP: For the *cardinality version* of WGFP, i.e., the WGFP where each edge is assigned weight 1, Dudycz and Paluch introduced a polynomial-time algorithm for this problem when all degree constraints have gaps of length at most 1. This result also leads to a pseudo-polynomial-time algorithm for the WGFP with non-negative integral edge weights [11].

The algorithm in [11] is based on a structural result showing that if a factor is not optimal, then a factor of larger weight can be found by a local search, which can be done in polynomial-time. However, it is not clear how much larger the weight of the new factor is. In order to get an optimal factor, the algorithm needs to repeat local searches iteratively until no better factors can be found, and the number of local searches is bounded by the total edge weight, which makes the algorithm pseudo-polynomial-time. Later, in an updated version [12], by carefully assigning edge weights, the algorithm was improved to be weakly polynomial-time with a running time $O(\log Wmn^6)$, where W is the largest edge weight, m is the number of edges and n is the number of vertices. Later, Kobayashi extended the algorithm to a more general setting called jump system intersections [27]. When degree constraints do not have gap integers of different parities,[3] Szabó gives a strongly polynomial-time algorithm for the WGFP with rational weights [39]. The algorithm is based on a reduction to the *weighted H-factor problem*, which can be solved by a primal-dual algorithm [34].[4]

1.2 Our algorithm

This paper focuses on the WGFP with real-valued edge weights defined on *subcubic graphs* (i.e., the maximum degree is at most 3). We establish a strongly polynomial-time algorithm for the WGFP on subcubic graphs, except for the known NP-hard case where a degree constraint with a gap of length at least 2 (specifically, $D = \{0, 3\}$) is allowed. Our algorithm runs in $O(n^6)$ time for a graph with n vertices. Thus, we obtain the following complexity dichotomy result for the WGFP on subcubic graphs.

Theorem 1.2 *A* WGFP *on subcubic graphs is parameterized by a set* $\mathscr{D}$ *of degree constraints of arity at most* 3. *The problem* WGFP($\mathscr{D}$) *is strongly polynomial-time solvable if*

1. *the degree constraint* $\{0, 3\}$ *of arity* 3 *is not in* $\mathscr{D}$*, or*
2. *for every degree constraint* $D \in \mathscr{D}$ *of arity* k, $D \subseteq \{0, k\}$.

Otherwise, the problem is NP-hard.

The tractability of Condition 2 follows directly from the complexity dichotomy for valued constraint satisfaction problems. Meanwhile, the tractability of Condition 1 is established by our algorithm. Notably, our algorithm handles degree constraints $\{0, 1, 3\}$ and $\{0, 2, 3\}$ which are neither interval constraints nor parity interval constraints. By Theorem 1.1, these are the "smallest" degree constraints that are provably not matching realizable. Furthermore, our algorithm extends to a slightly more general setting, allowing two additional types of degree constraints: $\{p, p+1, p+3\}$ and $\{p, p+2, p+3\}$ of arbitrary arities for some integer $p \geq 0$. We will call them *type-1* and *type-2* respectively. While these degree constraints may appear highly specific, they encompass interesting problems from various domains, including the *simplex*

[3] An integer p is a gap integer of a constraint D if $p \notin D$ and there exist some integers p_1 and p_2 such that $p_1 < p < p_2$ and $p_1, p_2 \in D$.

[4] [39] claims the result for real weights while Pap's result only works for rational weights.

cover problem and its equivalent *terminal backup problem*, which is a variant of the well-known *Steiner tree problem*.

Definition 1.3 *(Terminal backup problem, [43])* Given a graph consisting of terminal (required) nodes, Steiner (optional) nodes, and edges with non-negative costs, the goal is to find a subgraph with the minimum total cost such that each terminal node is connected to at least one other terminal node (for the purpose of backup in applications).

It is known that in an optimal solution to the terminal backup problem, each terminal node has degree 1 and each Steiner node has degree 0, 2 or 3 [2, 43]. (Technically, a Steiner node may have degree larger than 3, but in this case it can be decomposed into several disconnected Steiner nodes of degree 0, 2 or 3.) Thus, the terminal backup problem can be formulated as a WGFP with degree constraints $\{1\}$ and $\{0, 2, 3\}$. By an elaborate reduction from the terminal backup problem to the *simplex matching problem*, a weakly polynomial-time algorithm was provided in [2] by solving the more general simplex matching problem. Our result, however, establishes a strongly polynomial-time algorithm for this problem by directly formulating it as a WGFP. More details on the terminal backup problem and related problems are given in Appendix A.2.

Corollary 1.4 *There is a strongly polynomial-time algorithm for the terminal backup problem.*

1.3 A refined structural result

We note that for all type-1 and type-2 constraints, there are no gap integers of different parities. Thus, by Szabó's result [39], the WGFP with type-1 and type-2 constraints and rational edge weights is known to be strongly polynomial-time solvable via a reduction to the weighted H-factor problem and a primal-dual algorithm. However, our result provides a direct algorithm for the WGFP with real-valued edge weights by establishing a structural result for the optimality of weighted graph factors (Theorem 3.2), which constitutes our main technical contribution. Here, we give an informal statement for this result. We use $\Omega = (G, \pi, \omega)$ to denote a WGFP instance defined on the graph $G = (V, E)$ together with a function π mapping each vertex in V to a interval, parity interval, type-1, or type-2 constraint and an edge weight function $\omega : E \to \mathbb{R}$. For a subgraph H of G, we define its weight $\omega(H)$ as $\sum_{e\in E(H)} \omega(e)$.

Theorem 1.5 *Consider a* WGFP *instance* $\Omega = (G, \pi, \omega)$ *on* $G = (V, E)$*, where* $\pi(u) = \{p, p+2, p+3\}$ *for some* $u \in V$*. Suppose that* F *is a factor of* Ω *satisfying* $\deg_F(u) \in \{p, p+2\}$ *such that* F *is optimal among all factors* F' *with* $\deg_{F'}(u) \in \{p, p+2\}$*), i.e.,* $\omega(F) \geq \omega(F')$ *for every such factor* F'*. If* F *is not an optimal factor of* Ω*, then there exists a subgraph* H *of* G *with exactly two vertices of odd degree including* u *such that* $F \Delta H$ *is an optimal factor of* Ω*.*

Remark 1.6 The existence of such an H ensuring that $F \Delta H$ is optimal is a subtle and highly non-trivial property. In fact, this result *fails* to hold if the condition "$\deg_F(u) \in$

$\{p, p+2\}$" is slightly altered to " $\deg_F(u) \in \{p+3\}$" or any other subset of $\{p, p+2, p+3\}$. For more details and a concrete counterexample, see Remark 3.3.

Equipped with this structural result, our algorithm can directly find an *optimal* factor (rather than just an improved one) for an instance of larger size by performing only one local search from an optimal factor of a smaller instance. Thus, our algorithm is recursive, reducing the problem to a smaller sub-problem by fixing the parity of degree constraints on vertices. The crucial aspect of our approach is not merely finding a better factor via a local search (as in [12]) but rather ensuring that the new factor obtained through a single local search is actually optimal under specific conditions. This is the key insight that enables our algorithm to be strongly polynomial-time. Additionally, as a by-product, we provide an alternate proof of the result of [12] for the special case of WGFPs with interval, parity interval, type-1 and type-2 degree constraints by reducing the problem to the WGFP on subcubic graphs and exploiting the equivalence between *2-vertex connectivity* and *2-edge connectivity* in subcubic graphs.

1.4 Organization

In Sect. 2, we introduce fundamental definitions and notation. In Sect. 3, we present our algorithm and the structural result which guarantees both the correctness and strongly polynomial-time complexity of our algorithm. In Sect. 4, we develop the concept of basic augmenting subgraphs, which serves as an analogy to augmenting paths in weighted matchings, and we use it to prove the structural result. The proof relies on a property regarding the existence of specific basic factors in subcubic graphs, which is proved in Sect. 5. Finally, in Appendix A.1, we explore the notion of matching realizability and its connection to Δ-matroids. Additionally, in Appendix A.2, we discuss the application of our algorithm to the terminal backup problem and other related optimization problems.

2 Preliminaries

Let $\mathscr{D}$ be a (possibly infinite) set of degree constraints.

Definition 2.1 The *weighted general factor problem* parameterized by a set $\mathscr{D}$ is the following computational problem, denoted by WGFP($\mathscr{D}$): An instance of this problem is a triple $\Omega = (G, \pi, \omega)$, where $G = (V, E)$ is a graph, $\pi : V \to \mathscr{D}$ assigns to every $v \in V$ a degree constraint $D_v \in \mathscr{D}$ of arity $\deg_G(v)$, and $\omega : E \to \mathbb{R}$ assigns to every $e \in E$ a real-valued weight $\omega(e) \in \mathbb{R}$. The task is to find a subgraph F of G, if one exists, such that the total weight of edges in F is maximized. We say F is a π-factor of G. The *general factor problem* GFP($\mathscr{D}$) is the decision version of WGFP($\mathscr{D}$); i.e., deciding whether a π-factor exists.

Suppose that $\Omega = (G, \pi, \omega)$ is a WGFP instance. If F is a π-factor of G, then we say that F is a (general) factor of Ω, denoted by $F \in \Omega$. In terms of this inclusion relation, Ω can be viewed as a set of subgraphs of G. We extend the edge weight function ω to subgraphs of G. For a subgraph H of G, its weight $\omega(H)$ is $\sum_{e\in E(H)} \omega(e)$ ($\omega(H) = 0$

if H is the empty graph). If H contains an isolated vertex v, then its weight remains unchanged after removing v, i.e., $\omega(H) = \omega(H')$, where H' is obtained from H by removing v. Moreover, $H \in \Omega$ if and only if $H' \in \Omega$. In the following, without other specification, we always assume that a factor does not contain any isolated vertices. The optimal value of Ω, denoted by $\mathrm{Opt}(\Omega)$, is $\max_{F\in\Omega} \omega(F)$. We define $\mathrm{Opt}(\Omega) = -\infty$ if Ω has no factor. A factor F of Ω is *optimal* in Ω if $\omega(F) = \mathrm{Opt}(\Omega)$. For a WGFP instance $\Omega' = (G', \pi', \omega')$, where $G' \subseteq G$[5] and ω' is the restriction of ω on the edges of G', if for every $F \in \Omega'$ we have $F \in \Omega$, then we say that Ω' is a *sub-instance* of Ω, denoted by $\Omega' \subseteq \Omega$, In particular, Ω' is a subset of Ω by viewing them as two sets of subgraphs of G. If $\Omega' \subseteq \Omega$, then $\mathrm{Opt}(\Omega') \leq \mathrm{Opt}(\Omega)$.

For two WGFP instances $\Omega_1 = (G, \pi_1, \omega)$ and $\Omega_2 = (G, \pi_2, \omega)$ defined on the same graph G, we use $\Omega_1 \cup \Omega_2$ to denote the union of factors of these two instances, i.e., $\Omega_1 \cup \Omega_2 = \{F \subseteq G \mid F \in \Omega_1 \text{ or } F \in \Omega_2\}$, and $\Omega_1 \cap \Omega_2$ to denote the intersection, i.e., $\Omega_1 \cap \Omega_2 = \{F \subseteq G \mid F \in \Omega_1 \text{ and } F \in \Omega_2\}$. Note that $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2$ are just sets of subgraphs of G and may not define WGFP instances on G. We use $\mathscr{G}_1$ and $\mathscr{G}_2$ to denote the set of degree constraints that are intervals and parity intervals respectively, and $\mathscr{T}_1$ and $\mathscr{T}_2$ to denote the set of degree constraints that are type-1 and type-2 respectively. Let $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ and $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. In this paper, we study the problem WGFP($\mathscr{G} \cup \mathscr{T}$).

Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two subgraphs of G. The symmetric difference graph $H_1 \Delta H_2$ is the induced subgraph of G induced by the edge set $E_1 \Delta E_2$. Note that there are no isolated vertices in a symmetric difference graph. When $E_1 \cap E_2 = \emptyset$, $H_1 \Delta H_2$ is a union of two disconnected graphs, denoted by $H_1 \cup H_2$. When $E_2 \subseteq E_1$, $H_1 \Delta H_2$ is the graph obtained from H_1 by removing all edges in H_2, denoted by $H_1 \backslash H_2$. A *subcubic* graph is defined to be a graph where every vertex has degree 1, 2 or 3. Unless stated otherwise, we use V_G and E_G to denote the vertex set and the edge set of a graph G, respectively.

Definition 2.2 *(2-vertex-connectivity)* A connected graph G is 2-vertex-connected (or 2-connected) if it has at least 3 vertices and remains connected by removing any single vertex.

Menger's Theorem gives an equivalent definition of 2-connectivity, cf. [8] for a proof.

Theorem 2.3 (Menger's Theorem) *A connected graph G is 2-connected if and only if for any two vertices of G, there exists two vertex disjoint paths connecting them (i.e., there is a cycle containing these two vertices).*

Definition 2.4 *(Bridge and 2-edge-connectivity)* A bridge of a connected graph is an edge whose deletion makes the graph disconnected. A connected graph is 2-edge-connected if it has no bridge.

The following theorem is the edge version of Menger's Theorem.

[5] We use the term "subgraph" and notation $G' \subseteq G$ throughout for the standard meaning of a "normal" subgraph i.e., if $G = (V', E')$ and $G = (V, E)$ then $G' \subseteq G$ means $V' \subseteq V$ and $E' \subseteq E$.

Theorem 2.5 *A connected graph G is 2-edge-connected if and only if for any two vertices of G, there exists at least two edge disjoint paths connecting them.*

If two paths connecting a pair of vertices are vertex-disjoint, then they are also edge-disjoint. Thus, 2-vertex-connectivity implies 2-edge-connectivity. For subcubic graphs, one can check that two edge-disjoint paths are also vertex-disjoint. Thus, for subcubic graphs, 2-vertex-connectivity is equivalent to 2-edge-connectivity. In particular, we have the following result.

Lemma 2.6 *If a connected subcubic graph is not 2-connected, then it contains a bridge.*

The following fact regarding 2-connected graphs will also be used.

Lemma 2.7 *Let $G = (V_G, E_G)$ be a 2-connected graph, $H = (V_H, E_H) \subseteq G$, and $u \in V_H$. If $\deg_H(u) = 2 < \deg_G(u) = 3$, then there exists a simple path $p_{uw} = (V_{p_{uw}}, E_{p_{uw}}) \subseteq G$ with distinct endpoints u and w for some $w \in V_H$ such that $E_{p_{uw}} \cap E_H = \emptyset$.*

Proof Since $\deg_H(u) = 2 < \deg_G(u) = 3$, there is an edge $e_{vu} = (v, u) \in E_G$ incident to u such that $e_{vu} \notin E_H$. If $v \in V_H$, then the edge e_{vu} is the desired path. Thus, we may assume that $v \notin V_H$. Since G is 2-connected, there is a path p_{vu} with endpoints v and u such that $e_{vu} \notin E_{p_{vu}}$. Since $u \in V_H$, $V_{p_{vu}} \cap V_H \neq \emptyset$. Let w be the first vertex in the path p_{vu} (within the order of traversing the path from v to u) belonging to V_H. Then, $w \neq u$ since $e_{vu} \notin E_{p_{vu}}$ and $\deg_G(u) = 3$. Also, $w \neq v$ since $v \notin V_H$. Let $p_{vw} \subsetneq p_{vu}$ be the segment with endpoints v and w. Then, $E_{p_{vw}} \cap E_H = \emptyset$. Let p_{uw} be the path consisting of e_{vu} and p_{vw}. It has endpoints $u, w \in V_H$, and $E_{p_{uw}} \cap E_H = \emptyset$. □

3 Algorithm

We present a recursive algorithm for WGFP($\mathscr{G} \cup \mathscr{T}$), utilizing WGFP($\mathscr{G}$) and the decision problem GFP($\mathscr{G} \cup \mathscr{T}$) as oracles. Given an instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$), we define the following sub-instances of $\Omega = (G, \pi, \omega)$ that will be used in the recursion. Recall that V_G denotes the vertex set of the underlying graph G. Let T_Ω denote the set $\{v \in V_G \mid \pi(v) \in \mathscr{T}\}$. (We may omit the subscript Ω in T_Ω when the context is clear.)

For every vertex $v \in T_\Omega$, we split the instance Ω into two sub-instances by splitting the degree constraint $\pi(v)$ into two parity intervals. More precisely, we define

$$D_v^0 = \{p_v + 1, p_v + 3\} \text{ and } D_v^1 = \{p_v\} \quad \text{if} \quad \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathscr{T}_1;$$
$$D_v^0 = \{p_v, p_v + 2\} \text{ and } D_v^1 = \{p_v + 3\} \quad \text{if} \quad \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathscr{T}_2.$$

Note that $D_v^0, D_v^1 \in \mathscr{G}_2$ (the set of parity intervals).[6] For $i \in \{0, 1\}$ and $v \in T_\Omega$, we define $\Omega_v^i = (G, \pi_v^i, \omega)$ to be the sub-instance of Ω where $\pi_v^i(x) = \pi(x)$ for every

[6] A degree constraint consisting of a single integer is also a parity interval.

$x \in V_G \backslash \{v\}$ and $\pi_v^i(v) = D_v^i$. Then, for every $v \in T_\Omega$, we have $\Omega_v^0 \cap \Omega_v^1 = \emptyset$ and $\Omega_v^0 \cup \Omega_v^1 = \Omega$. Moreover, we have $T_{\Omega_v^0} = T_{\Omega_v^1} = T_\Omega \backslash \{v\}$.

Let F be a factor of Ω. Similarly to above, one can partition Ω into $2^{|T_\Omega|}$ many sub-instances according to F such that each one is an instance of WGFP($\mathscr{G}$): For each $v \in T_\Omega$, we choose one of the two splits of $\pi(v)$ as above.[7] In detail, for every vertex $v \in T_\Omega$, we define $D_v^F = D_v^i$ where $\deg_F(v) \in D_v^i$ as follows:

$$
\begin{aligned}
&D_v^F = \{p_v\} && \text{if} \;\; \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathscr{T}_1 \;\text{ and }\; \deg_F(v) = p_v,\\
&D_v^F = \{p_v + 1, p_v + 3\} && \text{if} \;\; \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathscr{T}_1 \;\text{ and }\; \deg_F(v) \neq p_v;\\
&D_v^F = \{p_v + 3\} && \text{if} \;\; \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathscr{T}_2 \;\text{ and }\; \deg_F(v) = p_v + 3,\\
&D_v^F = \{p_v, p_v + 2\} && \text{if} \;\; \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathscr{T}_2 \;\text{ and }\; \deg_F(v) \neq p_v + 3.
\end{aligned}
$$

By definition, $\deg_F(v) \in D_v^F \subseteq \pi(v)$ and $D_v^F \in \mathscr{G}_2$. In fact, D_v^F is the maximal set such that $\deg_F(v) \in D_v^F \subseteq \pi(v)$ and $D_v^F \in \mathscr{G}_2$. One can also check that for every $v \in T$, $\pi(v) \backslash D_v^F \in \mathscr{G}_2$, and moreover for every $p \in D_v^F$ and $q \in \pi(v) \backslash D_v^F$, $p \not\equiv q \mod 2$.

For every $W \subseteq T_\Omega$, we define $\Omega_W^F = (G, \pi_W^F, \omega)$ to be the sub-instance of Ω where

$$
\begin{aligned}
\pi_W^F(v) &= \pi(v) \backslash D_v^F && \text{for} \;\; v \in W,\\
\pi_W^F(v) &= D_v^F && \text{for} \;\; v \in T_\Omega \backslash W,\\
\pi_W^F(v) &= \pi(v) && \text{for} \;\; v \in V \backslash T_\Omega.
\end{aligned}
\tag{1}
$$

By definition, for every W, Ω_W^F is an instance of WGFP($\mathscr{G}$). Moreover, we have $\cup_{W \subseteq T_\Omega} \Omega_W^F = \Omega$ and $\Omega_{W_1}^F \cap \Omega_{W_2}^F = \emptyset$ for every $W_1 \neq W_2$. Thus, $\{\Omega_W^F\}_{W \subseteq T_\Omega}$ is a partition of Ω (viewed as a set of subgraphs of G). When $W = \emptyset$, we write Ω_W^F as Ω^F, and when $W = \{s\}$ or $W = \{s, t\}$, we write Ω_W^F as Ω_s^F or $\Omega_{s,t}^F$ respectively for simplicity.

Our algorithm is given in Algorithm 1.

The key to making our algorithm run in strongly polynomial-time is the following structural result (Theorem 3.2). It states that given a factor F of Ω that is optimal in Ω_u^0 for some $u \in T_\Omega$, if F is not optimal in Ω, then an optimal factor of Ω can be directly obtained by a local search operation which involves searching at most n sub-instances of Ω that belong to WGFP($\mathscr{G}$). Note that the number of searches is independent of the edge weights. Thus, finding an optimal factor in Ω can be reduced to finding an optimal factor in Ω_u^0, where one fewer vertex u has constraints in $\mathscr{T}$. By recursively reducing an instance to one with fewer vertices with constraints in $\mathscr{T}$, we eventually obtain an instance of WGFP($\mathscr{G}$) which can be solved in polynomial-time. The crucial aspect of our approach is not merely finding a better factor through a local search (as in [12]) but rather ensuring that, under specific conditions, the factor obtained through a single local search is actually optimal. This distinction is essential and is precisely what enables our algorithm to run in strongly polynomial-time.

[7] We note that the algorithm will not consider all exponentially many sub-instances.

Algorithm 1: Finding an optimal factor for an instance of WGFP($\mathscr{G} \cup \mathscr{T}$)

Function `Decision`:
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$).
  Output: A factor of Ω, or "No" if Ω has no factor.

Function `Optimization`:
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G}$).
  Output: An optimal factor of Ω, or "No" if Ω has no factor.

Function `Main`:
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$).
  Output: An optimal factor $F \in \Omega$, or "No" if Ω has no factor.
  $T_\Omega \leftarrow \{v \in V \mid \pi(v) \in \mathscr{T}\}$;
  if T_Ω is the empty set **then**
    return `Optimization` (Ω);
  else
    Arbitrarily pick $u \in T_\Omega$;
    if `Decision` (Ω_u^0) returns "No" **then**
      return `Main` (Ω_u^1);
    else
      $F^{\text{opt}} \leftarrow$ `Main` (Ω_u^0);
      foreach $v \in T_\Omega$ **do**
        // Elements of T_Ω can be traversed in an arbitrary order.
        $W \leftarrow \{u\} \cup \{v\}$;
        if `Optimization`$(\Omega_W^{F^{\text{opt}}}) \neq$ "No" **then** $F' \leftarrow$ `Optimization`$(\Omega_W^{F^{\text{opt}}})$;
        if $\omega(F') > \omega(F^{\text{opt}})$ **then** $F^{\text{opt}} \leftarrow F'$;
      end
      return F^{opt};
    end
  end

To provide a clear comparison between our result and [12], we restate the main result of [12] for the setting of WGFP($\mathscr{G} \cup \mathscr{T}$).

Theorem 3.1 ([12]) *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G} \cup \mathscr{T}$)*, and F is a factor of Ω. If F is not optimal in Ω, then there is a* better *factor F' of Ω (i.e., $\omega(F') > \omega(F)$) which belongs to Ω_W^F for some $W \subseteq T_\Omega$ where $|W| \leq 2$.*

Theorem 3.2 *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G} \cup \mathscr{T}$)*, and F is a factor of Ω. If F is not optimal in Ω but F is optimal in Ω_u^0 for some $u \in T_\Omega$, then there is an* optimal *factor of Ω which belongs to Ω_W^F for some W where $u \in W \subseteq T_\Omega$ and $|W| \leq 2$.*

Remark 3.3 Theorem 3.1 is a special case of the main result (Theorem 2) of [12] where the authors consider the WGFP for all constraints with gaps of length at most 1. Theorem 3.2 is more refined than Theorem 3.1 and is the key result that ensures our algorithm runs in strongly polynomial-time. In this paper, to complete the proof of Theorem 3.2, we provide an alternate proof of Theorem 3.1 for the special case WGFP($\mathscr{G} \cup \mathscr{T}$) based on certain properties of cubic graphs. However, our main tech-

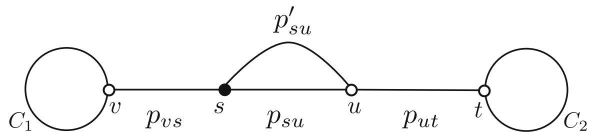


Fig. 1 An example showing Theorem 3.2 fails when F is optimal in Ω_u^1 instead of Ω_u^0

nical contribution is the proof of Theorem 3.2, which is *not* simply implied by [12]. To clarify this, we give a brief proof outline of Theorem 3.2.

Proof outline of Theorem 3.2: In order to prove Theorem 3.2, it suffices to prove the following statement. Given a factor F' of Ω, if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2, then F' is optimal in Ω.

We prove this by contradiction. Suppose that F' is not optimal in Ω, and that F^* is an optimal factor of Ω. Then, $\omega(F^*) > \omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. Also, $F^* \notin \Omega_u^0$ since $\omega(F^*) > \omega(F) = \mathrm{Opt}(\Omega_u^0)$. Thus, we have $\deg_{F^*}(u) \not\equiv \deg_F(u) \mod 2$. By [12], there exists a *canonical path*[8] $M \subseteq F \Delta F^*$ with positive weight. Then, $F \Delta M$ is a factor of Ω with larger weight than F and $F \Delta M \in \Omega_W^F$ for some $W \subseteq T_\Omega$ where $|W| \leq 2$. However, this alone does not lead to a contradiction. To get a contradiction, we need to further show that the positive weighted canonical path M (a basic augmenting subgraph) satisfies $\deg_M(u) \equiv 0 \mod 2$. Then, $\deg_{F \Delta M}(u) \equiv \deg_F(u) \mod 2$. Thus, $F \Delta M$ is a factor with larger weight than F and $F \Delta M \in \Omega_u^0$, which contradicts the optimality of F in Ω_u^0.

The existence of a basic augmenting subgraph M satisfying $\deg_M(u) \equiv 0 \mod 2$ is formally stated in the second property of Lemma 4.4. Establishing this property is the main technical contribution of the paper, and a significant portion of the paper is devoted to its proof. Importantly, this property is highly non-trivial. In fact, it no longer holds if we make the subtle change of replacing the condition "F is optimal in $\Omega_u^{\mathbf{0}}$" with "F is optimal in $\Omega_u^{\mathbf{1}}$". In this case, Theorem 3.2 fails. We give the following example (see Fig. 1) to demonstrate this. $\square$

In this instance, we have $\pi(u) = \pi(v) = \pi(t) = \{0, 1, 3\}$ (denoted by hollow nodes) and $\pi(s) = \{0, 2, 3\}$ (denoted by the solid node), and $\omega(C_1) = \omega(p_{vs}) = \omega(p_{su}) = \omega(p'_{su}) = \omega(p_{ut}) = \omega(C_2) = 1$. Inside the cycles C_1 and C_2, and the paths p_{vs}, p_{su}, p_{ut}, and p'_{su}, there are additional vertices of degree 2 with the degree constraint $\{0, 2\}$ to ensure that the graph G is simple. These vertices of degree 2 are omitted in Fig. 1 for clarity. In this case, we have $T_\Omega = \{u, v, s, t\}$. Consider the sub-instance $\Omega_u^1 = (G, \pi_u^1, \omega)$. We have $\pi_u^1(u) = D_u^1 = \{0\}$ since $\pi(u) = \{0, 1, 3\}$. Then, the only factor F of Ω_u^1 is the empty graph (assuming that factors do not contain isolated vertices), and F is not optimal in Ω. Furthermore, the only optimal factor of Ω is the graph G, and we have $G \in \Omega_{T_\Omega}^F$ where $|T_\Omega| = 4$. Clearly, $\deg_G(u) \not\equiv \deg_F(u) \mod 2$. One can verify that for any factor F' of Ω with larger weight than F, it holds that $\deg_{F'}(u) \not\equiv \deg_F(u) \mod 2$. In other words, there exists no basic augmenting subgraph M such that $\deg_M(u) \equiv 0 \mod 2$. Moreover, one can also verify that, in this case, Theorem 3.2 does not hold. This highlights the crucial role of the existence of a basic augmenting subgraph satisfying $\deg_M(u) \equiv 0 \mod 2$ (implied by $F \in \Omega_u^{\mathbf{0}}$) in ensuring the correctness of Theorem 3.2.

[8] See Definition 3 of [12]. In this paper, these are referred to as basic augmenting subgraphs (Definition 4.3).

Remark 3.4 The above example also provides evidence that Theorem 3.2 is not simply implied by [12] since the proofs in [12] do *not* rely on condition that $F \in \Omega_u^{\mathbf{0}}$. However, this condition is crucial for the correctness of Theorem 3.2.

Using Theorem 3.2, we now prove that Algorithm 1 is correct.

Lemma 3.5 *Given an instance $\Omega = (G, \pi, \omega)$ of* WGFP$(\mathscr{G}, \mathscr{T})$*, Algorithm 1 returns either an optimal factor of Ω, or "No" if Ω has no factor.*

Proof Recall that for an instance $\Omega = (G, \pi, \omega)$, we define $T_\Omega = \{v \in V_G \mid \pi(v) \in \mathscr{T}\}$ where V_G is the vertex set of G. We prove the correctness by induction on the $|T_\Omega|$. If $|T_\Omega| = 0$, Ω is an instance of WGFP$(\mathscr{G})$. Algorithm 1 simply returns `Optimization` (Ω). By the definition of the function `Optimization`, the output is correct.

Suppose that Algorithm 1 returns correct results for all instances Ω' of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_{\Omega'}| = k$. We consider an instance Ω of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_\Omega| = k+1$. Algorithm 1 first calls the function `Decision` (Ω_u^0) for some arbitrary $u \in T$.

We first analyze the case where `Decision` (Ω_u^0) returns "No". By the definition, Ω_u^0 has no factor. Moreover, since $\Omega = \Omega_u^0 \cup \Omega_u^1$, we have $F \in \Omega$ if and only if $F \in \Omega_u^1$. Then, a factor $F \in \Omega_u^1$ is optimal in Ω if and only if it is optimal in Ω_u^1. Note that Ω_u^1 is an instance of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_{\Omega_u^1}| = k$. By the induction hypothesis, Algorithm 1 returns a correct result `Main` (Ω_u^1) for the instance Ω_u^1, which is also a correct result for the instance Ω.

Now, we consider the case that `Decision` (Ω_u^0) returns a factor of Ω_u^0. Since Ω_u^0 is an instance of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_{\Omega_u^0}| = k$, by the induction hypothesis, `Main` (Ω_u^0) returns an optimal factor F of Ω_u^0. After the loop (lines 13 to 17) in Algorithm 1, we get a factor F^{opt} of Ω such that $\omega(F^{\text{opt}}) \geq \text{Opt}(\Omega_W^F)$ for every $u \in W \subseteq T_\Omega$ where $|W| = 1$ (when $u = v$) or $|W| = 2$ (when $u \neq v$) and $\omega(F^{\text{opt}}) \geq \omega(F)$. By Theorem 3.2, F^{opt} is an optimal factor of Ω. Thus, Algorithm 1 returns a correct result. □

Now, we consider the time complexity of Algorithm 1. The size of an instance is defined to be the number of vertices of the underlying graph of the instance.

Lemma 3.6 *Run Algorithm 1 on an instance $\Omega = (G, \pi, \omega)$ of size n. Then,*

- *the algorithm will stop the recursion after at most n recursive steps;*
- *the algorithm will call* `Decision` *at most n many times, call* `Optimization` *at most $\frac{n(n+1)}{2} + 1$ many times, and perform at most $\frac{n(n+1)}{2}$ many comparisons;*
- *the algorithm runs in time $O(n^6)$.*

Proof Let $\Omega^k = \{G, \pi^k, \omega\}$ be the instance after k many recursive steps. Here $\Omega^0 = \Omega$. Recall that $T_{\Omega^k} = \{v \in V \mid \pi^k(v) \in \mathscr{T}\}$. For an instance Ω^k with $|T_{\Omega^k}| > 0$, the recursive step will then go to the instance $(\Omega^k)_u^0$ or $(\Omega^k)_u^1$ for some $u \in T_{\Omega^k}$. Thus, $\Omega^{k+1} = (\Omega^k)_u^0$ or $(\Omega^k)_u^1$. In both cases, $T_{\Omega^{k+1}} = T_{\Omega^k} \backslash \{u\}$ and hence $|T_{\Omega^{k+1}}| = |T_{\Omega^k}| - 1$. By design, the algorithm will stop the recursion and return `Optimization`

(Ω^m) when it reaches an instance Ω^m with $|T_{\Omega^m}| = 0$. Thus, #recursive steps $= m = |T_\Omega| - 0 \le |V| = n$.

To prove the second item, we consider the number of operations inside the recursive step for the instance $\Omega^k = \{G, \pi^k, \omega\}$. Note that $k \le n$ and $|T_{\Omega^k}| = |T_\Omega| - k \le n - k$. If $|T_{\Omega^k}| = 0$, then the algorithm will simply call `Optimization` once. If $|T_{\Omega^k}| > 0$, then inside the recursive step, the algorithm will call `Decision` once, and call `Optimization` once or $|T_{\Omega^k}|$ many times depending on the answer of `Decision`. Moreover, in the later case, the algorithm will also perform $|T_{\Omega^k}|$ many comparisons. Thus,

$$\#\text{calls of } \texttt{Decision} = \sum_{|T_{\Omega^k}|>0} 1 = \sum_{i=1}^{|T_\Omega|} 1 = |T_\Omega| \le n.$$

$$\#\text{calls of } \texttt{Optimization} \le 1 + \sum_{|T_{\Omega^k}|>0} |T_{\Omega^k}| = 1 + \sum_{i=1}^{|T_\Omega|} i \le \frac{n(n+1)}{2} + 1.$$

$$\#\text{comparisons} \le \sum_{|T_{\Omega^k}|>0} |T_{\Omega^k}| \le \frac{n(n+1)}{2}$$

Let $t_{\texttt{Main}}(n)$ denote the running time of Algorithm 1 on an instance of size n, and $t_{\texttt{Dec}}(n)$ and $t_{\texttt{Opt}}(n)$ denote the running time of algorithms for the functions `Decision` and `Optimization`, respectively. Then, $t_{\texttt{Dec}}(n) = O(n^4)$ by the algorithm in [7] and $t_{\texttt{Opt}}(n) = O(n^4)$ by the algorithm in [11]. Thus, $t_{\texttt{Main}}(n) \le nt_{\texttt{Dec}}(n) + \frac{n(n+1)+2}{2} t_{\texttt{Opt}}(n) + \frac{n(n+1)}{2} = O(n^6)$. $\square$

4 Proof of Theorem 3.2

In this section, we give a proof of Theorem 3.2. The general strategy is that starting with a non-optimal factor F of an instance $\Omega = (G, \omega, \pi)$, we want to find a subgraph H of G such that by taking the symmetric difference $F \Delta H$, we get another factor of Ω with larger weight. The existence of such subgraphs is trivial (Lemma 4.2). However, the challenge is how to find one efficiently. As an analogy of augmenting paths in the weighted matching problem, we introduce basic augmenting subgraphs (Definition 4.3) for the weighted graph factor problem, which can be found efficiently. We will show that given a non-optimal factor F, a basic augmenting subgraph always exists (Lemma 4.4, property 1). Then, we can efficiently improve the factor F to another factor with larger weight. As shown in [11], this already gave a weakly-polynomial-time algorithm. However, the existence of basic augmenting subgraphs is not enough to get a strongly polynomial-time algorithm, which requires the number of improvement steps being independent of edge weights. Thus, in order to prove Theorem 3.2, which leads to a strongly polynomial-time algorithm, we further establish that there exists a basic augmenting subgraph that satisfies certain stronger properties under suitable assumptions (Lemma 4.4, property 2). This result will imply Theorem 3.2.

Definition 4.1 *(F-augmenting subgraphs)* Suppose that F is a factor of an instance $\Omega = (G, \pi, \omega)$. A subgraph H of G is F-augmenting if $F \Delta H \in \Omega$ and $\omega(F \Delta H) - \omega(F) > 0$.

Lemma 4.2 *Suppose that F is a factor of an instance Ω. If F is not optimal in Ω, then there exists an F-augmenting subgraph.*

Proof Since F is not optimal, there is some $F' \in \Omega$ such that $\omega(F') > \omega(F)$. Let $H = F \Delta F'$. We have $F \Delta H = F' \in \Omega$ and $\omega(H) = \omega(F') - \omega(F) > 0$. Thus, H is F-augmenting. □

Recall that for an instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$), T_Ω is the set $\{v \in V_G \mid \pi(v) \in \mathscr{T}\}$. For two factors $F, F^* \in \Omega$, we define $T_\Omega^{F \Delta F^*} = \{v \in T_\Omega \mid \deg_{F \Delta F^*}(v) \equiv 1 \mod 2\} = \{v \in T_\Omega \mid \deg_F(v) \not\equiv \deg_{F^*}(v) \mod 2\}$.

Definition 4.3 *(Basic augmenting subgraphs)* Suppose that $\Omega = (G, \pi, \omega)$ is an instance of WGFP($\mathscr{G}, \mathscr{T}$), and F and F^* are factors of Ω with $\omega(F) < \omega(F^*)$. An F-augmenting subgraph $H = (V_H, E_H)$ is (F, F^*)-*basic* if $H \subseteq F \Delta F^*$, $|V_H^{\text{odd}}| \leq 2$, and $V_H^{\text{odd}} \cap T_\Omega \subseteq T_\Omega^{F \Delta F^*}$ where $V_H^{\text{odd}} = \{v \in V_H \mid \deg_H(v) \equiv 1 \mod 2\}$.

Lemma 4.4 *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G} \cup \mathscr{T}$)*, and F and F^* are two factors of Ω.*

1. *If $\omega(F^*) > \omega(F)$, then there exists an (F, F^*)-basic subgraph.*
2. *If $\omega(F^*) > \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F \Delta F^*}$ with $|W| \leq 2$, and $T_\Omega^{F \Delta F^*}$ contains a vertex u such that $F \in \Omega_u^0$ (i.e., $\deg_F(u) \in D_u^0$), then there exists an (F, F^*)-basic subgraph H where $\deg_H(u) \equiv 0 \mod 2$.*

Remark 4.5 The first property of Lemma 4.4 implies Theorem 3.1, a special case of the main result (Theorem 2) of [11]. The second property of Lemma 4.4 is more refined than the first property and it implies our main result Theorem 3.2.

Using the second property of Lemma 4.4, we can prove Theorem 3.2.

Theorem (Theorem 3.2 restated) *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G} \cup \mathscr{T}$)*, F is a factor of Ω and F is optimal in Ω_u^0 for some $u \in T_\Omega$. Then a factor F' is optimal in Ω if and only if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2.*

Proof If F' is optimal in Ω, then clearly $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2. Thus, to prove the theorem, it suffices to prove the other direction. Since $\omega(F') \geq \omega(F)$ and F is optimal in Ω_u^0, we have $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $u \notin W$ and $|W| \leq 2$. Also, since $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2, we have $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$.

For a contradiction, suppose that F' is not optimal in Ω. Let F^* be an optimal factor of Ω. Then, $\omega(F^*) > \omega(F')$. Thus, $\omega(F^*) > \omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. Also, $\omega(F^*) \notin \Omega_u^0$ since $\omega(F^*) > \omega(F)$ and F is optimal in Ω_u^0. Thus, $\deg_{F^*}(u) \not\equiv \deg_F(u) \mod 2$. Then, $T_\Omega^{F\Delta F^*}$ contains the vertex u such that $F \in \Omega_u^0$. By Lemma 4.4, there exists an (F, F^*)-basic subgraph H where $\deg_H(u) \equiv 0 \mod 2$. Let $F'' = F\Delta H$. Then $F'' \in \Omega$ and $\omega(F'') > \omega(F)$. Also, $F'' \in \Omega_u^0$ since $\deg_{F''}(u) \equiv \deg_F(u) \mod 2$. This is a contradiction with F being optimal in Ω_u^0. ◻

Now it suffices to prove Lemma 4.4. By a type of normalization maneuver, we can transfer any instance of WGFP($\mathscr{G}$, $\mathscr{T}$) to an instance of WGFP($\mathscr{G}$, $\mathscr{T}$) defined on subcubic graphs, called a key instance (Definition 4.6). Recall that a subcubic graph is a graph where every vertex has degree 1, 2 or 3. For key instances, there are five possible forms of basic augmenting subgraphs, called basic factors (Definition 4.7). Then, the crux of the proof of Lemma 4.4 is to establish the existence of certain basic factors of key instances (Theorem 4.8).

Definition 4.6 *(Key instance)* A *key instance* $\Omega = (G, \pi, \omega)$ is an instance of WGFP($\mathscr{G}$, $\mathscr{T}$) where G is a subcubic graph, and for every $v \in V_G$, $\pi(v) = \{0, 1\}$ if $\deg_G(v) = 1$, $\pi(v) = \{0, 2\}$ if $\deg_G(v) = 2$, and $\pi(v) = \{0, 1, 3\}$ (i.e., type-1) or $\{0, 2, 3\}$ (i.e., type-2) if $\deg_G(v) = 3$. We say a vertex $v \in V_G$ of degree 3 is of type-1 or type-2 if $\pi(v)$ is type-1 or type-2 respectively. We say a vertex $v \in V_G$ of any degree is 1-feasible or 2-feasible if $1 \in \pi(v)$ or $2 \in \pi(v)$ respectively.

Definition 4.7 *(Basic factor)* Let Ω be a key instance. A factor of Ω is a *basic factor* if it is in one of the following five forms.

1. A path, i.e., a tree with two vertices of degree 1 (called endpoints) and all other vertices, if there exists any, of degree 2.
2. A cycle, i.e., a graph consisting of two vertex disjoint paths with the same two endpoints.
3. A tadpole graph, i.e., a graph consisting of a cycle and a path such that they intersect at one endpoint of the path.
4. A dumbbell graph, i.e., a graph consisting of two vertex disjoint cycles and a path such that the path intersects with each cycle at one of its endpoints.
5. A theta graph (i.e., a graph consisting of three internally vertex disjoint paths with the same two endpoints) where one vertex of degree 3 is of type-1, and the other vertex of degree 3 is of type-2.

Theorem 4.8 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance.*

1. *If $\omega(G) > 0$, then there is a basic factor F of Ω such that $\omega(F) > 0$.*
2. *If $\omega(G) > 0$, $\omega(G) > \omega(F)$ for every basic factor F of Ω, and G contains a vertex u with $\deg_G(u) = 1$ or $\deg_G(u) = 3$ and $\pi(u) = \{0, 2, 3\}$, then there is a basic factor F^* of Ω such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. (Recall that $\deg_{F^*}(u) = 0$ if $u \notin V_{F^*}$).*

Remark 4.9 For the second property of Theorem 4.8, the requirement of $\pi(u) = \{0, 2, 3\}$ when $\deg_G(u) = 3$ is crucial. Consider the instance $\Omega = (G, \pi, \omega)$ as shown in Fig. 1. It is easy to that Ω is a key instance. In this case, it can be checked that $\omega(G) = 6 > 0$ and $\omega(G) > \omega(F)$ for every basic factor F of Ω. However, there is no basic factor F^* of Ω such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. Thus, the second property does *not* hold for a vertex u where $\deg_G(u) = 3$ and $\pi(u) = \{0, 1, 3\}$.

We will now describe the normalization maneuver, and use Theorem 4.8 to prove Lemma 4.4.

Proof of Lemma 4.4 Recall that F and F^* are two factors of the instance $\Omega = (G, \pi, \omega)$ (not necessarily a key instance). Consider the subgraph $G_\Delta = F \Delta F^*$ of G. $G_\Delta = (V_{G_\Delta}, E_{G_\Delta})$ is not necessarily a subcubic graph. In order to invoke Theorem 4.8, we modify G_Δ to a subcubic graph G^s, and construct a key instance $\Omega^s = (G^s, \pi^s, \omega^s)$ on it.

For every $v \in V_{G_\Delta}$, we consider the set of edges incident to v in G_Δ, denoted by E_v. Since $G_\Delta = F \Delta F^*$, we have $E_v \subseteq E_{G_\Delta} = E_F \Delta E_{F^*}$, where E_F and E_{F^*} are the edge sets of the factors F and F^* respectively. If there is a pair of edges $e, e^* \in E_v$ such that $e \in E_F$ and $e^* \in E_{F^*}$, then we perform the following *separation* operation for this pair of edges. Suppose that $e = (v, u)$ and $e^* = (v, u^*)$; we add a new vertex v^1 to the graph, and replace the edges e and e^* by (v^1, u) and (v^1, u^*) respectively. We label the vertex v^1 (of degree 2) by $\pi^s(v^1) = \{0, 2\}$. With a slight abuse of notation, we may still use e and e^* to denote these two new edges, and also use E_{G_Δ} to denote the set of all edges of the new graph.

For each E_v, keep doing the separation operations for pairs of edges of which one is in E_F and the other is in E_{F^*} until all the remaining edges in E_v are in E_F or in E_{F^*} We use $E_v^{\mathfrak{r}}$ to denote the set of remaining edges. It is possible that $E_v^{\mathfrak{r}}$ is empty. Let $P_v^1, \ldots, P_v^k$ be the pairs of edges that have been separated, and $v^1, \ldots, v^k$ be the added vertices (k can be zero). Note that all these new vertices are of degree 2, and are labeled by $\{0, 2\}$. Now, we have the partition $E_v = P_v^1 \cup \cdots \cup P_v^k \cup E_v^{\mathfrak{r}}$. Let $r = |E_v^{\mathfrak{r}}|$. Then $r = |\deg_F(v) - \deg_{F^*}(v)|$. Note that r is even if $\pi(v) \in \mathscr{G}_2$, and $r \leq 3$ if $\pi(v) \in \mathscr{T}$. We deal with edges in $E_v^{\mathfrak{r}}$ according to r and $\pi(v)$.

- If $r = 0$, then v is an isolated vertex in the current graph, and we simply remove it. Consider an arbitrary subgraph H of the original G_Δ induced by a union of some pairs of edges in $P_v^1, \ldots, P_v^k$. Then, for the subgraph $F \Delta H$ of G_Δ, we have

$$\deg_{F\Delta H}(v) = \deg_F(v) \in \pi(v).$$

- If $r \neq 0$ and $\pi(v) \in \mathscr{G}_1$, then we replace the vertex v with r many new vertices, and replace the r many edges incident to v by r many edges incident to these new vertices such that each vertex has degree 1. We label every new vertex by $\{0, 1\}$. Suppose that $L = \min\{\deg_F(v), \deg_{F^*}(v)\}$ and $U = \max\{\deg_F(v), \deg_{F^*}(v)\}$. Since $\pi(v) \in \mathscr{G}_1$, $\{L, L+1, \ldots, U\} \subseteq \pi(v)$. Consider an arbitrary subgraph $H \subseteq G_\Delta$ induced by a union of some pairs of edges in $P_v^1, \ldots, P_v^k$ and a subset of $E_v^{\mathfrak{r}}$. Then, for the subgraph $F \Delta H$ of G_Δ, we have

$$\deg_{F\Delta H}(v) \in \{L, L+1, \ldots, U\} \subseteq \pi(v).$$

- If $r \neq 0$ and $\pi(v) \in \mathscr{G}_2 \backslash \mathscr{G}_1$, then we replace the vertex v with $r/2$ many vertices, and replace the r many edges incident to v by r many edges incident to these new vertices such that each vertex has degree 2. (We can partition these r many edges into arbitrary pairs.) We label every new vertex by $\{0, 2\}$. Suppose that $L = \min\{\deg_F(v), \deg_{F^*}(v)\}$ and $U = \max\{\deg_F(v), \deg_{F^*}(v)\}$. Since $\pi(v) \in \mathscr{G}_2$,

$\{L, L+2, \ldots, U\} \subseteq \pi(v)$. Consider an arbitrary subgraph $H \subseteq G_\Delta$ induced by a union of some pairs of edges in $P_v^1, \ldots, P_v^k$ and an even-size subset of E_v^r. Then, for the subgraph $F \Delta H$ of G_Δ, we have

$$\deg_{F\Delta H}(v) \in \{L, L+2, \ldots, U\} \subseteq \pi(v).$$

- If $r \neq 0$ and $\pi(v) \in \mathscr{T}$, then there are three subcases. If $r = 1$, then v has degree 1 in the current graph. We label it by $\pi^s(v) = \{0, 1\}$. If $r = 2$, then v has degree 2 in the current graph. We label it by $\pi^s(v) = \{0, 2\}$. If $r = 3$, then v has degree 3 in the current graph. We label it by $\pi^s(v) = \{0, 1, 3\}$ if $\deg_F(v) \in D_v^1$, and $\pi^s(v) = \{0, 2, 3\}$ if $\deg_F(v) \in D_v^0$. Consider an arbitrary subgraph $H \subseteq G_\Delta$ induced by a union of some pairs of edges in $P_v^1, \ldots, P_v^k$ and a subset I of E_v^r where $|I| \in \pi^s(v)$. Then, for the subgraph $F\Delta H$ of G_Δ, we have

$$\deg_{F\Delta H}(v) \in \pi(v).$$

Now, we get a subcubic graph $G^s = (V_{G^s}, E_{G^s})$ from G_Δ. Each vertex v in G_Δ is replaced by a set of new vertices in G^s, denoted by $S(v)$.

- If $\pi(v) \in \mathscr{G}_1$, then $S(v)$ consists of vertices of degree 2 or 1.
- If $\pi(v) \in \mathscr{G}_2$, then $S(v)$ consists of vertices of degree 2.
- If $\pi(v) \in \mathscr{T}$, then $S(v)$ consists of vertices of degree 2 and possibly a vertex of degree r where $r = |\deg_F(v) - \deg_{F^*}(v)| \leq 3$ (there is no such a vertex if $r = 0$). In particular, if $\deg_F(v) - \deg_{F^*}(v) \equiv 0 \mod 2$, then $S(v)$ consists of vertices of degree 2.

In all cases, we have $\deg_{G_\Delta}(v) = \sum_{x \in S(v)} \deg_{G^s}(x)$. Each edge (u, v) in G_Δ is replaced by an edge $(u^s, v^s) \in G_\Delta$ where $u^s \in S(u)$ and $v^s \in S(v)$. Once we get G^s from G_Δ, it is clear that there is a natural one-to-one correspondence between edges in G^s and edges in G_Δ. Without causing ambiguity, when we say an edge or an edge set in G^s, we may also refer it to the corresponding edge or edge set in G_Δ.

As we constructed G^s, we have already defined the mapping π^s which labels each vertex in G^s with a degree constraint. For $x \in V_{G^s}$, we have $\pi^s(x) = \{0, 1\}$ if $\deg_{G^s}(x) = 1$, $\pi^s(x) = \{0, 2\}$ if $\deg_{G^s}(x) = 2$, and $\pi^s(x) = \{0, 1, 3\}$ or $\{0, 2, 3\}$ if $\deg_{G^s}(x) = 3$. Moreover, as we have discussed above, for a vertex $v \in V_{G_\Delta}$ and a subgraph $H \subseteq G_\Delta$ induced by a set E of edges incident to v in G_Δ, we have $\deg_{F\Delta H}(v) \in \pi(v)$ if $\deg_{H^s}(x) \in \pi^s(x)$ for every $x \in S(v)$ where H^s is the subgraph of G^s induced by the edge set E (viewed as edges in G^s).

Now, we define the function ω^s for edges in G^s as follows. Recall that for every edge in G^s, its corresponding edge in G_Δ is either in the factor F or the factor F^* but not in both since $G_\Delta = F \Delta F^*$. For $e \in E_{G^s}$, we define $\omega^s(e) = \omega(e)$ if $e \in E_{F^*}$ and $\omega^s(e) = -\omega(e)$ if $e \in E_F$. We can extend ω^s to any subgraph of G^s by defining its weight to be the total weight of all its edges. Then, for any subgraph $H^s \subseteq G^s$, $\omega^s(H^s) = \omega(F \Delta H) - \omega(F)$ where H is the subgraph of G_Δ corresponding to H^s. In particular, $\omega^s(G^s) = \omega(F^*) - \omega(F) > 0$. Thus, we get a key instance $\Omega^s = (G^s, \pi^s, \omega^s)$ where $\omega^s(G^s) > 0$.

Suppose that F^s is a basic factor of G^s with $\omega^s(F^s) > 0$. We consider the subgraph H of G_Δ induced by the edge set E_{F^s} (viewed as edges in G_Δ). We show that H is an (F, F^*)-basic subgraph of G. We have $H \subseteq G_\Delta = F\Delta F^*$. As we have discussed above, for every vertex $v \in V_{F\Delta H}$, $\deg_{F\Delta H}(v) \in \pi(v)$. Thus, $F\Delta H \in \Omega$. Also, $\omega^s(F^s) = \omega(F\Delta H) - \omega(F) > 0$. Then, H is an F-augmenting subgraph. For every $v \in V_H$, $\deg_H(v) = \sum_{x\in S(v)} \deg_{F^s}(x)$. Then, $\deg_H(v)$ is odd only if there is a vertex $x \in S(v)$ such that $\deg_{F^s}(x)$ is odd. Thus, the number of odd vertices in H is no more than the number of odd vertices in F^s. Since F^s is a basic factor, it has at most 2 vertices of odd degree. Thus, H has at most 2 vertices of odd degree. Moreover, for a vertex $v \in V_H \cap T_\Omega$, if $\deg_F(v) \equiv \deg_{F^*}(v) \mod 2$, then $S(v)$ consists of vertices of degree 2. Thus, $\deg_{F^s}(x) \in \{0, 2\}$ for every $x \in S(v)$. Then, $\deg_H(v) = \sum_{x\in S(v)} \deg_{F^s}(x)$ is even. Thus, for a vertex $v \in V_H \cap T_\Omega$, $\deg_H(v)$ is odd only if $\deg_F(v) \not\equiv \deg_{F^*}(v) \mod 2$. Then, $V_H^{\text{odd}} \cap T_\Omega \subseteq T_\Omega^{F\Delta F^*}$ where $V_H^{\text{odd}} = \{v \in V_H \mid \deg_H(v) \equiv 1 \mod 2\}$. Thus, H is an (F, F^*)-basic subgraph of G.

By the first part of Theorem 4.8, there exists a basic factor $F^s \in \Omega^s$ with $\omega^s(F^s) > 0$. Thus, there exists an (F, F^*)-basic subgraph $H \subseteq G$ induced by the edge set E_{F^s}. The first part is done.

Now, we prove the second part. Suppose that $\omega(F^*) > \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F\Delta F^*}$ where $|W| \le 2$, and $T_\Omega^{F\Delta F^*}$ contains a vertex u where $\deg_F(u) \in D_u^0$. Consider the instance Ω^s. First, we prove that $\omega^s(G^s) > \omega(F^s)$ for every basic factor F^s of Ω^s. For a contradiction, suppose that there is some $F^s \in \Omega^s$ such that $\omega^s(G^s) \le \omega(F^s)$. Still consider the subgraph H of G_Δ inducted by E_{F^s}. We know that H is an (F, F^*)-basic subgraph of G and $\omega^s(F^s) = \omega(F\Delta H) - \omega(F)$. Let $W = V_H^{\text{odd}} \cap T_\Omega$. Then, $W \subseteq T_\Omega^{F\Delta F^*}$ and $|W| \le 2$. For every $x \in W$, since $\deg_H(x)$ is odd, we have $\deg_{F\Delta H}(x) \not\equiv \deg_F(x) \mod 2$, and then $\deg_{F\Delta H}(x) \in \pi(x)\backslash D_v^F$. For every $x \in T_\Omega\backslash W$, since $\deg_H(x)$ is even, we have $\deg_{F\Delta H}(x) \equiv \deg_F(x) \mod 2$ and then $\deg_{F\Delta H}(x) \in D_v^F$. Consider the sub-instance $\Omega_W^F = (G, \pi_W^F, \omega)$ of Ω (see Equation (1) for the definition of Ω_W^F). Then, $F\Delta H \in \Omega_W^F$. Thus, $\omega(F\Delta H) \le \mathrm{Opt}(\Omega_W^F)$. Since

$$\omega(F^*) - \omega(F) = \omega^s(G^s) \le \omega^s(F^s) = \omega(F\Delta H) - \omega(F),$$

we have $\omega(F^*) \le \omega(F\Delta H)$. Then, $\omega(F^*) \le \mathrm{Opt}(\Omega_W^F)$. A contradiction with the assumption that $\omega(F^*) > \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F\Delta F^*}$ where $|W| \le 2$. Thus, $\omega^s(G^s) > \omega^s(F^s)$ for every basic factor F^s of Ω^s.

Note that $T_\Omega^{F\Delta F^*}$ contains a vertex u where $\deg_F(u) \in D_u^0$. Consider the vertex set $S(u)$ in G^s that corresponds to u. Since $u \in T_\Omega^{F\Delta F^*}$, $\deg_F(u) \not\equiv \deg_{F^*}(u) \mod 2$. Thus, $S(u)$ consists of vertices of degree 2 and a vertex u^s of degree $\deg_{G^s}(u^s) = |\deg_F(u) - \deg_{F^*}(u)|$ which is 1 or 3. If $|\deg_F(u) - \deg_{F^*}(u)| = 3$, then $\pi^s(u^s) = \{0, 2, 3\}$ since $\deg_F(u) \in D_u^0$. Thus, G^s contains a vertex u^s where $\deg_{G^s}(u^s) = 1$ or $\deg_{G^s}(u^s) = 3$ and $\pi^s(u^s) = \{0, 2, 3\}$. Then, by the second part of Theorem 4.8, there is a basic factor $F^s \in \Omega^s$ such that $\omega^s(F^s) > 0$ and $\deg_{F^s}(u^s) \equiv 0 \mod 2$. Again, consider the subgraph H of G_Δ induced by E_{F^s}. We have proved that H is an

(F, F^*)-basic subgraph of G. Also,

$$\deg_H(u) = \sum_{x \in S(u)\backslash\{u^s\}} \deg_{F^s}(x) + \deg_{F^s}(u^s) \equiv 0 \mod 2$$

since $\deg_{F^s}(x) \in \pi^s(x) = \{0, 2\}$ for every $x \in S(u)\backslash\{u^s\}$, and $\deg_{F^s}(u^s) \equiv 0 \mod 2$. Thus, there is an (F, F^*)-basic subgraph H of G such that $\deg_H(u) \equiv 0 \mod 2$. □

5 Proof of Theorem 4.8

We first prove the first property (restated in Lemma 5.6), and then prove the second property (restated in Lemma 5.7) using the first property. In this section, for two points x and y, we use p_{xy}, p'_{xy} or p''_{xy} to denote a path with endpoints x and y. Recall that $V_{p_{xy}}$ and $E_{p_{xy}}$ denotes the vertex set and the edge set of p_{xy} respectively.

5.1 Proof of the first property

Lemma 5.1 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$. If G is not connected, then there is a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G|$.*

Proof Suppose that G_1 is a connected component of G, and $G_2 = G\Delta G_1$ is the rest of the graph. Note that G_1 and G_2 are both factors of G. By the definition of subcubic graphs, there are no isolated vertices in G. Thus, neither G_1 nor G_2 is a single vertex. Then, $|E_{G_1}|, |E_{G_2}| \geq 1$. Since E_G is the disjoint union of E_{G_1} and E_{G_2}, $|E_{G_1}|, |E_{G_2}| < |E_G|$, and $\omega(G) = \omega(G_1) + \omega(G_2)$. Since $\omega(G) > 0$, among $\omega(G_1)$ and $\omega(G_2)$, one is positive. Thus, we are done. □

Lemma 5.2 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$. Then, there is a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G|$ if one of the following conditions holds:*

1. *There is a path $p_{uv} \subseteq G$ with endpoints u and v where u and v are the only two vertices in p_{uv} of type-2 (i.e., $\deg_G(u) = \deg_G(v) = 3$ and $\pi(u) = \pi(v) = \{0, 2, 3\}$) and $\omega(p_{uv}) \leq 0$.*
2. *There is a cycle $C \subseteq G$ such that no vertex in C is of type-2 and $\omega(C) \leq 0$.*

Proof Suppose that the first condition holds. Consider the subgraph $F = G\backslash p_{uv}$ of F. Then, $|E_F| = |E_G| - |E_{p_{uv}}| < |E_G|$, and $\omega(F) = \omega(G) - \omega(p_{uv}) \geq \omega(G) > 0$. Now we only need to show that F is a factor of Ω. The vertex set V_F consists of three parts:

$$V_1 = V_G\backslash V_{p_{uv}}, \quad V_2 = \{x \in V_{p_{uv}}\backslash\{u, v\} \mid \deg_G(x) = 3\}, \quad \text{and} \quad V_3 = \{u, v\}.$$

Since u and v are the only two vertices of type-2 in p_{uv}, for every $x \in V_2$, x is of type-1 (i.e., $\pi(x) = \{0, 1, 3\}$). Then, for every $x \in V_F$, we have $\deg_F(x) = \deg_G(x) \in \pi(x)$

if $x \in V_1$, $\deg_F(x) = 1 \in \pi(x)$ if $x \in V_2$, and $\deg_F(x) = 2 \in \pi(x)$ if $x \in V_3$. Thus, F is a factor of G. We are done.

Suppose that the second condition holds. Consider the subgraph $F = G \backslash C$. Then $|E_F| < |E_G|$ and $\omega(F) > 0$. Similar to the above proof, one can check that F is a factor Ω. We are done. □

Lemma 5.3 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$, G is not a basic factor of Ω and $C \subseteq G$ is a cycle. Let k be the number of type-1 vertices and ℓ be the number of type-2 vertices in C. If $k \neq 1$ and $\ell \neq 1$, then there is a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G|$.*

Proof We prove this lemma in two cases depending on whether $\omega(C) > 0$ or $\omega(C) \leq 0$.

We first consider the case that $\omega(C) > 0$. If $k = 0$, then all vertices in C are 2-feasible (see Definition 4.6). Thus, C is a factor of Ω. Since G is not a basic factor of Ω, we have $G \neq C$. Also, since G has no isolated vertices, $C \subsetneq G$ implies that $|E_C| < |E_G|$. We are done. Thus, we may assume that $k \geq 2$. Suppose that $\{u_1, u_2, \ldots, u_k\}$ are the type-1 vertices in C. We list them in the order of traversing the cycle starting from u_1 in an arbitrary direction. Then, these k many vertices split the cycle into k many paths $p_{u_1u_2}, \ldots, p_{u_ku_{k+1}}$ ($u_{k+1} = u_1$). For each path, all its vertices are 2-feasible except for its two endpoints which are 1-feasible. Thus, each path is a basic factor of G. We have $|E_{p_{u_iu_{i+1}}}| < |E_G|$ for every $i \in [k]$. Since

$$\omega(C) = \sum_{i=1}^{k} \omega(p_{u_iu_{i+1}}) > 0,$$

there is a path $p_{u_iu_{i+1}}$ such that $\omega(p_{u_iu_{i+1}}) > 0$. Thus, we are done.

Then we consider the case that $\omega(C) \leq 0$. If $\ell = 0$, then $C \subseteq G$ is cycle with no type-2 vertices. By Lemma 5.2, we are done. Thus, we may assume that $\ell \geq 2$. Suppose that $\{v_1, v_2, \ldots, v_\ell\}$ are the type-2 vertices in C. We list them in the order of traversing the cycle starting from v_1 in an arbitrary direction. Then, these ℓ many vertices split the cycle into ℓ many paths $p_{v_1v_2}, \ldots, \ldots, p_{v_\ell v_{\ell+1}}$ ($v_{\ell+1} = v_1$). For each path, it has no vertex of type-2 except for its two endpoints which are of type-2. Since

$$\omega(C) = \sum_{i=1}^{\ell} \omega(p_{v_iv_{i+1}}) \leq 0,$$

there is a path $p_{v_iv_{i+1}}$ such that $\omega(p_{v_iv_{i+1}}) \leq 0$. Thus, there is a path $p_{v_iv_{i+1}} \subseteq G$ where v_i and v_{i+1} are the only two vertices of type-2 in $p_{v_iv_{i+1}}$ and $\omega(p_{v_iv_{i+1}}) \leq 0$. Then, by Lemma 5.2, we are done. □

Lemma 5.4 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$, and G is not a basic factor of Ω. If G is 2-connected, then there is a factor $F \in \Omega$ such that $\Omega(F) > 0$ and $|E_F| < |E_G|$.*

Proof Since G is 2-connected, it contains at least three vertices and it contains no vertex of degree 1. Consider the number of type-1 vertices in G. There are three cases.

- G has no type-1 vertex.
  Since G is 2-connected, there is a cycle $C \subseteq G$. Clearly, C has no type-1 vertex. If C has exactly one type-2 vertex, denoted by v, then v is the only vertex in C such that $\deg_G(v) = 3$. Then, there is an edge $e \in E_G$ incident to v such that $e \notin E_C$. It is easy to see that e is a bridge of G, a contradiction with G being 2-connected. Thus, C has no type-2 vertex, or it has at least two type-2 vertices. Then, by Lemma 5.3, we are done.
- G has exactly one type-1 vertex.
  Let u be the type-1 vertex of G. Since G is 2-connected, there is a cycle $C \subseteq G$ containing the vertex u. Since $\deg_C(u) = 2 < \deg_G(u) = 3$, by Lemma 2.7, there is a path $p_{uw} \subseteq G$ with endpoints $u, w \in V_C$ such that $E_{p_{uw}} \cap E_C = \emptyset$.
  Consider the subgraph $H = p_{uw} \cup C$ of G. H is a theta graph where $\deg_H(u) = \deg_H(w) = 3$. All vertices of H are 2-feasible except for u which is 1-feasible. Note that H is a basic factor of Ω. Since G is not a basic factor of Ω, $H \neq G$. Also since G is connected, there exists an edge $e_{ts} = (t, s)$ incident to a vertex $s \in V_H$ such that $e_{ts} \notin E_H$. Clearly, s is a vertex of type-2, $\deg_G(s) = 3$ and $\deg_H(s) = 2$. Then, by Lemma 2.7, there is a path p_{sr} with endpoints $s, r \in V_H$ such that $E_{p_{sr}} \cap E_H = \emptyset$. Clearly, $\deg_G(r) = 3$ and r is a vertex of type-2. Since $s, r \in V_H$ and H is a theta graph which is 2-connected, we can find a path $p'_{sr} \subseteq H$ with endpoints s and r such that the only type-1 vertex u in H is not in p'_{sr}. Consider the cycle $C' = p_{sr} \cup p'_{sr}$. It has no vertex of type-1, and it has at least two vertices s and r of type-2. By Lemma 5.3, we are done.
- G has at least two type-1 vertices.
  Since G is 2-connected and it contains at least two type-1 vertices, we can find a cycle $C \subseteq G$ that contains at least two type-1 vertices. Consider the number of type-2 vertices in C. If the number is not 1, then by Lemma 5.3, we are done. Thus, we may assume that C contains exactly one vertex of type-2, denoted by v. Since G is 2-connected and $\deg_G(v) = 3 > \deg_C(v) = 2$, we can find a path p_{vu} for some $u \in V_C$ such that $E_{p_{vu}} \cap E_C = \emptyset$. We have $\deg_G(u) = 3$. Since v is the only vertex of type-2 in C, u is a vertex of type-1. Vertices v and u split C into two paths p'_{vu} and p''_{vu}. Since C contains at least two type-1 vertices, there exists some $w \in V_C$ where $w \neq u$ such that w is of type-1. Also, $w \neq v$ since v is of type-2. Since $w \in V_C = V_{p'_{vu}} \cup V_{p''_{vu}}$ and $V_{p'_{vu}} \cap V_{p''_{vu}} = \{u, v\}$, without loss of generality, we may assume that $w \in V_{p'_{vu}}$.
  Consider the path p_{vu}. If p_{vu} contains at least two vertices of type-2, then the cycle $C' = p_{vu} \cup p'_{vu}$ contains at least two vertices of type-2 and at least two vertices u and w of type-1. Then, by Lemma 5.3, we are done. Thus, we may assume that v is the only vertex of type-2 in p_{vu}. Consider the theta graph $H = p_{vu} \cup C$. Then v is the only vertex of type-2 in H. Note that $w \in V_H$, $\deg_H(w) = 2 < \deg_G(w) = 3$. Since G is 2-connected, by Lemma 2.7, we can find a path p_{ws} for some $s \in V_H$ such that $E_{p_{ws}} \cap E_H = \emptyset$. Clearly $s \neq v$. Then, s is of type-1 since v is the only vertex of type-2 in H.

Consider the number of type-2 vertices in p_{ws}. Suppose that there is no vertex of type-2 in p_{ws}. Since H is 2-connected and H contains only one vertex v of type 2, we can find a path $p'_{ws} \subseteq H$ such that p'_{ws} does not contain the vertex v of type-2. Then, the cycle $p_{ws} \cup p'_{ws}$ has no vertex of type-2 and at least two vertices w and s of type-1. By Lemma 5.3, we are done. Otherwise, there is at least one vertex of type-2 in p_{ws}. Since H is 2-connected, we can find a path $p''_{ws} \subseteq H$ such that p''_{ws} contains the vertex v of type-2. Then, the cycle $p_{ws} \cup p''_{ws}$ has at least two vertices of type-2 and at least two vertices w and s of type-1. By Lemma 5.3, we are done.

□

Definition 5.5 *(Induced sub-instance)* For a key instance $\Omega = (G, \pi, \omega)$, and a factor $F \in \Omega$, the sub-instance of Ω induced by F, denoted by Ω_F, is a key instance (F, π_F, ω_F) defined on the subgraph F of G where $\pi_F(x) = \pi(x) \cap [\deg_F(x)] \subseteq \pi(x)$ for every $x \in V_F$ and ω_F is the restriction of ω on E_F (we may write ω_F as ω for simplicity).

We are now ready to prove the first property of Theorem 4.8 as restated in the next lemma.

Lemma 5.6 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance. If $\omega(G) > 0$, then there is a basic factor F of Ω such that $\omega(F) > 0$.*

Proof We prove this lemma by induction on the number of edges in G.

If $|E_G| = 1$, then G is a single edge. Thus, G is a basic factor of Ω, and $\omega(G) > 0$. We are done.

We assume that the lemma holds for all key instances where the underlying graph has no more than n many edges. We consider a key instance $\Omega = (G, \pi, \omega)$ where $|E_G| = n + 1$.

If G is a basic factor of Ω, then clearly we are done. Thus, we may assume that G is not a basic factor of Ω. Suppose that we can find a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G| = n + 1$. Then, consider the sub-instance Ω_F of Ω induced by F. Since $|E_F| < n + 1$ and $\omega(F) > 0$, by the induction hypothesis, there is basic factor $F' \in \Omega_F$ such that $\omega(F') > 0$. Since $\Omega_F \subseteq \Omega$, $F' \in \Omega$. Then, we are done. Thus, in order to establish the inductive step, it suffices to prove that there is a factor $F \in \Omega$ such that $|E_F| < |E_G|$ and $\omega(F) > 0$.

By Lemmas 5.1 and 5.4, if G is not connected or G is 2-connected, then we are done. Thus, we may assume that G is a connected graph but not 2-connected. By Lemma 2.6, G contains at least one bridge. Fix such a bridge of G. Let p_{uv} be the path containing the bridge such that for every vertex $x \in V_{p_{uv}} \backslash \{u, v\}$, $\deg_G(x) = 2$ and $\deg_G(u), \deg_G(v) \neq 2$; observe that such a path exists and it is unique. In fact, the whole path can be viewed as a "long bridge" of the graph G. Then, $G \backslash p_{uv}$ is not connected and it has two connected components. Let $G_u \subseteq G \backslash p_{uv}$ be the part that contains u and $G_v \subseteq G \backslash p_{uv}$ be the part that contains v.

If both G_u and G_v are single vertices, then the graph G is a path. If both G_u and G_v are cycles, then G is a dumbbell graph. If one of G_u and G_v is a single vertex and the other one is a cycle, then G is a tadpole graph. In all these cases, G is a basic factor of Ω. A contradiction with our assumption. Thus, among G_u and G_v, at least one is neither a cycle nor a single vertex. Without loss of generality, we may assume that G_u is neither a cycle nor a single vertex.

Since G_u is not a single vertex, $\deg_G(u) \neq 1$. By assumption, $\deg_G(u) \neq 2$. Then $\deg_G(u) = 3$, and hence $\deg_{G_u}(u) = 2$. Let $e_1 = (u, w_1)$ and $e_2 = (u, w_2)$ be the two edges incident to u in G_u. We slightly modify G_u to get a new graph. We replace the vertex u in G_u by two vertices u_1 and u_2, and replace the edges (u, w_1) and (u, w_2) in G_u by two new edges (u_1, w_1) and (u_2, w_2) respectively. We denote the new graph by G'. With a slight abuse of notations, we still use e_1 and e_2 to denote the edges (u_1, w_1) and (u_2, w_2) in G' respectively, and we say $E_{G_u} = E_{G'}$. Then, the edge weight function ω can be adapted to $E_{G'}$. We define the following instance $\Omega' = (G', \pi', \omega')$ where $\pi'(u_1) = \pi'(u_2) = \{0, 1\}$ and $\pi'(x) = \pi(x)$ for every $x \in V_{G'} \backslash \{u_1, u_2\}$, and $\omega'(e_1) = \omega(e_1) + \omega(G \backslash G_u)$, and $\omega'(e) = \omega(e)$ for every $e \in E_{G'} \backslash \{e_1\}$. In other words, we add the total weight of the subgraph $G \backslash G_u$ to the edge e_1. Then, $\omega'(G') = \omega(G) > 0$ and $|E_{G'}| = |E_{G_u}| < |E_G|$. By the induction hypothesis, there is a basic factor $F \in \Omega'$ such that $\omega'(F) > 0$. We will recover a factor of Ω from F such that it has positive weight and fewer edges than G. This will finish the proof of the inductive step.

There are four cases depending on the presence of e_1 and e_2 in F.

- $e_1, e_2 \notin E_F$. Then, $u_1, u_2 \notin V_F$. For every $x \in V_F$, $\deg_F(x) \in \pi'(x) = \pi(x)$. Thus, F is a basic factor of Ω. Clearly, $\omega(F) = \omega'(F) > 0$ and $|E_F| = |E_{F'}| < |E_G|$. We are done.
- $e_1 \in E_F$ and $e_2 \notin E_F$. We can view F as a subgraph of G_u by changing the edge (u_1, w_1) in G' back to the edge (u, w_1) in G_u. Then, the edge $(u, w_2) \notin E_F$. Consider the subgraph $H = F \cup (G \backslash G_u)$ of G. Since $(u, w_2) \notin E_F$, we have $(u, w_2) \notin E_H$. Then, $|E_H| < |E_G|$. Also, we have

  $$\omega(H) = \omega(F) + \omega(G \backslash G_u) = \omega'(F) > 0.$$

  The vertex set V_H consists of three parts $V_1 = V_F \backslash \{u\}$, $V_2 = \{u\}$, and $V_3 = V_{G \backslash G_u} \backslash \{u\}$. For every $x \in V_1$, $\deg_H(x) = \deg_F(x) \in \pi(x)$. For every $x \in V_3$, $\deg_H(x) = \deg_{G \backslash G_u}(x) = \deg_G(x) \in \pi(x)$. Now, we consider the vertex u.

  - If u is 2-feasible, then $\deg_H(u) = 2 \in \pi(x)$. Thus, H is a factor of Ω where $\omega(H) > 0$ and $|E_H| < |E_G|$.
  - If u is 1-feasible, then F and $G \backslash G_u$ both are factors of Ω since $\deg_F(u) = \deg_{G \backslash G_u}(u) = 1 \in \pi(u)$. Since $\omega(H) = \omega(F) + \omega(G \backslash G_u) > 0$, among $\omega(F)$ and $\omega(G \backslash G_u)$, at least one is positive. Also, $|E_F|, |E_{G \backslash G_u}| < |E_H| < |E_G|$. We are done.

- $e_2 \in E_F$ and $e_1 \notin E_F$. Again, we can view F as a subgraph of G_u where $(u, w_2) \in E_F$ and $(u, w_1) \notin E_F$. Then, we have $|E_F| < |E_{G_u}| < |E_G|$, and $\omega(F) = \omega'(F) > 0$.

  - If u is 1-feasible, then F is a factor of G where $|E_F| < |E_G|$ and $\omega(F) > 0$. We are done.
  - If u is 2-feasible, then G_u is a factor of Ω since $\deg_{G_u}(u) = 2$. If $\omega(G_u) > 0$, then we are done. Thus, we may assume that $\omega(G_u) \leq 0$. Then, $\omega(G \backslash G_u) =$

$\omega(G)-\omega(G\backslash G_u) \geq \omega(G) > 0$. Still consider the subgraph $H = F \cup (G\backslash G_u)$. Then, H is a factor of Ω since $\deg_H(u) = 2 \in \pi(u)$. Also, $\omega(H) = \omega(F) + \omega(G\backslash G_u) > 0$ and $|E_H| < |E_G|$. We are done.

- $e_1, e_2 \in E_F$. Then, F (as a subgraph of G') contains two vertices u_1 and u_2 of degree 1. Since F is a basic factor, it is a path. Still we can view F as a subgraph of G_u by changing edges (u_1, w_1) and (u_2, w_2) in G' to edges (u, w_1) and (u, w_2) in G. Then, F is a cycle in G_u. Since G_u is not a cycle and it has no isolated vertices, $|E_F| < |E_{G_u}|$. Consider the subgraph $H = F \cup (G\backslash G_u)$ of G. We have $|E_H| < |E_G|$ and $\omega(H) = \omega(F) + \omega(G\backslash G_u) = \omega'(F) > 0$. Also, one can check that H is a factor of G no matter whether u is 1-feasible or 2-feasible since $\deg_H(u) = 3 \in \pi(u)$. We are done.

□

5.2 Proof of the second property

Now we prove the second property of Theorem 4.8 using the first property (Lemma 5.6).

Lemma 5.7 *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance, and u is a vertex of G where* $\deg_G(u) = 1$ *or* $\deg_G(u) = 3$ *and* $\pi(u) = \{0, 2, 3\}$. *If* $\omega(G) > 0$ *and* $\omega(G) > \omega(F)$ *for every basic factor F of* Ω, *then there is a basic factor* F^* *of* Ω *such that* $\omega(F^*) > 0$ *and* $\deg_{F^*}(u) \equiv 0 \mod 2$. *(Recall that we agree* $\deg_{F^*}(u) = 0$ *if* $u \notin V_{F^*}$.*)*

Proof By Lemma 5.6, there exists at least one basic factor of Ω such that its weight is positive. Among all such basic factors, we pick an F such that $\omega(F)$ is the largest. We have $0 < \omega(F) < \omega(G)$. If $\deg_F(u)$ is even, then we are done. Thus, we may assume that $\deg_F(u)$ is odd. Since F is a basic factor and it contains a vertex u of odd degree, F is not a cycle. By the definition of basic factors, F contains exactly one more vertex v of odd degree. Since F is a factor of Ω, $\deg_F(u) \in \pi(u)$. Recall that $\deg_G(u) = 1$ or 3. If $\deg_G(u) = 1$, then $\pi(u) = \{0, 1\}$, and hence $\deg_F(u) = 1$. If $\deg_G(u) = 3$, then $\pi(u) = \{0, 2, 3\}$, and hence $\deg_F(u) = 3$. Thus, $\deg_F(u)$ always equals $\deg_G(u)$.

Consider the graph $G' = G\backslash F$, i.e., the subgraph of G induced by the edge set $E_G\backslash E_F$. Consider the instance $\Omega' = (G', \pi', \omega')$ where for every $x \in V_{G'}$, $\pi'(x) = \{0, 1\}$ if $\deg_{G'}(x) = 1$, $\pi'(x) = \{0, 2\}$ if $\deg_{G'}(x) = 2$ and $\pi'(x) = \pi(x)$ if $\deg_{G'}(x) = 3$, and ω' is the weight function ω restricted to G'. Note that Ω' is also a key instance, but it is not necessarily a sub-instance of Ω. Since $\omega(G) > \omega(F)$, we have $\omega'(G') = \omega(G') = \omega(G) - \omega(F) > 0$. Without causing ambiguity, we may simply write ω' as ω in the instance Ω'. By Lemma 5.6, there exists a basic factor F' of Ω' such that $\omega(F') > 0$. Since $E_{F'} \subseteq E_G\backslash E_F$, F and F' are edge-disjoint. Let $H = F \cup F'$, which is the subgraph of G induced by the edge set $E_F \cup E_{F'}$. We show that H is a factor of Ω.

Let $V_\cap = V_F \cap V_{F'}$. First we show that for every $x \in V_H\backslash V_\cap$, $\deg_H(x) \in \pi(x)$. If $x \in V_F\backslash V_\cap$, then $\deg_H(x) = \deg_F(x)$. Since $F \in \Omega$, $\deg_F(x) \in \pi(x)$. Then, $\deg_H(x) \in \pi(x)$. If $x \in V_{F'}\backslash V_\cap$, then $\deg_H(x) = \deg_{F'}(x)$. Since $x \notin V_F$ and

$G' = G \backslash F$, $\deg_{G'}(x) = \deg_G(x)$. Then, by the definition of Ω', we have $\pi'(x) = \pi(x)$. Since F' is a factor of Ω', $\deg_{F'}(x) \in \pi'(x)$. Thus, $\deg_H(x) \in \pi(x)$. Now, we consider vertices in $V_\cap$. Since F and F' are edge disjoint, for every $x \in V_\cap$ we have $\deg_H(x) = \deg_F(x) + \deg_{F'}(x) \leq \deg_G(x) \leq 3$. Also, $\deg_F(x), \deg_{F'}(x) \geq 1$ since F and F' are subcubic graphs which have no isolated vertices.

- If $\deg_F(x) = 1$, then $1 \in \pi(x)$. The vertex x is 1-feasible. Thus, $\deg_G(x) \neq 2$. Since $\deg_G(x) > \deg_F(x) = 1$, $\deg_G(x) = 3$. Then, $\deg_{G'}(x) = \deg_G(x) - \deg_F(x) = 2$, $\pi'(x) = \{0, 2\}$ and $\deg_{F'}(x) = 2$.
- If $\deg_F(x) = 2$, then $\deg_G(x) = 3$ since $\deg_G(x) > \deg_F(x)$. Then, $\deg_{G'}(x) = \deg_G(x) - \deg_F(x) = 1$, $\pi'(x) = \{0, 1\}$ and $\deg_{F'}(x) = 1$.

Thus, for every $x \in V_\cap$, $\deg_H(x) = \deg_F(x) + \deg_{F'}(x) = 3 \in \pi(x)$. Thus, H is a factor of Ω.

Consider the sub-instance $\Omega_H = (H, \pi_H, \omega_H)$ of Ω induced by H (we will write ω_H as ω for simplicity). We will show that we can find a basic factor F^* of Ω_H such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. Clearly, F^* is also a factor of Ω.

Consider the set $V_\cap$ of intersection points. If $V_\cap = \emptyset$, then for every $x \in V_{F'}$, $\deg_{F'}(x) = \deg_H(x) \in \pi(x)$. Thus, F' is a basic factor of Ω where $\omega(F') > 0$ and $\deg_{F'}(u) = 0$. That is, F' is the desired F^*. We are done. Thus, we may assume that $V_\cap$ is non-empty. For every $x \in V_\cap$, $\deg_F(x) = 1$ and $\deg_{F'}(x) = 2$, or $\deg_F(x) = 2$ and $\deg_{F'}(x) = 1$. Recall that F is a basic factor containing two vertices u, v of odd degree, and $\deg_F(u) = \deg_G(u)$. Clearly, $u \notin V_\cap$.

We consider the possible forms of F and F'. Recall that F is not a cycle. We show that F' is also not a cycle. For a contradiction, suppose that F' is a cycle. Then, all vertices of F' have degree 2. Thus, the only possible vertex in $V_\cap$ is v. Since $V_\cap$ is non-empty, $V_\cap = \{v\}$. Then, $\deg_F(v) = 1$ and $\deg_{F'}(v) = 2$. If $\deg_F(u) = 1$, then F is a path. The graph H is a tadpole graph where v is the only vertex of degree 3. If $\deg_F(u) = 3$, then F is a tadpole graph. The graph H is a dumbbell graph where v and u are the two vertices of degree 3. In both cases, H is a basic factor of Ω. Since $\omega(F') > 0$, we have $\omega(H) = \omega(F) + \omega(F') > \omega(F)$ which leads to a contradiction with F being a basic factor with the largest weight. Thus, F' is a basic factor which is not a cycle. Then, it contains exactly two vertices s, t of odd degree. Then, $V_\cap \subseteq \{v, s, t\}$.

We consider the graph H depending on the forms of F and F', and the vertices in $V_\cap$. There are 5 main cases.

I. F is a path.
II. F is a tadpole graph and $\deg_F(u) = 3$.
III. F is a tadpole graph and $\deg_F(u) = 1$.
IV. F is a dumbbell graph.
V. F is a theta graph.

Recall that for two points x and y, we use p_{xy}, p'_{xy} or p''_{xy} to denote a path with endpoints x and y. We also use q_{xy^3} or q'_{xy^3} to denote a tadpole graph where x is the vertex of degree 1 and y is the vertex of degree 3. In the following Figs. 2 to 15, we use hollow nodes to denote 1-feasible vertices, solid nodes to denote 2-feasible vertices, semisolid nodes to denote vertices that are possibly 1-feasible or 2-feasible, red-colored lines to denote paths in F, and blue-colored lines to denote paths in F'.

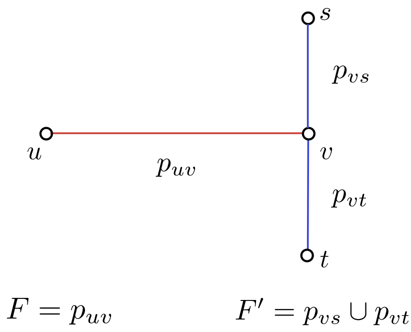


Fig. 2 The graph H in Case I.1.(a)

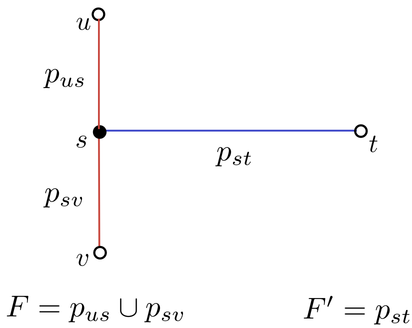


Fig. 3 The graph H in Case I.1.(b)

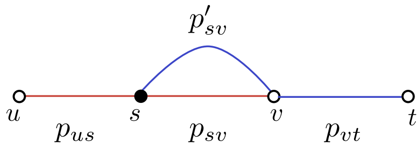
Case I: F is a path. There are 4 subcases depending on the form of F'.

I.1 F and F' are both paths. Then, $V_\cap \subseteq \{v, s, t\}$. There are 5 subcases: $V_\cap = \{v\}$, $V_\cap = \{s\}$ or $\{t\}$, $V_\cap = \{v, s\}$ or $\{v, t\}$, $V_\cap = \{s, t\}$, and $V_\cap = \{v, s, t\}$.

(a) $V_\cap = \{v\}$.
In this case, $\deg_H(u) = \deg_H(s) = \deg_H(t) = 1$, $\deg_H(v) = 3$, and $\pi(v) = \{0, 1, 3\}$ since $\deg_F(v) = 1 \in \pi(v)$. The graph H consists of three edge-disjoint paths p_{uv}, p_{vs} and p_{vt}. Then, $F = p_{uv}$ and $F' = p_{vs} \cup p_{vt}$. (See Fig. 2.)
Since $\omega(F') = \omega(p_{vs}) + \omega(p_{vt}) > 0$, among $\omega(p_{vs})$ and $\omega(p_{vt})$, at least one is positive. Without loss of generality, we may assume that $\omega(p_{vs}) > 0$. Since u does not appear in p_{vs}, we have $\deg_{p_{vs}}(u) = 0$. For every vertex x in p_{vs} where $x \neq v$, $\deg_{p_{vs}}(x) = \deg_H(x) \in \pi(x)$. Also, $\deg_{p_{vs}}(v) = 1 \in \pi(v)$. Thus, the path p_{vs} is a basic factor of Ω where $\omega(p_{vs}) > 0$ and $\deg_{p_{vs}}(u) = 0$.

(b) $V_\cap = \{s\}$ or $\{t\}$.
These two cases are symmetric. We only consider the case that $V_\cap = \{s\}$.
In this case, $\deg_H(s) = 3$, $\deg_H(u) = \deg_H(v) = \deg_H(t) = 1$, and $\pi(s) = \{0, 2, 3\}$ since $\deg_{F'}(s) = 1$ and $\deg_F(s) = 2 \in \pi(s)$. The graph H consists of three edge-disjoint paths p_{us}, p_{sv} and p_{st}. Then, $F' = p_{st}$ and $F = p_{us} \cup p_{sv}$. (See Fig. 3.)
Note that $\omega(p_{st}) = \omega(F') > 0$. Let $p_{ut} = p_{us} \cup p_{st}$ be the path with endpoints u and t. For every vertex x in p_{ut} where $x \neq s$, we have $\deg_{p_{ut}}(x) = \deg_H(x) \in \pi(x)$. Also, $\deg_{p_{ut}}(s) = 2 \in \pi(s)$. Thus, p_{ut} is a basic factor of Ω. Then,

Fig. 4 The graph H in Case I.1.(c)



$\omega(F) \geq \omega(p_{ut})$ since F is a basic factor of Ω with the largest weight $\omega(F)$. Then,

$$\omega(p_{us}) + \omega(p_{sv}) = \omega(F) \geq \omega(p_{ut}) = \omega(p_{us}) + \omega(p_{st}).$$

Thus, $\omega(p_{sv}) \geq \omega(p_{st}) > 0$. Let $p_{vt} = p_{sv} \cup p_{st}$ be the path with endpoints v and t. Then,

$$\omega(p_{vt}) = \omega(p_{sv}) + \omega(p_{st}) > 0.$$

Since u is not in p_{vt}, $\deg_{p_{vt}}(u) = 0$. Similar to the proof of $p_{ut} \in \Omega$, we have $p_{vt} \in \Omega$. Thus, the path p_{vt} is a basic factor of Ω where $\omega(p_{vt}) > 0$ and $\deg_{p_{vt}}(u) = 0$.

(c) $V_\cap = \{v, s\}$ or $\{v, t\}$.
These two cases are symmetric. We only consider the case that $V_\cap = \{v, s\}$. In this case, $\deg_H(v) = \deg_H(s) = 3$, $\deg_H(u) = \deg_H(t) = 1$, $\pi(v) = \{0, 1, 3\}$ since $\deg_F(v) = 1 \in \pi(v)$, and $\pi(s) = \{0, 2, 3\}$ since $\deg_F(s) = 2 \in \pi(s)$. The point s splits F into two paths p_{us} and p_{sv}. Then, $F = p_{us} \cup p_{sv}$. The point v splits F' into two paths p'_{sv} and p_{vt}. Then, $F' = p'_{sv} \cup p_{vt}$. (See Fig. 4.)
Consider the path $p'_{uv} = p_{us} \cup p'_{sv}$. Note that $\deg_{p'_{uv}}(s) = 2 \in \pi(s)$. Then, p'_{uv} is a basic factor of Ω. Since, F is a basic factor of Ω with the largest weight, we have

$$\omega(p_{us}) + \omega(p_{sv}) = \omega(F) \geq \omega(p'_{uv}) = \omega(p_{us}) + \omega(p'_{sv}).$$

Thus, $\omega(p_{sv}) \geq \omega(p'_{sv})$. Let F^* be the tadpole graph $q_{tv^3} = p_{sv} \cup p'_{sv} \cup p_{vt}$. Note that $\deg_{F^*}(s) = 2 \in \pi(s)$ and $\deg_{F^*}(v) = 3 \in \pi(v)$. Then, F^* is a basic factor of Ω and $\deg_{F^*}(u) = 0$. Also, the path p_{vt} is a basic factor of Ω since $\deg_{p_{vt}}(v) = 1 \in \pi(v)$, and $\deg_{p_{vt}}(u) = 0$. Then,

$$\begin{aligned}\omega(F^*) + \omega(p_{vt}) &= \omega(p_{sv}) + \omega(p'_{sv}) + \omega(p_{vt}) + \omega(p_{vt}) \\ &\geq 2(\omega(p'_{sv}) + \omega(p_{vt})) = 2\omega(F') > 0.\end{aligned}$$

Thus, among $\omega(F^*)$ and $\omega(p_{vt})$, at least one is positive. Thus, F^* or p_{vt} is a desired basic factor of Ω that satisfies the requirements.

(d) $V_\cap = \{s, t\}$.
In this case, $\deg_H(u) = \deg_H(v) = 1$, $\deg_H(s) = \deg_H(t) = 3$, and $\pi(s) = \pi(t) = \{0, 2, 3\}$. The points s and t split F into three paths. Without loss of

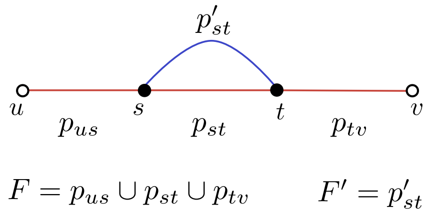


Fig. 5 The graph H in Case I.1.(d)

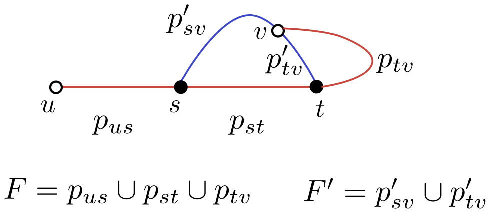


Fig. 6 The graph H in Case I.1.(e)

generality, we may assume that s is closer to u and t is closer to v. Then, the three paths are p_{us}, p_{st}, and p_{tv}, and $F = p_{us} \cup p_{st} \cup p_{ts}$. Also, F' is a path with endpoints s and t, which is disjoint with p_{st}. (See Fig. 5.)

Consider the path $p'_{uv} = p_{us} \cup F' \cup p_{tv}$. One can check that p'_{uv} is a basic factor of Ω. Then, $\omega(F) \geq \omega(p'_{uv})$. Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Consider the cycle $F^* = F' \cup p_{st}$. Also, one can check that F^* is a basic factor of Ω. Moreover, $\omega(F^*) = \omega(F') + \omega(p_{st}) > 0$ and $\deg_{F^*}(u) = 0$. We are done.

(e) $V_\cap = \{v, s, t\}$.

In this case, $\deg_H(u) = 1$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, $\pi(v) = \{0, 1, 3\}$, and $\pi(s) = \pi(t) = \{0, 2, 3\}$. The points s and t split F into three paths. Without loss of generality, we assume that they are p_{us}, p_{st}, and p_{tv}. Then, $F = p_{us} \cup p_{st} \cup p_{tv}$. The point v splits F' into two paths, p'_{sv} and p'_{tv}. Then, $F' = p'_{sv} \cup p'_{tv}$. (See Fig. 6.)

Consider the path $p'_{uv} = p_{us} \cup p'_{sv}$. One can check that it is a basic factor of Ω. Since $\omega(F) \geq \omega(p'_{uv})$, we have

$$\omega(p_{st}) + \omega(p_{tv}) \geq \omega(p'_{sv}).$$

Consider the path $p''_{uv} = p_{us} \cup p_{st} \cup p'_{tv}$. One can check that it is also a basic factor of Ω. Since $\omega(F) \geq \omega(p''_{uv})$, we have

$$\omega(p_{tv}) \geq \omega(p'_{tv}).$$

Consider the tadpole graph $q_{uv^3} = p_{us} \cup p'_{sv} \cup p'_{tv} \cup p_{tv}$. One can check that it is also a basic factor of Ω. Since $\omega(F) \geq \omega(q_{uv^3})$, we have

$$\omega(p_{st}) \geq \omega(p'_{sv}) + \omega(p'_{tv}).$$

Sum up the above three inequalities, and we have

$$2(\omega(p_{st}) + \omega(p_{tv})) \geq 2(\omega(p'_{sv}) + \omega(p'_{tv})) = 2\omega(F') > 0.$$

Consider the theta graph $F^* = p_{st} \cup p_{tv} \cup p'_{sv} \cup p'_{tv}$. Still one can check that it is a basic factor of Ω. Moreover,

$$\omega(F^*) = \omega(p_{st}) + \omega(p_{tv}) + \omega(p'_{sv}) + \omega(p'_{tv}) > 0$$

and $\deg_{F^*}(u) = 0$. We are done.

We are done with Case I.1 where F and F' are both paths.

I.2. F is a path and F' is a tadpole graph. Without loss of generality, we may assume that $\deg_{F'}(s) = 1$ and $\deg_{F'}(t) = 3$. In other words, F' consists of a path with endpoints s and t, and a cycle C containing the vertex t. Then, $V_\cap \subseteq \{v, s\}$. There are three subcases: $V_\cap = \{v\}$, $V_\cap = \{s\}$, and $V_\cap = \{v, s\}$.

(a) $V_\cap = \{v\}$.
In this case, $\deg_H(u) = \deg_H(s) = 1$, $\deg_H(v) = \deg_H(t) = 3$, $\pi(v) = \{0, 1, 3\}$, and $\pi(t) = \{0, 1, 3\}$ or $\{0, 2, 3\}$. There are two subcases depending on whether the intersection point v appears in the path part or the cycle part of F'.

i. v appears in the path part.
Note that for every $x \in V_C \backslash \{t\}$, $\deg_H(x) = 2$, and $\deg_H(t) = 3$. We say such a cycle with exactly one vertex of degree 3 in H is a *dangling* cycle in H. Let e_t be the edge incident to t where $e_t \notin E_C$. We call the vertex t the *connecting point* of C, and the edge e_t the *connecting bridge* of C. Recall that $\Omega_H = (H, \pi_H, \omega)$ is a sub-instance of Ω induced by H.
Consider the graph $H' = H \backslash C$. Notice that $\deg_{H'}(x) = \deg_H(x)$ for every $x \in V_{H'} \backslash \{t\}$ and $\deg_{H'}(t) = 1$. Consider the instance $\Omega_{H'} = (H', \pi_{H'}, \omega_{H'})$ where $\pi_{H'}(x) = \pi_H(x)$ for every $x \in V_{H'} \backslash \{t\}$ and $\pi_{H'}(t) = \{0, 1\}$, and $\omega_{H'}(e) = \omega(e)$ for every $e \in E_{H'} \backslash \{e_t\}$ and $\omega_{H'}(e_t) = \omega(e_t) + \omega(C)$. In other words, the instance $\Omega_{H'}$ is obtained from Ω_H by contracting the dangling cycle C to its connecting point t and adding the total weight of C to its connecting bridge e_t. Clearly, $\Omega_{H'}$ is a key instance and $\omega_{H'}(H') = \omega(H') + \omega(C) = \omega(H) > 0$.
For every factor $K' \in \Omega_{H'}$, we can recover a factor $K \in \Omega_H$ from K' as follows: $K = K'$ if $e_t \notin E_{K'}$ and $K = K' \cup C$ if $e_t \in E_{K'}$. One can check that K is a factor of Ω_H, and $\omega(K) = \omega_{H'}(K')$. If $e_t \notin K$, then $K' = K$. Clearly, K' is a basic factor of $\Omega_{H'}$ if and only if K is a basic factor of Ω. Now, suppose that $e_t \in K$. Remember that $\deg_{H'}(t) = 1$. Then, K' is a path with t as an endpoint if and only if $K = K' \cup C$ is a tadpole graph with t as the vertex of degree 3, and K' is a tadpole with t as the vertex of degree 1 if and only if $K = K' \cup C$ is a dumbbell graph. Thus, K' is a basic factor of $\Omega_{H'}$ if and only if K is a basic factor of Ω_H.
Notice that the instance Ω'_H has a similar structure to the instance Ω_H in Case I.1.(a). By replacing the vertex v in Case I.1.(a) by the cycle C (and re-arranging the weights between the cycle C and its connecting bridge), one can check that the proof of Case I.1.(a) works here. Note that after this replacement, the path p_{vt} in Case I.1.(a) becomes a tadpole graph q_{vt^3} which is still a basic factor.

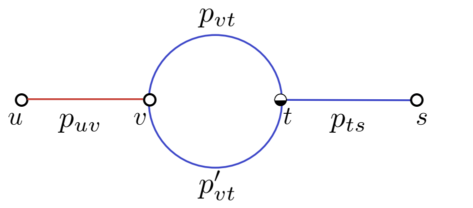


Fig. 7 The graph H in Case I.2.(a)

ii. v appears in the cycle part.
Together with the point t, the point v splits the cycle in F' into two paths p_{vt} and p'_{vt}. Let p_{ts} denote the path in F' with endpoints t and s. Then, $F' = p_{vt} \cup p'_{vt} \cup p_{ts}$. (See Fig. 7.)

- If $\pi(t) = \{0, 1, 3\}$, then the paths p_{vt}, p'_{vt} and p_{ts} are all basic factors of Ω. Moreover, the vertex u does not appear in any of these paths. Also, since $\omega(F') = \omega(p_{vt}) + \omega(p'_{vt}) + \omega(p_{ts}) > 0$, there is at least one path with positive weight. We are done.
- If $\pi(t) = \{0, 2, 3\}$, then the tadpole graph $q_{uv^3} = F \cup p_{vt} \cup p'_{vt}$ is a basic factor or Ω. Since $\omega(F) \geq \omega(q_{uv^3})$, we have $\omega(p_{vt}) + \omega(p'_{vt}) \leq 0$. Without loss of generality, we assume that $\omega(p'_{vt}) \leq 0$. Consider the path $F^* = p_{vt} \cup p_{ts}$. We have $\deg_{F^*}(u) = 0$, and F^* is a basic factor of Ω. Since

$$\omega(F') = \omega(p_{vt}) + \omega(p'_{vt}) + \omega(p_{ts}) = \omega(F^*) + \omega(p'_{vt}) > 0$$

and $\omega(p'_{vt}) \leq 0$, we have $\omega(F^*) > 0$. We are done.

(b) $V_\cap = \{s\}$.
Still, the cycle C in F' is a dangling cycle with the connecting point t. We can contract C to t and add the weight $\omega(C)$ to its connecting bridge. Then, this case is similar to Case I.1.(b). By replacing the vertex t in Case I.1.(b) by the C, one can check that the proof of Case I.1.(b) works here. Note that the path p_{st} in Case I.1.(b) is replaced by a tadpole graph q_{st^3} which is still a basic factor.

(c) $V_\cap = \{v, s\}$.
In this case, $\deg_H(u) = 1$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, $\pi(v) = \{0, 1, 3\}$, $\pi(s) = \{0, 2, 3\}$, and $\pi(t) = \{0, 1, 3\}$ or $\{0, 2, 3\}$. There are two subcases depending on whether the intersection point v appears in the path part or the cycle part of the tadpole graph F'. Note that there is only one way for the intersection point s to appear in the path F, and s always splits F into two paths p_{us} and p_{st}.

i. v appears in the path part.
Still, the cycle C is a dangling cycle with the connecting point t. This case is similar to Case I.1.(c). By replacing the vertex t in Case I.1.(c) by the cycle C, one can check that the proof of Case I.1.(c) works here. Note that the path p_{vt} and the tadpole graph $F^* = q_{tv^3}$ in Case I.1.(c) are replaced

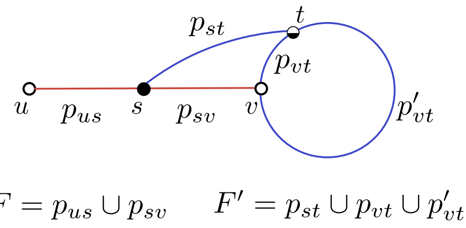


Fig. 8 The graph H in Case I.2.(c)

by a tadpole graph and a dumbbell graph respectively. Both are still basic factors.

ii. v appears in the cycle part.
Together with the point t, the point v splits the cycle in F' into two parts p_{vt} and p'_{vt}. Let p_{st} denote the path in F' with endpoints s and t. Then, $F' = p_{st} \cup p_{vt} \cup p'_{vt}$. (See Fig. 8.)

- If $\pi(t) = \{0, 1, 3\}$, then the paths p_{vt} and p'_{vt} are both basic factors of Ω. If $\omega(p_{vt}) > 0$ or $\omega(p'_{vt}) > 0$, then we have a basic factor satisfying the requirements. Thus, we may assume $\omega(p_{vt}), \omega(p'_{vt}) \leq 0$. Since $\omega(F') = \omega(p_{st}) + \omega(p_{vt}) + \omega(p'_{vt}) > 0$, we have $\omega(p_{st}) > 0$. Consider the path $p_{ut} = p_{us} \cup p_{st}$. It is a basic factor of Ω. Since F is a basic factor of Ω with the largest weight,

$$\omega(p_{us}) + \omega(p_{sv}) = \omega(F) \geq \omega(p_{ut}) = \omega(p_{us}) + \omega(p_{st}).$$

  Then, $\omega(p_{sv}) \geq \omega(p_{st}) > 0$.
  Consider the path $p''_{vt} = p_{sv} \cup p_{st}$. It is a basic factor of Ω and $\deg_{p''_{vt}}(u) = 0$. Also, $\omega(p''_{vt}) = \omega(p_{sv}) + \omega(p_{st}) > 0$. We are done.
- If $\pi(t) = \{0, 2, 3\}$, then the tadpole graph $q_{uv^3} = F \cup p_{vt} \cup p'_{vt}$ is a basic factor of Ω. Since $\omega(F) \geq \omega(q_{uv^3})$, we have $\omega(p_{vt}) + \omega(p'_{vt}) \leq 0$. Since $\omega(F') > 0$, we have $\omega(p_{st}) > 0$. Consider the path $p'_{uv} = p_{us} \cup p_{st} \cup p_{vt}$. It is a basic factor of Ω. Since $\omega(F) \geq \omega(p'_{uv})$, we have

$$\omega(p_{sv}) \geq \omega(p_{st}) + \omega(p_{vt}).$$

  Similarly, consider the path $p''_{uv} = p_{us} \cup p_{st} \cup p'_{vt}$. We have

$$\omega(p_{sv}) \geq \omega(p_{st}) + \omega(p'_{vt}).$$

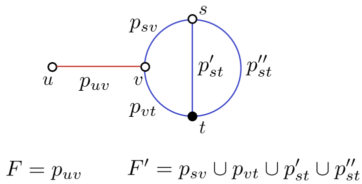
  Sum up the above two inequalities, we have

$$2\omega(p_{sv}) \geq 2\omega(p_{st}) + \omega(p_{vt}) + \omega(p'_{vt}) = \omega(p_{st}) + \omega(F') > 0.$$

  Consider the theta graph $F^* = p_{sv} \cup F'$. Note that F^* is a basic factor of Ω and $\deg_{F^*}(u) = 0$. Also, $\omega(F^*) = \omega(p_{sv}) + \omega(F') > 0$. We are done.

We are done with Case I.2 where F is a path and F' is a tadpole graph.

Fig. 9 The graph H in Case I.4



I.3. F is a path and F' is a dumbbell graph. Then, $V_\cap = \{v\}$.
Let C_s and C_t be the two cycles in F' that contain vertices s and t respectively. Clearly, among V_{C_s} and V_{C_t}, there exists at least one such that it does not contain the intersection point v. Notice that vertices s and t are symmetric in this case. Without loss of generality, we may assume that $v \notin V_{C_s}$. Then, $V_{C_s} \cap V_F = \emptyset$. Thus, C_s is a dangling cycle with the connecting point s. Then, this case is similar to Case I.2.(a). By replacing the vertex s in Case I.2.(a) by the cycle C_s, one can check that the proof of Case I.2.(a) works here.

I.4. F is a path and F' is a theta graph. Then, $V_\cap = \{v\}$.
In this case, $\deg_H(u) = 1$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, and $\pi(v) = \{0, 1, 3\}$. Since F' is a theta graph and $\deg_{F'}(s) = \deg_{F'}(t) = 3$, without loss of generality, we may assume that $\pi(s) = \{0, 1, 3\}$ and $\pi(t) = \{0, 2, 3\}$. F' consists of three paths p_{st}, p'_{st} and p''_{st}. Without loss of generality, we may assume that v appears in the path p_{st} and it splits the path into two paths p_{sv} and p_{vt}. (See Fig. 9.)
Consider the paths $p'_{sv} = p'_{st} \cup p_{vt}$ and $p''_{sv} = p''_{st} \cup p_{vt}$, and the tadpole graph $q_{vs^3} = p_{sv} \cup p'_{st} \cup p''_{st}$. It can be checked that p_{sv}, p'_{sv}, p''_{sv} and q_{vs^3} are all basic factors of H. The vertex u does not appear in any of them. Also,

$$\omega(p_{sv}) + \omega(p'_{sv}) + \omega(p''_{sv}) + \omega(q_{vs^3}) = 2\omega(F') > 0.$$

Then, among them at least one is positive. Thus, we can find a basic factor of Ω satisfying the requirements.

We are done with Case I where F is a path.

Case II: F is a tadpole graph and $\deg_F(u) = 3$. By assumption, $\pi(u) = \{0, 2, 3\}$. Also, since $\deg_F(v) = 1 \in \pi(v)$, v is 1-feasible. Let C be the cycle part of F. Consider $\{s, t\} \cap V_C$. Here, we discuss possible cases depending on intersection vertices belonging to V_C instead of the entire set $V_\cap$ of vertices points as in Case I. There are three subcases.

II.1 $\{s, t\} \cap V_C = \emptyset$.
In this case, $\deg_H(x) = 2$ for every $x \in V_C \backslash \{u\}$. Thus, C is a dangling cycle with in connecting point u in H. Then, the case is similar to Case I. By replacing the vertex u in Case I by the cycle C, one can check that the proof of Case I works here. Note that after the above replacement, a path containing u as an endpoint in Case I becomes a tadpole graph containing the cycle C, and a tadpole graph containing u as the vertex of degree 1 in Case I becomes a dumbbell graph.

II.2 $\{s, t\} \cap V_C = \{s\}$ or $\{t\}$.

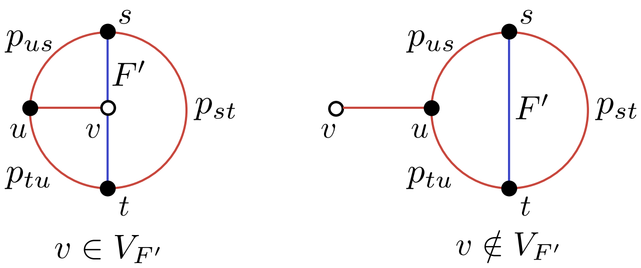


Fig. 10 The two possible forms of graph H in Case II.3

Without loss of generality, we may assume that $s \in V_C$. Then, $\deg_H(u) = \deg_H(s) = 3$ and $\pi(u) = \pi(s) = \{0, 2, 3\}$. If $\omega(C) > 0$, then we are done since C is a basic factor of Ω and $\deg_C(u) = 2$. Thus, we may assume that $\omega(C) \leq 0$. Vertices s and u split C into two paths p_{us} and p'_{us}. Since $\omega(C) = \omega(p_{us}) + \omega(p'_{us}) \leq 0$, among them at least one is non-positive. Without loss of generality, we assume that $\omega(p_{us}) \leq 0$.

Consider the graph $H' = H \backslash p_{us}$. Note that $V_{H'} = (V_H \backslash V_{p_{us}}) \cup \{u, s\}$. For every $x \in V_{H'} \backslash \{u, s\}$, we have $\deg_{H'}(x) = \deg_H(x) \in \pi(x)$ since H is a factor of Ω. Also, $\deg_{H'}(u) = 2 \in \pi(u)$ and $\deg_{H'}(s) = 2 \in \pi(s)$. Thus, H' is a factor of Ω. Also, $\omega(H') = \omega(H) - \omega(p_{us}) > 0$. However, it is not clear whether H' is a *basic* factor of Ω. Consider the sub-instance $\Omega'_H = (H', \pi_{H'}, \omega)$ of Ω induced by the factor H'. Since $\omega(H') > 0$, by Lemma 5.6, there is a basic factor $F^* \in \Omega_{H'}$ such that $\omega(F^*) > 0$. Then, $\deg_{F^*}(u) \in \pi_{H'}(u) = \{0, 2\}$. Clearly, F^* is also a basic factor of Ω. We are done.

Note that this proof works no matter whether F' is a path or a tadpole graph, and whether $v \in V_\cap$ or $t \in V_\cap$. In fact, this proof also works when F is a dumbbell graph as long as s (or symmetrically t) is the only vertex in $V_{F'}$ appearing in the cycle C of F that contains the vertex u.

II.3 $\{s, t\} \subseteq V_C$.

In this case, $\deg_H(u) = \deg_H(s) = \deg_H(t) = 3$ and $\pi(u) = \pi(s) = \pi(t) = \{0, 2, 3\}$. Also, $\deg_{F'}(s) = \deg_{F'}(t) = 1$. Thus, F' is a path with endpoints s and t. Note that in this case, it is possible that $v \in V_{F'}$. If $v \in V_{F'}$, then $\deg_H(v) = 3$ and $\pi(v) = \{0, 1, 3\}$; otherwise, $\deg_H(v) = 1$ and $\pi(v) = \{0, 1\}$. The points u, s, and t split C into three paths, p_{us}, p_{st}, p_{tu}. Then, $C = p_{us} \cup p_{st} \cup p_{tu}$. (See Fig. 10.) If $\omega(C) > 0$, then we are done. Thus, we may assume that $\omega(C) \leq 0$.

Consider the graph $H_1 = H \backslash p_{st} = (F \backslash p_{st}) \cup F'$. Similar to the above Case II.2, one can check that H_1 is a factor of Ω. Also, H_1 is a tadpole graph if $\deg_H(v) = 1$ or a theta graph if $\deg_H(v) = 3$. Thus, in both cases, H_1 is a *basic* factor of Ω. Since F is a basic factor of Ω with the largest weight, we have

$$\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{st}) + \omega(F').$$

Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Since $\omega(C) = \omega(p_{st}) + \omega(p_{us}) + \omega(p_{tu}) \leq 0$, we have $\omega(p_{us}) + \omega(p_{tu}) < 0$. Without loss of generality, we may assume that $\omega(p_{us}) < 0$. Then, consider the graph $H_2 = H \backslash p_{us}$. Still, one can check that H_2

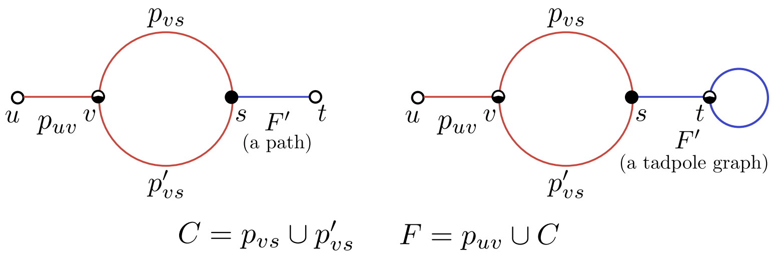


Fig. 11 The two possible forms of graph H in Case III.2 where $t \notin V_{p_{uv}}$

is a factor of Ω, and $\deg_{H_2}(u) = 2$. Also, H_2 is a tadpole graph if $\deg_H(v) = 1$, or a theta graph if $\deg_H(v) = 3$. Thus, H_2 is a basic factor of Ω. Moreover, $\omega(H_2) = \omega(H) - \omega(p_{us}) > 0$. We are done.

Case III: F is a tadpole graph and $\deg_F(v) = 3$. In this case, $\deg_F(u) = 1$, $\pi(u) = \{0, 1\}$, $\deg_F(v) = 3$, and $\pi(v) = \{0, 1, 3\}$ or $\{0, 2, 3\}$. Recall that $\deg_H(u) = \deg_F(u) = 1$, and $u \notin V_\cap$. Let C be the cycle part of F. Still consider $\{s, t\} \cap V_C$. There are three subcases.

III.1 $\{s, t\} \cap V_C = \emptyset$.
In this case, $\deg_H(x) = 2$ for every $x \in V_C \backslash \{v\}$. Thus, in the graph H, the cycle C is a dangling cycle with the connecting point v. Then, the case is similar to Case I. For a graph H in Case I where $\deg_H(v) = 1$ (i.e., $v \notin V_\cap$), by replacing the vertex v by the cycle C, one can check that the proof of Case I works here.

III.2 $\{s, t\} \cap V_C = \{s\}$ or $\{t\}$.
Without loss of generality, we may assume that $s \in V_C$. Then $\deg_H(s) = 3$ and $\pi(s) = \{0, 2, 3\}$. Vertices s and v split C into two paths p_{vs} and p'_{vs}. Let p_{uv} be the path part in the tadpole graph F. There are two subcases depending on whether $t \in V_F$. Since $t \notin V_C$, $t \in V_F$ implies $t \in V_{p_{uv}}$.

(a) $t \notin V_{p_{uv}}$.
In this case, $\deg_H(t) = 1$ or 3 depending on whether F' is a path or a tadpole graph respectively (See Fig. 11). If F' is a tadpole graph, then the cycle in F' containing t is a dangling cycle in H with the connecting point t.

- If $\pi(v) = \{0, 1, 3\}$, then p_{uv} is a basic factor of Ω (this is true no matter whether $t \in V_{p_{uv}}$). Since F is a basic factor of Ω with the largest weight, $\omega(F) \geq \omega(p_{uv})$. Thus, $\omega(p_{vs}) + \omega(p'_{vs}) = \omega(C) = \omega(F) - \omega(p_{uv}) \geq 0$. Without loss of generality, we may assume that $\omega(p_{vs}) \geq 0$. Consider the graph $H' = F' \cup p_{vs}$. It is a path if F' is a path, or a tadpole graph of F' is a tadpole graph. Note that $\deg_{H'}(u) = 0 \in \pi(u)$, $\deg_{H'}(v) = 1 \in \pi(v)$, $\deg_{H'}(s) = 2 \in \pi(s)$, and $\deg_{H'}(t) = \deg_H(t) \in \pi(t)$. Also, for every $x \in V_{H'} \backslash \{u, v, s, t\}$, $\deg_{H'}(x) = \deg_H(x) \in \pi(x)$. Thus, H' is a basic factor of Ω. Also, $\omega(H') = \omega(F') + \omega(p_{vs}) > 0$. We are done.
- If $\pi(v) = \{0, 2, 3\}$, then the cycle C is a basic factor of Ω. Consider $H_1 = H \backslash p_{vs}$. Note that $\deg_{H_1}(u) = 1 \in \pi(u)$, $\deg_{H_1}(v) = 2 \in \pi(v)$, $\deg_{H_1}(s) = 2 \in \pi(s)$, and $\deg_{H_1}(t) = \deg_H(t) \in \pi(t)$. One can check

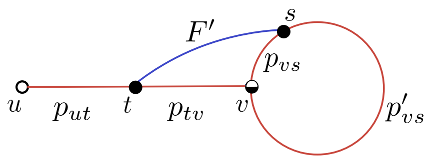


Fig. 12 The graph H in Case III.2 where $t \in V_{p_{uv}}$

that H_1 is a factor of Ω. Also, H_1 is either a path with endpoints u and s if F' is a path, or a tadpole graph with u being the vertex of degree 1 and t being the vertex of degree 3 if F' is a tadpole graph. Thus, H_1 is a basic factor of Ω. Since F is a basic factor with the largest weight,

$$\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{vs}) + \omega(F').$$

Thus, $\omega(p_{vs}) \geq \omega(F') > 0$. Similarly, by considering $H_2 = H \backslash p'_{vs}$, we have $\omega(p'_{vs}) > 0$. Then, $\omega(C) = \omega(p_{vs}) + \omega(p'_{vs}) > 0$. Thus, C is a basic factor of Ω with positive weight and $\deg_C(u) = 0$.

(b) $t \in V_{p_{uv}}$.
In this case, $\deg_H(t) = 3$ and $\pi(t) = \{0, 2, 3\}$. F' is a path with endpoints s and t. The vertex t splits p_{uv} into two parts p_{ut} and p_{tv} (see Fig. 12).

- If $\pi(v) = \{0, 1, 3\}$, then p_{uv} is a basic factor of Ω. Since $\omega(F) \geq \omega(p_{uv})$, we have $\omega(C) \geq 0$. Consider the path $p'_{uv} = p_{ut} \cup F' \cup p_{vs}$. It is also a basic factor of Ω. Still, since $\omega(F) \geq \omega(p'_{uv})$, we have

$$\omega(p_{tv}) \geq \omega(F') + \omega(p_{vs}).$$

Similarly, by considering the path $p''_{uv} = p_{ut} \cup F' \cup p'_{vs}$, we have

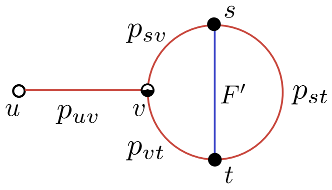
$$\omega(p_{tv}) \geq \omega(F') + \omega(p'_{vs}).$$

Thus, $2\omega(p_{tv}) \geq 2\omega(F') + \omega(p_{vs}) + \omega(p'_{vs}) = 2\omega(F') + \omega(C) > 0$. Consider the theta graph $F^* = p_{tv} \cup C \cup F'$. Clearly, $\omega(F^*) > 0$. Then, F^* is a basic factor of Ω with $\deg_{F^*}(u) = 0$. We are done.

- If $\pi(v) = \{0, 2, 3\}$, then C is a basic factor of Ω. Consider $H_1 = H \backslash p_{vs}$. It is a tadpole graph with the vertex u of degree 1 and the vertex t of degree 3. Note that $\deg_{H_1}(u) = 1 \in \pi(u)$, $\deg_{H_1}(v) = 2 \in \pi(v)$, $\deg_{H_1}(s) = 2 \in \pi(s)$, and $\deg_{H_1}(t) = 3 \in \pi(t)$. One can check that H_1 is a basic factor of Ω. Since F is a basic factor with the largest weight,

$$\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{vs}) + \omega(F').$$

Fig. 13 The graph H in Case III.3



$$C = p_{sv} \cup p_{vt} \cup p_{st} \qquad F = p_{uv} \cup C$$

Thus, $\omega(p_{vs}) \geq \omega(F') > 0$. Similarly, by considering $H_2 = H \backslash p'_{vs}$, we have $\omega(p'_{vs}) > 0$. Then, $\omega(C) = \omega(p_{vs}) + \omega(p'_{vs}) > 0$. Thus, C is a basic factor of Ω with positive weight and $\deg_C(u) = 0$.

III.3 $\{s, t\} \subseteq V_C$. In this case, $\deg_H(u) = 1$, $\pi(u) = \{0, 1\}$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, and $\pi(s) = \pi(t) = \{0, 2, 3\}$. Also, $\deg_{F'}(s) = \deg_{F'}(t) = 1$. Thus, F' is a path with endpoints s and t. Let $p_{st} \subseteq C$ be the path with endpoints t and s such that $v \notin V_{p_{st}}$. (See Fig. 13.)
Consider the tadpole graph $q_{uv^3} = (F \backslash p_{st}) \cup F'$. In other words, q_{uv^3} is the tadpole graph obtained from F by replacing the path p_{st} by F'. One can check that q_{uv^3} is also a basic factor of Ω. Since F is a basic factor of Ω with the largest weight,

$$\omega(F) \geq \omega(q_{uv^3}) = \omega(F) - \omega(p_{st}) + \omega(F').$$

Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Consider the cycle $C' = p_{st} \cup F'$. Note that it is a basic factor of Ω. Also, $\deg_{C'}(u) = 0$ and $\omega(C') = \omega(p_{st}) + \omega(F') > 0$. We are done.

Case IV: F is a dumbbell graph. Let C_u and C_v be the two cycles of F containing vertices u and v respectively.

If $\{s, t\} \cap C_v = \emptyset$, then C_v is a dangling cycle in H with the connecting point v. This case is similar to Case II. For a graph H in Case II where $\deg_H(v) = 1$ (i.e., $v \notin V_\cap$), by replacing the vertex v by the cycle C_v, one can check that the proof of Case II works here.

If $\{s, t\} \cap C_u = \emptyset$, then C_u is a dangling cycle in H with the connecting point u. This case is similar to Case III. By replacing the vertex u in Case III by the cycle C_u, one can check that the proof of Case III works here.

If $\{s, t\} \cap C_u$ and $\{s, t\} \cap C_v$ are both non-empty, then without loss of generality, we may assume that $s \in C_u$ and $t \in C_v$. Thus, F' is a path with endpoints s and t. As we have mentioned in Case II.2, one can check that the proof of Case II.2 works here.

Case V: F is a theta graph. In this case, $\deg_H(u) = \deg_H(v) = 3$. By assumption, $\pi(u) = \{0, 2, 3\}$. Also, by the definition of theta graphs, $\pi(v) = \{0, 1, 3\}$. Then, $V_\cap \subseteq \{s, t\}$. There are two subcases.

V.1 $V_\cap = \{s\}$ or $\{t\}$.
Without loss of generality, we assume that $V_\cap = \{s\}$. Then, $\deg_H(s) = 3$ and $\pi(s) = \{0, 2, 3\}$. The theta graph F consists of three paths p_{uv}, p'_{uv} and p''_{uv}.

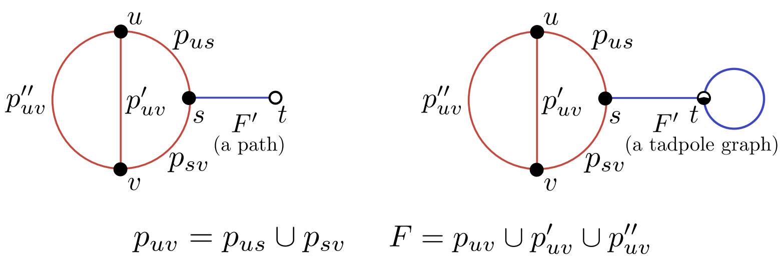


Fig. 14 The two possible forms of graph H in Case V.1

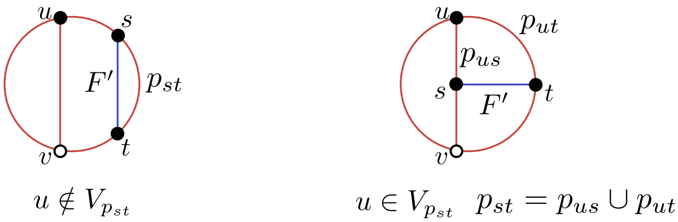


Fig. 15 The two possible forms of graph H in Case V.2 depending on whether $u \in V_{p_{st}}$

Without loss of generality, we may assume that s appears in the path p_{uv} and it splits p_{uv} into two paths p_{us} and p_{sv}. Note that F' is possibly a path or a tadpole graph. (See Fig. 14.)

Consider the paths p_{sv}, $p'_{sv} = p'_{uv} \cup p_{us}$ and $p''_{sv} = p''_{uv} \cup p_{us}$, and the tadpole graph $q_{sv^3} = p_{sv} \cup p'_{uv} \cup p''_{uv}$. They are not factors of H since the degree of s is 1 in all these four graphs. However, by taking the union of F' with any one of them, we can get a basic factor of H and the degree of u in it is even. Since

$$\omega(p_{sv}) + \omega(p'_{sv}) + \omega(p''_{sv}) + \omega(q_{sv^3}) = 2\omega(F) > 0,$$

among them at least one is positive. Also, $\omega(F') > 0$. Then, by taking the union of it with F', we can find a basic factor of Ω satisfying the requirements.

V.2 $V_\cap = \{s, t\}$.

In this case, F' is a path with endpoints s and t. Note that $\deg_F(s) = 2 \in \pi(s)$, and $\deg_F(t) = 2 \in \pi(t)$. Thus, $\pi(s) = \pi(t) = \{0, 2, 3\}$. Since F is a theta graph which is 2-connected, we can find a path $p_{st} \subseteq F$ such that $v \notin V_{p_{st}}$. Then, there are two cases depending on whether u appears in the path p_{st}. (See Fig. 15.)

If $u \notin V_{p_{st}}$, then one can check that the theta graph $H' = (F \backslash p_{st}) \cup F'$ is also a basic factor of Ω. Since $\omega(F) \geq \omega(H')$, we have $\omega(p_{st}) \geq \omega(F') > 0$. Then, the cycle $C = p_{st} \cup F'$ is a basic factor of Ω where $\omega(C) = \omega(p_{st}) + \omega(F') > 0$ and $\deg_C(u) = 0$. We are done. Otherwise, $u \in V_{p_{st}}$. The vertex u splits p_{st} into two paths p_{us} and p_{ut}. Consider the theta graph $H' = (F \backslash p_{us}) \cup F'$, where $\deg_{H'}(v) = \deg_{H'}(t) = 3$, $\pi(v) = \{0, 1, 3\}$, and $\pi(t) = \{0, 2, 3\}$. One can check that H' is a basic factor of Ω. Since $\omega(F) \geq \omega(H') = \omega(F) - \omega(p_{us}) + \omega(F')$, we have $\omega(p_{us}) \geq \omega(F') > 0$. Similarly, by considering the theta graph $H'' = (F \backslash p_{ut}) \cup F'$, we have $\omega(p_{ut}) \geq \omega(F') > 0$. Then the cycle $C = p_{st} \cup F' =$

$p_{us} \cup p_{ut} \cup F'$ is a basic factor of Ω where $\omega(C) = \omega(p_{us}) + \omega(p_{ut}) + \omega(F') > 0$ and $\deg_C(u) = 2$. We are done.

We have taken care of all possible cases and finished the proof. □

Combining Lemmas 5.6 and 5.7, we finished the proof of Theorem 4.8.

A Appendix

A.1 Δ-matroids and matching realizability

A Δ-matroid is a family of sets obeying an axiom generalizing the matroid exchange axiom. Formally, a pair $M = (U, \mathscr{F})$ is a Δ-matroid if U is a finite set and $\mathscr{F}$ is a collection of subsets of U satisfying the following: for any $X, Y \in \mathscr{F}$ and any $u \in X \Delta Y$ in the symmetric difference of X and Y, there exits a $v \in X \Delta Y$ such that $X \Delta \{u, v\}$ belongs to $\mathscr{F}$ [4]. A Δ-matroid is *symmetric* if, for every pair of $X, Y \subseteq U$ with $|X| = |Y|$, we have $X \in \mathscr{F}$ if and only if $Y \in \mathscr{F}$. A Δ-matroid is *even* if for every pair of $X, Y \in \mathscr{F}$, $|X| \equiv |Y| \mod 2$.

Suppose that $U = \{u_1, u_2, \ldots, u_n\}$. A subset $V \subseteq U$ can be encoded by a binary string α_V of n-bits where the i-th bit of α_V is 1 if $u_i \in V$ and 0 if $u_i \notin V$. Then, a Δ-matroid $M = (U, \mathscr{F})$ can be represented by a relation R_M of arity $|U|$ which consists of binary strings that encode all subsets in $\mathscr{F}$. Such a representation is unique up to a permutation of variables of the relation. A degree constraint D of arity n can be viewed as an n-ary symmetric relation which consists of binary strings with the Hamming weight d for every $d \in D$. By the definition of Δ-matroids, it is easy to check that a degree constraint D (as a symmetric relation) represents a Δ-matroid if and only if D has all gaps of length at most 1.

Definition A.1 *(Matching Gadget)* A *gadget* using a set $\mathscr{D}$ of degree constraints consists of a graph $G = (U \cup V, E)$ where $\deg_G(u) = 1$ for every $u \in U$ and there are no edges between vertices in U, and a mapping $\pi : V \to \mathscr{D}$. A *matching gadget* is a gadget where $\mathscr{D} = \{\{0, 1\}, \{1\}\}$. A degree constraint D of arity n is *matching realizable* if there exists a matching gadget $(G = (U \cup V, E), \pi : V \to \{\{0, 1\}, \{1\}\})$ such that $|U| = n$ and for every $k \in [n]$, $k \in D$ if and only if for every $W \subseteq U$ with $|W| = k$, there exists a perfect matching $F = (V_F, E_F)$ on the vertex set $V_F \subseteq U \cup V$ such that $V_F \cap U = W$ and for every $v \in V$ where $\pi(v) = \{1\}$, $v \in V_F$.

The definition of matching realizability can be extended to a relation R of arity n by requiring the set U of n vertices in a matching gadget to represent the n variables of R. If R is realizable by a matching gadget $G = (U \cup V, E)$, then for every $\alpha \in \{0, 1\}^n$, $\alpha \in R$ if and only if there is a matching $F = (V_F, E_F)$ of G such that $V_F \cap U$ is exactly the subset of U encoded by α (i.e., for every $u_i \in U$, $u_i \in V_F$ if and only if $\alpha_i = 1$), and for every $v \in V$ where $\pi(v) = \{1\}$, $v \in V_F$. Note that the matching realizability of a relation is invariant under a permutation of its variables. We say that a

Δ-matroid is matching realizable if the relation representing it is matching realizable,[9] The following is an equivalent definition for the matching realizability of Δ-matroids.

Lemma A.2 *If a Δ-matroid $M = (U, \mathscr{F})$ is matching realizable, then there is a graph $G = (U \cup W \cup X, E)$ where $\deg(v) = 1$ for every $v \in U \cup X$ and there are no edges between vertices in $U \cup X$, such that for every $V \subseteq U$, $V \in \mathscr{F}$ if and only if there exists $X_1 \subseteq X$ such that the induced subgraph of G induced by the vertex set $V \cup W \cup X_1$ (denoted by $G(V \cup W \cup X_1)$) has a* perfect *matching.*

With a slight abuse of notation, we also say the graph $G = (U \cup W \cup X, E)$ realizes M.

Proof Let $G = (U \cup W, E)$ be the matching gadget realizing $M = (U, \mathscr{F})$. We construct the following graph G' from G. For every $x \in W$ with $\pi(x) = \{0, 1\}$, we add a new edge incident to it. As the edge is added, a new vertex of degree of 1 is also added to the graph. We denote these new vertices by X and these new edges by E_X. Then, one can check that the graph $G' = (U \cup W \cup X, E \cup E_X)$ satisfies the requirements. □

The following result generalizes Lemma A.1 of [26].

Lemma A.3 *Suppose that $M = (U, \mathscr{F})$ is a matching realizable Δ-matroid, and $V_1, V_2 \in \mathscr{F}$. Then, $V_1 \Delta V_2$ can be partitioned into singletons $S_1, \ldots, S_k$ and pairs $P_1, \ldots, P_\ell$ of vertices such that for every $P = S_{i_1} \cup \cdots \cup S_{i_r} \cup P_{j_1} \cup \cdots \cup P_{j_t}$ ($\{i_1, \ldots, i_r\} \subseteq [k], \{j_1, \ldots, j_t\} \subseteq [\ell]$), $V_1 \Delta P \in \mathscr{F}$ and $V_2 \Delta P \in \mathscr{F}$.*

Proof By Lemma A.2, there is a graph $G = (U \cup W \cup X, E)$ realizing M. Since $V_1, V_2 \in \mathscr{F}$, there exists $X_1 \subseteq X$ and $X_2 \subseteq X$ such that the induced subgraph $G(V_1 \cup W \cup X_1)$ has a perfect matching M_1, and $G(V_2 \cup W \cup X_2)$ has a perfect matching M_2. Let E_1 and E_2 be the edge sets of M_1 and M_2 respectively. Consider the graph $G' = (U \cup W \cup X, E_1 \Delta E_2)$. Since E_1 covers each vertex in $V_1 \cup W \cup X_1$ exactly once, and E_2 covers each vertex in $V_2 \cup W \cup X_2$ exactly once, for every $v \in (V_1 \cap V_2) \cup W \cup (X_1 \cap X_2)$ in G', $\deg_{G'}(v) = 0$ or 2, and for every $v \in (V_1 \Delta V_2) \cup (X_1 \Delta X_2)$ in G', $\deg_{G'}(v) = 1$. Thus, G' is a union of cycles and paths, where each path connects two vertices in $(V_1 \Delta V_2) \cup (X_1 \Delta X_2)$. For every vertex $u \in V_1 \Delta V_2$, if it is connected to another vertex $v \in V_1 \Delta V_2$ by a path in G', then we make $\{u, v\}$ a pair. Otherwise (i.e., u is connected to a vertex in $X_1 \Delta X_2$ by a path in G'), we make $\{u\}$ a singleton. Then, $V_1 \Delta V_2$ can be partitioned into singletons $S_1, \ldots, S_k$ and pairs $P_1, \ldots, P_\ell$ according to the paths in G'.

Moreover, each path in G' is an alternating path with respect to both matchings M_1 and M_2. Pick a union of such paths (note that they are edge-disjoint). Suppose that there are r many paths that connect single vertices in $S_{i_1}, \ldots, S_{i_r}$ with vertices in X, and t many paths that connect pairs $P_{j_1}, \ldots, P_{j_t}$. Let $P = S_{i_1} \cup \cdots \cup S_{i_r} \cup P_{j_1} \cup \cdots \cup P_{j_t}$. After altering the matchings M_1 and M_2 according to these t many alternating paths, we obtain two new matchings that cover exactly $(V_1 \Delta P) \cup W \cup X_1'$ for some $X_1' \subseteq X$

[9] This definition of matching realizability for Δ-matroids is different with the one that is usually used for even Δ-matroids [5, 9, 26] in which the gadget is only allowed to use the constraint $\{1\}$ for perfect matchings, and hence the resulting Δ-matroid must be even.

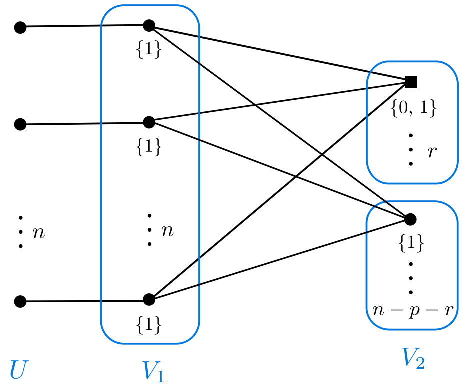


Fig. 16 A matching gadget realizing $D = \{p, p+1, \ldots, p+r\}$ of arity n

and $(V_2 \Delta P) \cup W \cup X'_2$ for some $X'_2 \subseteq X$ respectively. Thus, $V_1 \Delta P \in \mathscr{F}$ and $V_2 \Delta P \in \mathscr{F}$. □

Theorem A.4 *A degree constraint D of gaps of length at most* 1 *is matching realizable if and only if all its gaps are of the same length* 0 *or* 1.

Proof By the gadget constructed in the proof of [7, Theorem 2], if a degree constraint has all gaps of length 1 then it is matching realizable.[10] We give the following gadget (Fig. 16) to realize a degree constraint D with all gaps of length 0, which generalizes the gadget in [42]. Suppose that $D = \{p, p+1, \ldots, p+r\}$ of arity n where $n \geq p+r \geq p \geq 0$. Consider the following graph $G = (U \cup V, E)$: U consists of n vertices of degree 1, and V consists of two parts V_1 with $|V_1| = n$ and V_2 with $|V_2| = n - p$; the induced subgraph $G(V)$ of G induced by V is a complete bipartite graph between V_1 and V_2, and the induced subgraph $G(U \cup V_1)$ of G induced by $U \cup V_1$ is a bipartite perfect matching between U and V_1. Every vertex in V_1 is labeled by the constraint $\{1\}$. There are r vertices in V_2 labeled by $\{0, 1\}$ and the other $n - p - r$ vertices in V_2 labeled by $\{1\}$. One can check that this gadget realizes D.

For the other direction, without loss of generality, we may assume that $\{p, p+1, p+3\} \subseteq D$ and $p+2 \notin D$. Since D has gaps of length at most 1, it can be associated with a symmetric Δ-matroid $M = (U, \mathscr{F})$. Then, there is $V_1 \in \mathscr{F}$ with $|V_1| = p$ and $V_2 \in \mathscr{F}$ with $|V_2| = p+3$. Since M is symmetric, we may pick $V_2 = V_1 \cup \{v_1, v_2, v_3\}$ for some $\{v_1, v_2, v_3\} \cap V_1 = \emptyset$. Let $S = V_1 \Delta V_2 = \{v_1, v_2, v_3\}$. By Lemma A.3, S can be partitioned into singletons and/or pairs of vertices such that for any union P of them, $V_2 \backslash P \in \mathscr{F}$. Since $|S| = 3$, there exists at least one singleton $\{v_i\}$ in the partition of S such that $V_2 \backslash \{v_i\} \in \mathscr{F}$. Note that $|V_2 \backslash \{v_i\}| = p+2$. Thus, $p+2 \in D$. A contradiction. □

[10] We remark that [7] includes gadgets for other types of degree constraints, including type-1 and type-2, but only under a more general notion of gadget constructions that involve edges and triangles. The gadget that only involves edges is a matching gadget defined in this paper.

A.2 Application to the terminal backup problem

The terminal backup problem originates from network design. Similar to the Steiner tree problem, in this problem, we are given a graph consisting of terminal (required) nodes, Steiner (optimal) nodes, and edges with costs. The goal is to connect every terminal node to at least one other terminal node to ensure backup functionality. Specifically, we need to construct a minimum-cost forest where each connected component contains at least two terminal nodes. This ensures that the facilities connected within a component can back up their data, and if one facility fails, another will hold its data. By a detailed reduction, it has been shown that the terminal backup problem is equivalent to the simplex cover problem [43], which can further be reduced to the simplex matching problem.

For a hypergraph $H = (V, E)$ containing edges of sizes 2 and 3 with edge costs $c : E \to \mathbb{R}_{\geq 0}$, we say it satisfies the *simplex condition* if for every edge (u_1, u_2, u_3) of size 3, the three corresponding edges (u_1, u_2), (u_1, u_3) and (u_2, u_3) of size 2 exist, with the cost relation of

$$c(u_1, u_2) + c(u_1, u_3) + c(u_2, u_3) \leq 2c(u_1, u_2, u_3).$$

Definition A.5 *(Simplex cover and simplex matching)* Given a hypergraph $H = (V, E)$ satisfying the simplex condition, the simplex cover problem is to find an edge cover $C \subseteq E$ (i.e., every vertex of H is incident to at least one edge in C) that minimizes the total cost; the simplex matching problem is to find a perfect matching $M \subseteq E$ (i.e., every vertex of H is incident to exactly one edge in C) that minimizes the total cost.

Lemma A.6 *The terminal backup problem is equivalent to the simplex cover problem. Moreover, the simplex cover problem can be reduced to the simplex matching problem.*

A weakly polynomial-time algorithm for the terminal backup problem was provided by solving the simplex matching problem in [2]. However, by directly formulating the terminal backup problem as a WGFP, our approach provides a strongly polynomial-time algorithm for the problem.

Lemma A.7 ([43]) *There is an optimal solution of the terminal backup problem such that all its connected components are either*

- *paths connecting two terminal nodes, or*
- *stars with at least three terminals as the leaves and a Steiner node at the center.*

In other words, there is an optimal solution of the terminal backup problem in which each terminal node has degree 1 *and each Steiner node has degree not equal to* 1.

Consider a terminal backup problem defined on the graph G with the edge weight function $\omega : E \to \mathbb{R}_{\geq 0}$. We first reduce it to an equivalent terminal backup problem on a graph G' such that every Steiner node in G' has degree at most 3. For every Steiner node v of G with $\deg_G(v) = k > 4$, we replace v with a cycle C of length k, whose vertices are denoted by $v_1, v_2, \ldots, v_k$. Each edge in G incident to v is reconnected to exactly one vertex v_i in G'. All edges in the newly added cycle C are assigned weight zero. As a result, we obtain a graph G' in which every Steiner node has degree at most 3, together with an edge weight function ω' defined by $\omega'(e) = \omega(e)$ for original edges and $\omega'(e) = 0$ for edges added in the cycles. One can verify that the optimal value of the terminal backup problem on G' with respect to ω' is equal to that of the original instance.

Now, we consider the following WGFP instance $\Omega = (G', \pi, -\omega')$ where $\pi(u) = \{1\}$ (with arity $\deg_{G'}(v)$) for every terminal node u, and $\pi(v) = \{0, 2, 3\} \cap [\deg_{G'}(v)]$ for every Steiner node v. Note that $\deg_{G'}(v) \leq 3$. Under this formulation, the optimal value of the terminal backup problem equals $-\operatorname{Opt}(\Omega)$. Since the degree constraint $\{0, 2, 3\} \cap [\deg_{G'}(v)]$ is either a type-2 constraint or a parity interval, $-\operatorname{Opt}(\Omega)$ can be computed using our algorithm. Therefore, we obtain a strongly polynomial-time algorithm for the terminal backup problem.

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Data Availability All data is provided in full in the results section of this paper.

Declarations

Conflict of interest The authors declare no conflict of interest.

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